QUESTION (HD 1803): Is there a domain D such that $D[X] \subsetneq \cap D_P[X]$, where P ranges over the associated primes of principal ideal of D?

ANSWER. I am afraid the answer is no. I'd first answer the question in the same vein it is asked then give a. Yet before giving an answer, in that vein, it seems pertinent to give some introduction. An associated prime of a principal ideal is a prime ideal minimal over a proper nonzero ideal of the form $(a) : (b)$. It is well known that for an integral domain D we have $D = \cap D_P$ where P ranges over the associated primes of principal ideals of D. Moreover if Q is an associated prime of a principal ideal of the polynomial ring $D[X]$ over the domain D and $Q \cap D = P$ then $Q = P[X]$. All this information can be gleaned from Brewer and Heinzer's paper [Duke Math. J . 41(1974) 1-7]. Let's denote by $assoc(D)$ the set of associated primes of principal ideals of D

Thus $D[X] = \cap D[X]_Q$ where Q ranges over $assoc(D[X])$. Now there are two kinds of associated primes of $D[X]$: (a) $Q \in assoc(D[X])$ with $Q \cap D = P \neq (0)$ and (b) $Q \in assoc(D[X])$ with $Q \cap D = (0)$. Now if Q is an associated prime of a principal ideal with $Q \cap D = (0)$ then as $D[X]_Q \supseteq K[X] \supseteq D_P[X]$ for all $P \in assoc(D)$. This gives $D[X] = \cap D[X]_{P[X]} \cap (\cap D[X]_Q)$ where $P \in assoc(D)$ and $Q \in assoc(D[X])$ as in (b) above.

Now Q, as described in (b) above is such that $D[X]_Q \supseteq K[X] \supseteq D_P[X]$; $P \in assoc(D)$. So, if $R = \cap D[X]_Q$, where Q is as in (b), then $R \supseteq D_P[X]$ for all $P \in assoc(D)$.

We can write $D[X] = \cap D[X]_{P[X]} \cap R$ where P ranges over $assoc(D)$ and $D[X]$ can in turn be written as $D[X] = \cap(D[X]_{P[X]} \cap R)$ where $P \in assoc(D)$. Now for each $P \in assoc(D)$ we have $D[X]_{P[X]} \supseteq D_P[X]$ and since already $R \supseteq D_P[X]$ for all $P \in assoc(D)$ we have for each $P \in assoc(D)$, $D[X]_{P[X]} \cap R \supseteq D_P[X]$. But then $D[X] = \cap D[X]_{P[X]} \cap R \supseteq \cap D_P[X]$ where $P \in assoc(D)$. Since already, $D[X] \subseteq \cap D_P[X]$ where $P \in assoc(D)$ we conclude that $D[X] = \cap D_P[X]$ where $P \in assoc(D)$.

We have essentially established the following observation.

Observation A. Let D be a domain, let $assoc(D)$ be the set of all associated primes of principal ideals of D and let X be an indeterminate over D. Then $D[X] = \cap D_P[X]$ where P ranges over $assoc(D)$.

As we see below, Observation A is a corollary to a simpler more general result: Let D be a domain with quotient field K, let $\{D_\alpha\}_{\alpha \in I}$ be a family of overrings (rings between D and K) of D such that $D = \cap_\alpha D_\alpha$ and let X be an indeterminate over D. Then $D[X] = \cap_\alpha (D_\alpha[X])$.

(Obviously $D[X] \subseteq \cap_\alpha (D_\alpha[X])$. For the reverse inclusion let $f(X) = \sum_{i=0}^n f_i X^i$ $\in \cap_\alpha (D_\alpha[X])$. Then obviously $f(X) \in D_\alpha[X] \subseteq K[X]$ for each α, we have $f(X) \in K[X]$ and for each $\alpha \in I$ we have $A_f = (f_0, f_1, ..., f_n) \subseteq D_\alpha$ for each $\alpha \in I$, forcing $A_f \subset \cap_\alpha D_\alpha = D$ and ensuring $f(X) \in D[X]$.)

Evan Houston's remark: With essentially the same "proof" one can prove: Let D be a domain with quotient field K, let $\{D_\alpha\}_{\alpha \in I}$ be a family of overrings (rings between D and K) of D such that $D = \cap_\alpha D_\alpha$ and let X be an indeterminate over D. Then $D[[X]] = \cap_\alpha (D_\alpha[[X]])$.

Thus if $\{P_\alpha\}_{\in I}$ is a family of prime ideals of a domain D such that $D = \cap_\alpha D_{P_\alpha}$, then $D[X] = \cap_\alpha (D_{P_\alpha}[X])$ and $D[[X]] = \cap_\alpha (D_{P_\alpha}[[X]])$.