QUESTIONS (HD 1805):Q1. If D is a domain different from its quotient field K, must the ring $D+X\ K[X]$ be a TV domain? Q2. When is $D+XK[X]$ a TV domain? Q3. If $D+XK[X]$ is of finite t-character, how is D semilocal with every maximal ideal a t-ideal? Q4. If the v-class group of D is trivial must the v-class group of $D+XK[X]$ be trivial. Q5. When is $D+XK[X]$ divisorial? w-divisorial?

Professor Jesse Elliott, a dear friend of mine often puts to me interesting questions. Recently he put these quetions to me. I tried to answer them more or less as they came and the result was a jumbled mess. I could have made several questions and answers from this, but I am old and not so energetic as I used to be. In any case I have tried to keep it readable, though the order of answers may be a little bit off.

Before I begin to answer the questions I must make sure that we are all on the same page, that is we all know what I am talking about. If you haven't gotten the drift, yet, we are talking about the star operations called the v-operation and the t-operation. If you don't know much about the star operations or knew about them and need to brush up on them look up sections 32 and 34 of [G]. For now, let us look at it this way. Let D be an integral domain with quotient field K and let $F(D)$ be the set of all nonzero fractional ideals of D. (These are the D-submodules H of K such that $dH\subseteq D$ for some nonzero $d\in D$.) For $A\in F(D)$ the set $A^{-1}=\{x\in K|\ xA\subseteq D\}$, called the "inverse" of A, is again a fractional ideal and we can show that $A^{-1}=\frac{B}{h}$ where B is an ideal of D and h a nonzero element of D. Thus the set $A_v=(A^{-1})^{-1}$ is again a fractional ideal called the v-envelope or the v-image of A. A v-image indeed, because we can show that $A\mapsto A_v$ is a function on $F(D)$, with certain properties. The v-operation is an example of a star operation. Here a map $*:F(D)\to F(D)$ is a star operation if the following conditions hold for all $A,B\in F(D)$ and all $c\in K\setminus(0)$:

1. $(cA)^*=cA^*$ and $D^*=D$;
2. $A\subseteq A^*$, and, if $A\subseteq B$, then $A^*\subseteq B^*$; and
3. $A^{**}=A^*$.

The most frequently used star operations (as well as the most important for our purposes) are the v-operation, defined above and the t-operation, given by $A_t=\bigcup B_v$, where the union is taken over all nonzero finitely generated fractional subideals B of A. For any star operation $*$ on D, we have $d\leq *\leq v$, in the sense that $A=A_d\subseteq A^*\subseteq A_v$ for all nonzero fractional ideals A of D. (Here $A\mapsto A$ defines on $F(D)$ another star operation, the identity star operation denoted by d.) Next $A\in F(D)$ is called a $*$-ideal if $A=A^*$ and by definition every nonzero principal fractioal ideal is a $*$-ideal for every star operation $*$.

Let's also note that for $A,B\in F(D)$ we can show that $(AB)^*=(A^*B)^*=(A^*B^*)^*$ for any star operation $*$ and that every integral t-ideal of D is contained in an integral ideal that is maximal with respect to being a t-ideal. (Same as every nonzero ideal is contained in a maximal (integral) ideal. We can call maximal ideals the maximal d-ideals.)

Call, for $* = d$ or t, an integral domain D $*$-independent if no two maximal $*$ ideals of D share a nonzero prime ideal.

Lemma A. Let K be the quotient field of D, let X be an indeterminate over K and let $R = D + XK[X]$.

(1) If R is d-independent then D is local. Conversely if D is local then R is d-independent.

(2) If R is t-independent then R is d-independent and D is t-local. Conversely if D is t-local then R is t-independent.

Proof. By Theorem 4.21 of [CMZ] a maximal ideal M of R with $M \cap D = 0$ is principal given by $f(X)R$ such that $f(X)$ is irreducible with $f(0) = 1$ and that such a maximal ideal M is of height one. Next if N is a prime (maximal) ideal of R with $N \cap D \neq 0$ then $N = N \cap D + XK[X]$ where $N \cap D$ is a prime (maximal) ideal of D. So X belongs to every prime (resp., maximal) ideal P of R such that $P \cap D \neq 0$. It follows from Lemma 4.41 of [CMZ] that P is a t-ideal if and only if $0 \neq P \cap D$ is. Now R being t-(d-) independent requires that there is only one maximal t-(d-) ideal of D. If R is t-independent then D has only one maximal t-ideal and so only one maximal ideal too, making local. Thus R being t-independent implies that R is d-independent.

