QUESTION (HD 1804): I have this question about your paper, On $*$-semi homogeneous domains, that you posted at: https://arxiv.org/pdf/1802.08353.pdf

How can you justify introducing such huge machinery, to explain just unique factorization?

Answer. The paper you refer to has a very small amount of basic theory which involves a device that can be used to create a large number of parallel theories, with some imagination. It consists in assuming that there is a finite character star operation $*$ and in the definition of certain entities called $*$-homogeneous ($*$-homog) ideals. It seems unfair to the readers who do not know anything about the star operations on integral domains, so let me digress a little to give a working introduction to star operations. Readers who are familiar with star operations may skip this part of introduction.

Let D be an integral domain with quotient field K. Let $F(D)$ (resp., $f(D)$) be the set of nonzero fractional ideals (resp., nonzero finitely generated fractional ideals) of D. A star operation $*$ on D is a function on $F(D)$ that satisfies the following properties for every $I, J \in F(D)$ and $0 \neq x \in K$:

(i) $(x)^* = (x)$ and $(xI)^* = xI^*$,

(ii) $I \subseteq I^*$, and $I^* \subseteq J^*$ whenever $I \subseteq J$, and

(iii) $(I^*)^* = I^*$.

Now, an ideal $I \in F(D)$ is a $*$-ideal if $I^* = I$, so a principal ideal is a $*$-ideal for every star operation $*$. Moreover $I \in F(D)$ is called a $*$-ideal of finite type if $I = J^*$ for some $J \in f(D)$. To each star operation $*$ we can associate a star operation $*_s$ defined by $I^{*_s} = \bigcup\{ J^* \mid J \subseteq I \text{ and } J \in f(D) \}$. A star operation $*$ is said to be of finite type, if $I^* = I^{*_s}$ for all $I \in F(D)$. Indeed for each star operation $*$, $*_s$ is of finite character. Thus if $*$ is of finite character $I \in F(D)$ is a $*$-ideal if and only if for each finitely generated subideal J of I we have $J^* \subseteq I$. For $I \in F(D)$, let $I_d = I$, $I^{-1} = (D :_K I) = \{ x \in K \mid xI \subseteq D \}$, $I_v = (I^{-1})^{-1}$, $I_t = \bigcup\{ J_v \mid J \subseteq I \text{ and } J \in f(D) \}$, and $I_w = \{ x \in K \mid xJ \subseteq I$ for some $J \in f(D)$ with $J_v = D \}$. The functions defined by $I \mapsto I_d$, $I \mapsto I_v$, $I \mapsto I_t$, and $I \mapsto I_w$ are all examples of star operations A v-ideal is sometimes also called a divisorial ideal. Given two star operations $*_1, *_2$ on D, we say that $*_1 \leq *_2$ if $I^{*_1} \subseteq I^{*_2}$ for every $I \in F(D)$. Note that $*_1 \leq *_2$ if and only if $(I^{*_1})^{*_2} = (I^{*_2})^{*_1} = I^{*_2}$ for every $I \in F(D)$. The d-operation, t-operation, and w-operation all have finite character, $d \leq \rho \leq v$ for every star operation ρ, and $\rho \leq t$ for every star operation ρ of finite character. We will often use the facts that (a) for every star operation $*$ and $I, J \in F(D)$, $(IJ)^* = (IJ^*)^* = (I^*J^*)^*$, (the $*$-product), (b) $(I+J)^* = (I+J^*)^* = (I^*+J^*)^*$ (the $*$-sum) and (c) $I_v = I_t$ for every $I \in f(D)$. An $I \in F(D)$ is said to be $*$-invertible, if $(II^{-1})^* = D$. If I is $*$-invertible for $*$ of finite character, then both I^* and I^{-1} are $*$-ideals of finite type. An integral domain D is called a Prufer $*$-Multiplication Domain (P$*$MD), for a general star operation $*$, if A is $*_s$-invertible for every $A \in f(D)$. Now let D be a P$*$MD. Because in a P$*$MD D, $F^* = F_v$ for each $F \in f(D)$, we have $A^{*_s} = A_t$ for each $A \in F(D)$. (When $*$ is of finite character, $* = *_s$ and so in such a P$*$MD D, we have $A^* = A_t$ for each $A \in F(D)$ and so $* = t$. Moreover, in a PdMD $d = t$, making a PdMD a Prufer domain.) A PvMD is often written as PVMD. A reader in need of more introduction may consult [**?**] or [**?**, Sections 32 and 34].

