QUESTION (HD 1802): In a personal communication, Professor Gyu Whan Chang wrote: I have the following objection to the proof of your Theorem 1 of your paper, with Tiberiu Dumitrescu, on, "Characterizing domains of finite *-character" (JPAA 214 (11(2010) 2087-2091.)

In line -4 ~-1 of page 2088,

you said that "If V_n is not homogeneous, then V_n is contained in at least two *-comaximal elements

which are *-comaximal with V_1, ..., V_{n-1}. This contradicts the maximality of U."

But why is this a contradiction ? If W_1, W_2 are the two *-comaximal elements,

then U is contained in W = {V_1, ... , V_{n-1}, W_1, W_2} ?

(I think you thought that U is contained in W, which contradicts the maximality of U.

But U is not contained in W as a set.)

ANSWER: A very astute observation indeed and we stand corrected. But, before I answer and put forward the necessary correction, let me put the other readers up to speed on this by first giving below the statement of the theorem along with its proof, marking the under scrutiny part of the proof .

Theorem A (Theorem 1 of [1] . Let D be an integral domain, $*$ a finite character star operation on D and let Γ be a set of proper, nonzero, $*$-ideals of finite type of D such that every proper nonzero $*$-finite $*$-ideal of D is contained[1] in some member of Γ. Let I be a nonzero finitely generated ideal of D with $I^* \neq D$. Then I is contained in an infinite number of maximal $*$-ideals if and only if there exists an infinite family of mutually $*$-comaximal ideals in Γ containing I. Equivalently, with the same assumption on I, I is contained in at most a finite number of $*$-maximal ideals if and only if I is contained in at most a finite number mutually $*$-comaximal members of Γ.

Call a proper $*$-finite $*$-ideal A of D *homogeneous* if A is contained in a unique maximal $*$-ideal.

Lemma B (Lemma 2 of [1]). Let D be a domain, $*$ a finite character star operation on D and let Γ be a set of $*$-finite $*$-ideals of D as described in Theorem A. A proper $*$-finite $*$-ideal A of D is homogeneous if and only if whenever $B, C \in \Gamma$ are containing A, we get $(B, C)^* \neq D$.

Proof. ($\Rightarrow$). Suppose that M is the only maximal $*$-ideal containing A and $B, C \in \Gamma$ ideals containing A. Then $B, C \subseteq M$, so $(B, C)^* \neq D$. ($\Leftarrow$). Suppose that A is contained in two distinct maximal $*$-ideals M_1, M_2. Hence $(M_1, M_2)^* = D$, so we can choose finitely generated ideals $F_i \subseteq M_i$, $i = 1, 2$, such that $A \subseteq F_i^*$ and $(F_1, F_2)^* = D$. There exist $G_1, G_2 \in \Gamma$ such that $F_i \subseteq G_i$, $i = 1, 2$. Hence $A \subseteq G_1, G_2$ and $(G_1, G_2)^* = D$. ■

Proof. (of Theorem *A)* The implication ($\Leftarrow$) is clear since a maximal $*$-ideal cannot contain two $*$-comaximal $*$-ideals. ($\Rightarrow$). Deny. So the following condition holds: ($\sharp$) there is no infinite family of mutually $*$-comaximal ideals in Γ containing I, Γ as defined in Theorem A. First we show the following property: ($\sharp\sharp$) every proper $*$-finite $*$-ideal $I' \supseteq I$ is contained in some homogeneous ideal. Deny. As I' is not homogeneous, there exist $P_1, N_1 \in \Gamma$ such that

$I' \subseteq P_1, N_1$ and $(P_1, N_1)^* = D$ (cf. Lemma B). Since N_1 is not homogeneous, there exist $P_2, N_2 \in \Gamma$ such that $N_1 \subseteq P_2, N_2$ and $(P_2, N_2)^* = D$. Note that $(P_1, P_2)^* = (P_1, N_2)^* = D$. By induction, we can construct an infinite sequence $(P_k)_{k \geq 1}$ of mutually $*$-comaximal ideals in Γ with $I' \subseteq P_k$, $k \geq 1$. This fact contradicts condition ($\sharp$). So ($\sharp\sharp$) holds. To show that I is contained in at most a finite number of maximal $*$-ideals we proceed as follows. Let $\mathcal{S}$ be the family of sets of mutually $*$-comaximal members of Γ containing I. Then $\mathcal{S}$ is non-empty by ($\sharp\sharp$). Obviously $\mathcal{S}$ is partially ordered under inclusion. Let $A_{n_1} \subset A_{n_2} \subset ... \subset A_{n_r} \subset ...$ be an ascending chain of sets in $\mathcal{S}$. Consider $T = \cup A_{n_r}$. We claim that the members of T are mutually $*$-comaximal. For take $x, y \in T$, then $x, y \in A_{n_i}$, for some i, and hence are $*$-comaximal. Having established this we note that by ($\sharp$), T must be finite and hence must be equal to one of the A_{n_j}. Thus by Zorn's Lemma, $\mathcal{S}$ must have a maximal element $U = \{V_1, V_2, ..., V_n\}$. That each of V_i is homogeneous follows from the observation that ... begin under scrutiny part .... ■