On the other hand if D is local (t-local), then there is only one maximal (t-) ideal of R intersecting D and the rest are all principal, maximal of height one. Thus D being t-local (local) makes R t-(d-)-independent.

A domain whose nonzero ideals are all divisorial may be called a divisorial domain. A divisorial domain is known to be independent [H].

Proposition B. Let D be a valuation domain with principal maximal ideal and let R be defined as before then R is a t-independent Bezout domain and hence a divisorial domain.

Proof. That R is a Bezout domain follows from [CMZ] and because D is t-independent R is t-independent. Thus R is an independent Bezout domain with every maximal ideal principal and that makes R divisorial.

Lemma 4.11 of [CMZ] says that if I is an ideal of $R = D + XK[X]$ such that $I \cap D \neq 0$, then $I = I \cap D + XK[X]$. Moreover, Lemma 4.12 says that each ideal of R is of the form $f(X)(F + XK[X])$ where $f(X) \in K[X]$ and F is a D-submodule of K such that $f(0)F \subseteq D$. (Thus if $f(0) = 0$ then F can be any D-submodule of K).

Proposition C. If $R = D + XK[X]$ is divisorial then D is local divisorial and for every nonzero D-submodule F of K we have $F_v = F$ and $F_v = K$ implies $F = K$.

Proof. Because $XK[X]$ must be contained in $P + XK[X]$ for every maximal t-ideal of D and since R is divisorial, $XK[X]$ can be contained in only one maximal t-ideal we conclude that D has at most one maximal t-ideal.

Lemma C1. Let F be a D-submodule of $K = qf(D) \neq D$ and let X be an indeterminate over D. Then $F + XK[X]$ is a fractional ideal of $R = D + XK[X]$

and $(F+XK[X])^{-1}=(F^{-1}+XK[X])$.

Proof. If $D\neq K$ then $F+XK[X]$ is an R-submodule of $qf(R)$. Now $X(F+XK[X])$ is an ideal of R and so $F+XK[X]$ is a fractional ideal of R. Now as $F+XK[X]\supseteq XK[X]$ we have $(F+XK[X])^{-1}\subseteq (XK[X])^{-1}=K[X]$. Thus $(F+XK[X])^{-1}=\{f=a+Xg(X)|f(F+XK[X])\subseteq R\}=F^{-1}+XK[X]$.

Using Lemma C1 $(F+XK[X])_v=F_v+XK[X]$ and so if R is divisorial then every nonzero D-submodule of K is divisorial. Also as $A+XK[X]$ is an ideal of R and $(A+XK[X])_v=(A_v+XK[X])$, if R is divisorial then $A=A_v$ for each nonzero ideal A of D. But R being divisorial requires us to be certain about those D-submodules of K for which we cannot say much, that is those F for which $F^{-1}=0$. We must identify them with K to make sure that R is indeed divisorial.

Remark D. Even for D a local divisorial domain the conclusion that every nonzero D-submodule of K should be divisorial may seem highly unlikely, but if you reqire that $F^{-1}=0$ can happen only if $F=K$ then it is another story. Note here that "every D-submodule of $qf(D)$ being divisorial" boils down to "every D-submodule of $qf(D)$ properly contained in $qf(D)$ is a fractional ideal of D". For if F is a D-submodule of $K=qf(D)\neq F$ such that F is not a fractional ideal then $F^{-1}=(0)$. But then $F_v=K$ and F would be divisorial only if $F=K$. Next, there do exist domains satisfying the condition that F is a fractional ideal for every proper D-submodule F of K. These domains are called conducive domains and were first studied in [DF].

Let's call D super divisorial if every proper D-submodule of $qf(D)$ is divisorial. Then we can state the following result.

Proposition D1. If D is super divisorial local then $D+XK[X]$ is divisorial.

Proof. A typical ideal A of $D+XK[X]$ is given by $f(X)(F+XK[X])$ where F is a D-submodule of K such that $f(0)F\subseteq D$. As D is super divisorial, F is either K which can be assumed to be divisorial because $(K^{-1})=0$ and $0^{-1}=K$, or a fractional ideal which must be divisorial because D is divisorial. This indeed establishes that the ideal A is divisorial and as A is a typical nonzero ideal of R we have the conclusion.