For a star operation $*$, a maximal $*$-ideal is an integral $*$-ideal that is maximal among proper integral $*$-ideals. Let $*$-Max(D) be the set of maximal $*$-ideals of D. For a star operation $*$ of finite character, it is well known that a maximal $*$-ideal is a prime ideal; every proper integral $*$-ideal is contained in a maximal $*$-ideal; and $*$-Max$(D) \neq \emptyset$ if D is not a field. For a star operation $*$ two ideals A, B may be called $*$-comaximal if $(A, B)^* = D$. Indeed if $*$ is of finite character then two ideals A, B are $*$-comaximal if, and only if, A, B do not share (being in) any maximal $*$-ideal M. Thus integral ideals $A_1, A_2, ..., A_n$ are $*$-comaximal to an ideal B if and only if $(A_1A_2...A_n, B)^* = D$. Next, $I_w = \bigcap_{M \in t\text{-Max}(D)} ID_M$ for every $I \in F(D)$ and $I_wD_M = ID_M$ for every $I \in F(D)$ and $M \in t$-Max(D). A $*$-operation that gets defined in terms of maximal $*$-ideals is denoted by $*_w$ and it is defined as follows: For $I \in F(D)$, and $I^{*_w} = \bigcap\limits_{M \in *_s\text{-Max}(D)} ID_M$. This operation was introduced in [**?**] where it was established that for any star operation $*$, $*_w$ is a star operation of finite character and $*_w$-Max$(D) = *_s$-Max(D) and $*_w \leq *$, according to, again, [**?**]. An integral domain D is a P$*$MD if and only if D_M is a valuation domain for every maximal $*$-ideal M of D, [**?**]. Next, as the $*$-product $(IJ)^*$ of two $*$-invertible $*$-ideals is again $*$-invertible it is easy to see that $Inv_*(D) = \{I : I \text{ is a } *\text{-invertible } *\text{-ideal of } D\}$ is a group under $*$-multiplication with $P(D)$ the group of nonzero principal fractional ideals of D as its sub group. The quotient group $Inv_*(D)/P(D)$ is called the $*$-class group of D, denoted by $Cl_*(D)$. The $*$-class groups were introduced and studied by D.F. Anderson in [**?**] as a generalization of the t-class groups introduced in [**?**], [**?**] and further studied in [**?**]. It was shown in [**?**], in addition to many other insightful results, that if $*_1 \leq *_2$ are two star operations then $Cl_{*_1}(D) \subseteq Cl_{*_2}(D)$.

Now a $*$-homog ideal is a $*$-ideal I of finite type that is contained in a unique maximal $*$-ideal $M(I)$. What is interesting aand it was shown in [AZ, SR] that $M(I)$ the unique maximal $*$-ideal containing the $*$-homog ideal I is completely determined as $M(I) = \{x : (x, I)^* \neq D\}$. It turns out, and it is easy to see, that if I and J are two $*$-homog ideals that are similar, i.e. that belong to the same unique maximal $*$-ideal (i.e. $M(I) = M(J)$ in the terminology of [AZ, SR]) then $(IJ)^*$ is $*$-homog belonging to the same maximal $*$-ideal. With the help of some auxiliary results it is then shown that if an ideal K is a $*$-product of finitely many $*$-homog ideals then K can be uniquely expressed as a $*$-product of mutually $*$-comaximal $*$-homog ideals. Based on this a domain D is called a $*$-semi homogeneous ($*$-SH) domain if every proper principal ideal of D is expressible as a $*$-product of finitely many $*$-homog ideals. Now if you redefine a $*$-homog ideal so that the $*$-product of two similar, newly defined, $*$-homog ideals is a $*$-homog ideal meeting the requirements of the new definition, you have a new theory.