"if any of the V_i, say V_n by a relabeling, is nonhomogeneous then by Lemma B V_n is contained in at least two $*$-comaximal elements which by dint of containing V_n are $*$-comaximal with $V_1, ..., V_{n-1}$. This contradicts the maximality of U" .... end under scrutiny part.....

Next let M_i be the maximal $*$-ideal containing V_i for each i and M be a maximal $*$-ideal that contains I and suppose that M does not contain any one of V_i. Then M is $*$-comaximal with each of the M_i. But then there is $x \in M \backslash \cup M_i$. Clearly (x, V_i) is contained in no maximal $*$-ideals and so $(x, V_i)^* = D$. But then $(I, x) \subseteq M$ is $*$-comaximal with each of V_i and by ($\sharp\sharp$), (I, x) is contained in a homogeneous $*$-ideal of finite type which being $*$-comaximal with V_i again contradicts the maximality of U. Consequently I is contained exactly in $M_1, M_2, ..., M_n$. The Equivalently part does not need extra proof being a contrapositive of the result that we have just proven.

Looks like an impossible spot that we are in, but it can be easily remedied by switching to the number of mutually $*$-comaximal elements and saying: Let n be the largest number of mutually $*$-comaximal elements of Γ containing I, say $I \subseteq V_i \in \{V_1, V_2, ..., V_n\}$ and show as you have done above that assuming non-homogeneousness of any of the V_i would cause the number of mutually $*$-comaximal members of Γ containing I to go up, which would indeed be the desired contradiction that leads to the conclusion that the V_i are all homogeneous and then completing the proof as in the paper. But we can avoid entering the Zorn maze altogether and write the proof of the theorem as follows.

Proof. (Alternate proof of Theorem *A.)* The implication ($\Leftarrow$) is clear since a maximal $*$-ideal cannot contain two $*$-comaximal $*$-ideals. ($\Rightarrow$). Deny. So the following condition holds: ($\sharp$) there is no infinite family of mutually $*$-comaximal ideals in Γ containing I, Γ as defined in Theorem A. First we show the following property: ($\sharp\sharp$) every proper $*$-finite $*$-ideal $I' \supseteq I$ is contained in some homogeneous ideal. Deny. As I' is not homogeneous, there exist $P_1, N_1 \in \Gamma$ such that $I' \subseteq P_1, N_1$ and $(P_1, N_1)^* = D$ (cf. Lemma B, which is lemma 2 in the published paper). Since N_1 is not homogeneous, there exist $P_2, N_2 \in \Gamma$ such that $N_1 \subseteq P_2, N_2$ and $(P_2, N_2)^* = D$. Note that $(P_1, P_2)^* = (P_1, N_2)^* = D$.

By induction, we can construct an infinite sequence $(P_k)_{k\geq 1}$ of mutually $*$-comaximal ideals in Γ with $I' \subseteq P_k$, $k \geq 1$. This contradicts condition ($\sharp$). So ($\sharp\sharp$) holds.

From the above procedure we conclude that the ideal I is contained in at most a finite number n of mutually $*$-comaximal members of Γ and that each of them can be assumed to be homogeneous, because of Lemma B. Let $V_1, V_2...., V_n$ be all the mutually $*$-comaximal homogeneous ideals containing I and note that there can be only finitely many of them. Next let M_i be the maximal $*$-ideal containing V_i for each i and M be a maximal $*$-ideal that contains I and suppose that M does not contain any one of the V_i. Then M is $*$-comaximal with each of the M_i. But then there is $x \in M \backslash \cup M_i$. Clearly (x, V_i) is contained in no maximal $*$-ideals and so $(x, V_i)^* = D$. But then $(I, x) \subseteq M$ is $*$-comaximal with each of V_i and by ($\sharp\sharp$), (I, x) is contained in a homogeneous $*$-ideal of finite type which being $*$-comaximal with V_i increases the number of homogeneous $*$-ideals containing I, by one, a contradiction. Consequently I is contained exactly in $M_1, M_2, ..., M_n$. The Equivalently part does not need extra proof being a contrapositive of the result that we have just proven. ■