Let me now answer the other questions/concerns that our friend Jesse has.

Proposition E. If $R=D+XK[X]$ has finite t-character then D must be a semi local domain whose maximal ideals are t-ideals.

Proof. Because $XK[X]$ must be contained in $P+XK[X]$ for every maximal t-ideal of D and since R has finite t-character, $XK[X]$ can be contained in only finitely maximal t-ideals and we conclude that D has at most a finite number of maximal t-ideals.

Let $M_1,M_2,...,M_r$ be all the maximal t-ideals of D. Because every nonzero non unit belongs to at least one maximal t-ideal the set of nonzero non units of D is contained in $M_1\cup M_2\cup...\cup M_r$. Let M be a maximal ideal. Then $M\subseteq M_1\cup M_2\cup...\cup M_r$. By prime avoidance M must be contained in M_i for some i. But then $M=M_i$. Thus we are forced to conclude that $M_1,M_2,...,M_r$ are the only maximal ideals.

Let $F(D)$ denote the set of nonzero fractional ideals of D. A fractional ideal $A\in F(D)$ is said to be v-invertible if $(AA^{-1})_v=D$. Next let $Int_v(D)$ denote the set of v-invertible v-ideals of the integral domain D. Then $Int_v(D)$ is a group and obviously the group $P(D)$ of nonzero principal fractional ideals of D is a subgroup of $Int_v(D)$. The quotient group $Cl_v(D) = \frac{Int_v(D)}{P(D)}$ is called the v-class group of D. This group can be used to measure how far the v-invertible v-ideals are from being principal. Indeed if all the v-invertible v-ideals of D are principal, the v-class group of D is 0 or trivial.

Proposition F. Let D be an integral domain, K the quotient field of D and $R = D + XK[X]$. If the v-class group of D is zero then so is the v-class group of R.

Proof. By the v-class group of D being zero we mean that every integral v-invertible v-ideal of D is principal.

Now by 4.12 of [CMZ], as mentioned above, a typical nonzero ideal of R is given by $f(X)(F+XK[X])$ where $f \in K[X]$ and F is a D-submodule of K such that $f(0)F \subseteq D$. We have seen above (in Lemma C1) that $(F + XK[X])^{-1} = (F^{-1} + XK[X])$. If $F^{-1} = 0$ then $((F + XK[X])(F + XK[X])^{-1})^{-1} = ((F + XK[X])(XK[X]))^{-1} = (XK[X])^{-1} = K[X] \neq R$. So if $F^{-1} = 0$, then $F + XK[X]$ is not v-invertible. If on the other hand $F^{-1} \neq 0$ then F is a fractional ideal and the ideal $f(X)(F + XK[X])$ can be put in: $((f(X)(F + XK[X])(f(X)(F + XK[X]))^{-1})_v$ to check if it is v-invertible. If $f(X)(F + XK[X])$ is v-invertible then we have $R = ((f(X)(F + XK[X])(f(X)(F + XK[X]))^{-1})_v = (((F + XK[X])(F^{-1} + XK[X])))_v = ((FF^{-1} + XK[X]))_v = (FF^{-1})_v + XK[X] = D + XK[X]$. Now $(FF^{-1})_v = D$ implies that F_v is principal because the v-class group of D is trivial. But then if $(f(X)(F + XK[X]))$ is v-invertible, we conclude that $F + XK[X]$ is v-invertible which also means that F is v-invertible and that F_v is principal. To recap: If $Cl_v(D) = 0$ then for every v-invertible ideal $f(X)(F+XK[X])$ we have $(f(X)(F+XK[X]))_v = f(X)(F+XK[X])_v = f(X)(F_v+XK[X]) = f(X)(lD+XK[X]) = lf(X)(D+XK[X]) = lf(X)R$ principal. Thus $Cl_v(D) = 0$ implies $Cl_v(R) = 0$.

Now let's get down to the question: When is $D + XK[X]$ a TV domain? Let's call D a super TV domain if for every D-submodule F of K we have $F_v = K \Rightarrow F = K$ and $F_t = F \Rightarrow F_v = F$.