In any case each of the definitions of homogeneous elements can actually be studied in the same manner as the $*$-super potent domains of [HZ]. In [HZ], for a star operation $*$ of finite character a $*$-homog ideal is called $*$-rigid. The

$*$-maximal ideal containing a $*$-homogeneous ideal I may be called a $*$-potent maximal ideal. A domain D, with a finite character $*$-operation defined on it may be called $*$-potent if every maximal $*$-ideal of D contains a $*$-homogenous ideal. Next we may call the $*$-homogeneous ideal I $*$-super-homog if for each $*$-homog ideal $J \supseteq I; J$ is $*$-invertible and we may call a $*$- potent domain D $*$-super potent if every maximal $*$- ideal I of D contains a $*$-super homog ideal. But then one can study $*$-A-potent domains where A refers to a $*$-homog ideal that corresponds to a particular definition. For example a $*$-homog ideal J is said to be of type 1 in [AZ, SR] if $\sqrt{J} = M(J)$. So we can talk about $*$-type 1 potent domains as domains whose maximal $*$-ideals contain a $*$-homog ideal of type 1. The point is, to each suitable definition say A of a $*$-homog ideal we can study the $*$-A-potent domains as we studied the $*$-super potent domains in [HZ]. Of course the theory corresponding to definition A would be different from that of other $*$-super potent domains. For example each of the maximal $*$-ideal of the $*$-type 1 potent domain would be the radical of a $*$-homog ideal etc. Now as it is usual we start with some of the concepts that have some direct and obvious applications. For this we select the $*$-f-potent domains for a first study.

$*$-f-potent domains

Let $*$ be a finite type star operation defined on an integral domain D. Call a nonzero non unit element r of D $*$-factorial rigid ($*$-f-rigid) if r belongs to a unique maximal $*$-ideal and every finite type $*$-homog ideal containing r is principal. Indeed if r is a $*$-f-rigid element then rD is a $*$-f- homog ideal and hence a $*$-super homog ideal. So the terminology and the theory developed in [AZ, SR] applies. Note here that every non unit factor s of a $*$-f-rigid element r is $*$-f-rigid because of the definition. Note also that if r, s are similar $*$-f-rigid elements (i.e. rD, sD are similar $*$-f-homog ideals) then rs is a $*$-f-rigid element similar to r and s and so if r is $*$-f-rigid then r^n is $*$-f-rigid for any positive integer n.

Example A. Every prime element is a t-f-rigid element.

Call a maximal $*$-ideal M $*$-f-potent if M contains a $*$-f-rigid element and a domain D $*$-f-potent if every maximal $*$-ideal of D is $*$-f-potent.

Examples B. UFDs PIDs, Semirigid GCD domains, prime potent domains (domains in which every maximal t-ideal contains a prime element.)

The definition suggests right away that if r is $*$-f-rigid and x any element of D then $(r, x)^* = sD$ and applying the v-opertion to both sides we conclude that $GCD(r, x) = (r, x)_v$ of r exists with every nonzero element x of D and that for each pair of nonzero factors $u; v$ of r we have $u|v$ or $v|u$; that is r is a rigid element of D, in Cohn's terminology [Cohn]. Indeed it is easy to see, if necessary with help from [AZ, SR], that a finite product of $*$-f-rigid elements is uniquely expressible as a product of mutually $*$-comaximal $*$-f-rigid elements, up to order and associates and that if every nonzero non unit of D is expressible as a product of $*$-f-rigid elements then D is a semirigid GCD domain. Also, as we shall show below, a t-f-potent domain of t-dimension one is a GCD domain of finite t-character. But generally a t-f-potent domain is far from being a GCD domain. Before we delve into examples, let's prove a necessary lemma.