Remarks (1). It is worth noting that in some situations, it is not really necessary to use Zorn's Lemma and if we do end up using it, we can argue on the number of mutually $*$-comaximal being at most finite, if the situation allows it, as we have seen above. As some of my advisors would say if a big theorem does not directly apply, you do not have to drag it in. Instead create your own alternative theory. But of course your theory has to have a sound basis. For if you can't defend it, it's no theory. As usual with me, I have invited comment from some other Mathematicians. Will, hopefully, include them at the end, as I receive them.

(2). Professor Tiberiu Dumitrescu initially offered a response to Professor Chang's question. Now he has a new soluton. As his approach gives a different proof and not just one that includes the corrective patch, I had no choice but to put it at the end.

Professor Tiberiu Dumitrescu's approach

Prof. Gyu Whan Chang pointed some gaps in the proof of Theorem 1 in [1]. We repair.

Let $(B, \leq)$ be a partially ordered set whose every element is $\leq$ some maximal element. Let $Max(B)$ be the set of maximal elements.

• Call two elements $b_1, b_2 \in B$ *comaximal* if there is no $m \in Max(B)$ such that $b_1, b_2 \leq m$.

• Call $C \subseteq B$ a *comaximal subset* if every two distinct elements in C are comaximal.

• Say that $h \in B$ is *homogeneous* if $a \leq m$ for a unique $m \in Max(B)$.

Proposition 1 *With B as above, let A be a nonempty subset of B such that*

(1) *every non-homogeneous $a \in A$ is $\leq a_1, a_2$ for some comaximal elements $a_1, a_2 \in A$.*

(2) *If* $\{m_1, ..., m_n\}$ *is a proper subset of* $Max(B)$*, there exists some* $a \in A$ *which is not* $\leq m_i$ *for any* i*.*
Then the following are equivalent.

(a) $Max(B)$ *is finite.*

(b) *Every comaximal subset of* A *is finite.*

Proof. Implication $(a) \Rightarrow (b)$ follows easily from definitions.
$(b) \Rightarrow (a)$. We first prove:

Claim $(\sharp)$. *Every* $a_0 \in A$ *is* $\leq$ *some homogeneous element.* Deny. By (1), we have $a_0 \leq b_1, c_1$ for some comaximal elements $b_1, c_1 \in A$. Again by (1), we have $c_1 \leq b_2, c_2$ for some comaximal elements $b_2, c_2 \in A$. Note that b_1,b_2 are comaximal. Continuing in this way, we construct the infinite comaximal subset $\{b_n | n \geq 1\}$ of A, which is a contradiction.

From (b) and $(\sharp)$, there exists a finite comaximal set $\{h_1, ..., h_n\}$ consisting of homogeneous elements of A such that every $x \in A$ is not comaximal to some h_i. As h_i is homogeneous, $h_i \leq m_i$ for a unique $m_i \in Max(B)$. It follows that every $x \in A$ is $\leq$ some m_i, hence $Max(B) = \{m_1, ..., m_n\}$, cf. (2). ■

Theorem 2 *Let* D *be a domain and* $*$ *a finite character star operation on* D*. Let* E *be a nonempty set of ideals of* D *such that*

(i) $J^* \neq D$ *for each* $J \in E$*.*

(ii) *If* $J \in E$*,* $x \in D$ *and* $(J, x)^* \neq D$*, then* (J, x) *is contained in some* $H \in E$*.*

Then, for a fixed $I_0 \in E$*, the following are equivalent:*

(a) I_0 *is contained in only finitely many maximal* $*$*-ideals.*

(b) *Every* $C \subseteq E$ *of mutually* $*$*-comaximal ideals of* E *containing* I_0 *is finite.*

Proof. Apply Proposition 1 for $B := \{H |\ H \text{ ideal of } D,\ I_0 \subseteq H \text{ and } H^* \neq D\}$ and $A := B \cap E$.

■

References

[1] T. Dumitrescu and M. Zafrullah, Characterizing domains of finite $*$-character, J. Pure Appl. Algebra 214 (2010), 2087-2091.