Lemma B1. Let D be an integral domain and let L be an extension of the field of fractions K of D. Then every finitely generated integral ideal F of $D+XL[X]$ is of the form $f(X)J(D+XL[X])$ where $f(X)\in L[X]$ and J is an ideal of D.

Proof. First let's note that if $F=(f_1,f_2,...,f_n)(D+XL[X])$ is an ideal of $(D+XL[X])$ with $I=F\cap D\neq(0)$ then $F=(f_{10},f_{20},...,f_{n0})(D+XL[X])$, where f_{i0} are the constant terms of f_i. This is because, if $F\cap D\neq(0)$ then $XL[X]\subseteq F$. Thus the constant terms are all contained in F. This gives $F\supseteq(f_{10},f_{20},...,f_{n0})(D+XL[X])$. But for $f(X)\in F$ we have $f(X)=\sum a_i(X)f_i(X)=\sum a_{i0}f_{i0}+Xh(X)\in(f_{10},f_{20},...,f_{n0})(D+XL[X])$. Thus we have $F=(f_{10},f_{20},...,f_{n0})(D+XL[X])=f(X)J(D+XL[X])$ where $f(X)=1$.

If on the other hand $I=F\cap D=(0)$ $FL[X]=f(X)L[X]$ where $f(X)\in L[X]$ and $f_i(X)=f(X)h_i(X)$ with $h_i(X)\in L[X]$. Now suppose that $f(0)\neq 0$. Then we can assume that $f(0)=1$ and so $h_i(X)=(h_{i0}+Xk_i(X))$ where $h_{i0}\in D$ and so $F=(f_1,f_2,...,f_n)(D+XL[X]=f(X)(h_{10}+Xk_1(X),...,h_{n0}+Xk_n(X))(D+XL[X])=f(X)(h_{10},...,h_{n0})(D+XL[X])$.

Finally if $f(0)=0$ then $f(X)L[X]\subseteq XL[X]$ and so $F=f(X)J(D+XL[X])$ where $J=D$.

Lemma B2. Let D be an integral domain and let L be an extension of the field of fractions K of D. Then every t-f-homog element of D is also a t-f-homog element of $D+XL[X]$.

Proof. Let's first note that $D+XL[X]$ has the $D+M$ form. Thus if I is a nonzero ideal of D then $(I+XDL[X])_v=I_v+XL[X]=I_v(D+XL[X])$, by [AR, Proposition 2.4] and using this we can also conclude that $(I+XDL[X])_t=I_t+XL[X]=I_t(D+XL[X])$.

Example C. Let D be a UFD (GUFD, Semirigid GCD domain) and L an extension of the quotient field K of D, the ring $D+XL[X]$ is a t-f-potent domain.

Question D: Apart from the detail and some results about factorization what is so interesting about $*$-f-potent domains?

Answer: Off-hand all I can say is that they have the following interesting properties, in terms of the t-operation.

Observation E: A t-f-potent domain has the PSP property.

Verification: Let $f=\sum a_iX^i$ be primitive i.e. $(a_0;a_1;...;a_n)\subseteq aD$ implies a is a unit; here the ideal $(a_0;a_1;...;a_n)$ is called the content of f and is often denoted by A_f. On the other hand a polynomial $g=\sum b_jX^j$ is called super primitive if $(A_g)_v=D$. It is known that while a super primitive polynomial is primitive a primitive polynomial may not be super primitive, see e.g. Example 3.1 of [AZ, GL]. A domain D is said to have the PSP property if every primitive polynomial over D is super primitive.

Now consider a finitely generated ideal $(a_0,a_1,...,a_n)$ in a t-f-potent domain D. Then $(a_0,a_1,...,a_n)$ is contained in a maximal t-ideal M associated with a t-f-rigid element r (i.e. $M=M(rD)$) if and only if each of the a_i is divisible by r or some non unit factor of r. Thus $(a_0,a_1,...,a_n)\subseteq M(rD)$ if and only if $(a_0,a_1,...,a_n)\subseteq sD$ for a t-f-rigid element of D similar to r.Thus, in a t-f-potent domain D, $f=\sum a_iX^i$ primitive implies that A_f is contained in no maximal

t-ideal of D; giving $(A_f)_v = D$ which means that each primitive polynomial f in a t-f-potent domain D is actually super primitive.

Now PSP implies AP i.e. every atom is prime. So, in a t-f-potent domain every atom is a prime. If it so happens that a t-f-potent domain has no prime elements then the t-f-potent domain in question is atomless. Recently Atomless domains have been in demand. The atomless domains are also known as antimatter domains. One example (Example 2.11 [AZ, GL]) was laboriously constructed in [AZ, GL] and this example was atomless and not pre-Schreier, and it killed the whole paper [BR] by Brookfield and Rush, as the gentlement acknowledged in a subsequent papaer. As we indicate below, it is easy to establish a method of telling whether a t-f-potent domain is pre-Schreier or not and I am sure that Professor Rush would be very pleased to see a a lot of of non pre-Schreier atomless domains.

Cohn in [Cohn] called an element c in an integral domain D primal if (in D) $c|a_1a_2$ implies $c = c_1c_2$ where $c_i|a_i$. Cohn [Cohn] assumes that 0 and units are primal. We deviate slightly from this definition and call a nonzero element c of an integral domain D primal if $c|a_1a_2$, for all $a_1, a_2 \in D\backslash\{0\}$, implies $c = c_1c_2$ such that $c_i|a_i$. He called an integral domain D a Schreier domain if (a) every (nonzero) element of D is primal and (b) D is integrally closed. We have included nonzero in brackets because while he meant to include zero as a primal element, he mentioned that the group of divisibility of a Schreier domain is a Riesz group. Now the definition of the group of divisibility $G(D)(= \{\frac{a}{b}D : a, b \in D\backslash\{0\}$ ordered by reverse containment) of an integral domain D involves fractions of only nonzero elements of D, so it's permissible to restrict primal elements to be nonzero and to study domains whose nonzero elements are all primal. This is what McAdam and Rush did in [MR]. In [Z] integral domains whose nonzero elements are primal were called pre-Schreier. It turned out that pre-Schreier domains possess all the multiplicative properties of Schreier domains. So let's concentrate on the terminology introduced by Cohn as if it was actually introduced for pre-Schreier domains.

Cohn called an element c of a domain D completely primal if every factor of c is primal and proved, in Lemma 2.5 of [Cohn] that the product of two completely primal elements is completely primal and showed in Theorem 2.6 a Nagata type result that can be rephrased as: Let D be integrally closed and let S be a multiplicative set generated by completely primal elements of D. If D_S is a Schreier domain then so is D. This result was analyzed in [AZ, GL] and it was decided that the following version ([AZ,GL, Theorem 4.4]) of Cohn's Nagata type theorem works for pre-Schreier domains.

Theorem F (Cohn's Theorem for pre-Schreier domains). Let D be an integral domain and S a multiplicative set of D.
(i) If D is pre-Schreier, then so is D_S .
(ii) (Nagata type theorem) If D_S is a pre-Schreier domain and S is the saturation of the set generated by a set of completely primal elements of D, then D is a pre-Schreier domain.

Now we have already established above that if r is a t-f-rigid element then $(r, x)_v$ is principal for each $x \in D\backslash\{0\}$. But then $(r, x)_v$ is principal for each

$x \in D\backslash\{0\}$ if and only if $(r) \cap (x)$ is principal for each $x \in D\backslash\{0\}$. But then r is what was called in [AZ, CP] an extractor. Indeed it was shown [AZ, CP] that an extractor is completely primal. Thus we have the following statement.

Corollary G. Let D be a t-f-potent domain. Then D is pre-Schreier if and only if D_S is pre-Schreier for some multiplicative set S that is the saturation of a set generated by some t-f rigid elements.

(Pf. If D is pre-Schreier then D_S is pre-Schreier anyway. If on the other hand D_S is pre-Schreier and S is the saturation of a set multiplicatively generated by some t-factorial rigid elements. Then by Theorem F, D is pre-Schreier.)

One may note here that if D_S is not pre-Schreier for any multiplicative set S, then D is not pre-Schreier. So the decision making result of Cohn comes in demand only if D_S is pre-Schreier. Of course in the Corollary G situation, the saturation S of the multiplicative set generated by all the t-f-rigid elements of D, leading to: if D_S is not pre-Schreier then D is not pre-schreier for sure and if D_S is pre-Schreier then D cannot escape being a pre- Schreier domain.

The other property that can be mentioned "off hand" is given in the following statement.

Theorem H A t-f-potent domain of t-dimension one is a GCD domain of finite t-character.

A domain of t-dimension one that is of finite t-character is called a weakly Krull domain. (D is weakly Krull if $D = \cap D_P$ where P ranges over a family $\mathcal{F}$ of height one prime ideals of D and each nonzero non unit of D belongs to at most a finite number of members of $\mathcal{F}$.) A weakly Krull domain D is dubbed in [AZ,SR] as t-weakly Krull domain or as a type 1 t-SH domain. Here a $*$-homog ideal I is said to be of type 1 if $M(I) = \sqrt{I^*}$ and D is a type 1 t-SH domain if every nonzero nonunit of D is a t-product of finitely many t-homogeneous ideals of type 1.

Lemma K. A t-f-potent weakly Krull domain is a type 1 t-f-SH domain.

Proof. A weakly Krull domain is a type 1 t-SH domain. But then for every pair I, J of similar homogeneous ideals $I^n \subseteq J^*$ and $J^m \subseteq I^*$ for some positive integers m, n. So J is a t-f-homogeneous ideal if I is and vice versa. Thus in a t-f-potent weakly Krull domain the t-image of every t-homog ideal is principal whence every nozero nonunit of D is expressible as a product of t-f-homog elements which makes D a t-f-SH domain and hence a GCD domain.

Proof of Theorem H. Use Theorem 5.3 of [HZ] for $* = t$ to decide that D is of finite t-character and of t-dimension one. Indeed, that makes D a weakly Krull domain that is t-f-potent. The proof would be complete once we apply Lemma K and note that a t-f-SH domain is a GCD domain and of course of finite t-character.

Generally a domain that is t-f-potent and with w-dimension> 1, is not necessarily GCD nor of finite t-character.

Example L. $D = Z + XL[[X]]$ where Z is the ring of integers and L is a proper extension of Q the ring of rational numbers. Indeed D is prime potent and two dimensional but neither of finite t-character nor a GCD domain.

There are some special cases, in which a t-f-potent domain is GCD of finite t-character.

i) If every nonzero prime ideal contains a t-f- homog ideal. (Use (4) of Theorem 5 of [AZ, SR] along with the fact that D is a t-f-SH domains if and only if D is a t-SH domain with every t-homog ideal t-f-homog. Thus a t-f-potent domain of t-dim 1 is of finite character.

ii) If D is a t-f-potent PVMD of finite t-chracer that contains a set S multiplicatively generated by t-f-homog elements of D and if D_S is a GCD domain then so is D.

One may also include among the "offhand" remarks that a t-prime potent domain, i.e. a domain each of whose maximal t-ideal contains a prime element p such that p belongs to no other maximal t-ideal.

I'd be doing a grave injustice if I don't mention the fact that before there was any modern day multiplicative ideal theory there were prime potent domains as Z the ring of integers and the rings of polynomials over them. It is also worth mentioning that there are three dimensional prime potent Prufer domains that are not Bezout. The examples that I have in mind are due to Loper [L].

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