Theorem G. Let D be an integral domain with quotient field K, X an indeterminate over K and $R = D + XK[X]$. Then R is a TV domain if and only if D is a super TV domain.

Proof. Let $R = D + XK[X]$ be a TV domain. Then a typical nonzero ideal of R is given by $A = f(X)(F + XK[X])$ where $f(X) \in K[X]$ and F is a D-submodule of K such that $f(0)F \subseteq D$. Since $f(0)$ can be 0, F can be any D-submodule of K. Also, as $A_* = f(X)(F_* + XK[X])$ for $* = v$ or t, $A_t = f(X)(F_t + XK[X]) = A$ forces $F_t = F$ and as $A_t = A$ implies $A_v = A$ we have $F_t = F$ implies $F_v = F$ for every D-submodule F of K. Yet if F is a D-submodule of K such that $F_v = K$ then $F_t = F \Rightarrow F_v = F$ would be false without F being K. Thus R being a TV domain forces every proper D-submodule F of K that is a t-submodule to be a v-submodule and hence K or a fractional v-ideal along with the restriction that if $F_v = K$ then $F = K$ and that makes D a super TV domain.

Conversely let D be a super TV domain. Then a typical nonzero ideal of $D+XK[X]$ is given by $A = f(X)(F + XK[X])$ where $f \in K[X]$ and F is a D-submodule of K such that $f(0)F \subseteq D$ and $A_* = f(X)(F_* + XK[X])$, for $* = v$ or t. Clearly $A_t = A$ makes $F_t = F$. Since D is a super TV domain, $F_t = F$ implies that $F_v = F$, because F is a fractional ideal of the super TV domain D, or K. But then $A_t = A$ implies $A_v = A$. Thus $D + XK[X]$ is a TV domain.

Remark G1. Implications of Theorem G: (1) By Proposition E, a super TV domain must be semilocal with every maximal ideal a t-ideal.

(2) Let D be a super TV domain with maximal ideals say $M_1, M_2, ..., M_r$, let S_i be the set of all nonzero non unit elements of D that are not in M_i and let d_i be any element of D such that $d_i D_{M_i} \subseteq D$. Then none of the d_i can be multipliatively split, i.e, we cannot write $d_i = xy$ where x, y are v-coprime non units. For, being v-cooprime x, y cannot both share M_i.If y is not in M_i, then y ends up dividing x because all powers of y divide d_i. But that is impossible. So super TV domains are conducivwe domains, with a special restriction.

(3) From (2) it appers that a super TV domain may either be local or an intersection of an intertwined bunch of localizations. Intertwined in that the maximal ideals involved have some nonzero prime ideal(s) in common. But then we can at least use local super TV domains to construct examples saying that for D a local domain, $D + XK[X]$ is a TV domain if and only if D is a super TV domain. Indeed a super TV domain is onducive nd this makes the task a lot easier.

But there are other ways we can get a $D + XK[X]$ TV domain. A domain whose nonzero finitely generated ideals are v-invertible is called a v-domain. A PVMD is a v-domain.

Proposition H. Let D be a v-domain. Then the following are equivalent.

(1) D is a valuation domain with principal maximal ideal

(2) $D + XK[X]$ is a divsorial domain

(3) $D + XK[X]$ is a TV domain.

Proof. (1) $\Rightarrow$ (2) by Propsition D1 and (2) $\Rightarrow$ (3) because a divisorial domain is a TV domain.

(3) $\Rightarrow$ (1) $D + XK[X]$ being a TV domain, D is a semilocal TV domain with each maximal ideal a t-ideal, by Theorem G. Also, being a v-domain, D is a PVMD by Theorem 3.1of [HZ TV]. Being a semilocal TV PVMD D is a t-independent Bezout domain. This makes $D + XK[X]$ a Bezout TV domain which must again be t-independent, by Theorem 3.1 of [HZ TV]. But then Lemma A applies to force D to be a valuation TV domain.

Corollary J. (Example). Let D be a semilocal PID with more than one maximal ideals, let K be the quotient field of D and let X be an indeterminate over K. Then $R = D + XK[X]$ is not a TV domain, while D is.

Proof (illustration). Left to the reader.

Proposition K. Let D be an integral domain with quotient field K and let X be an indeterminate over K. Then the domain $R = D + XK[X]$ is a divisorial domain if and only if D is a local super TV divisorial domain.

Proof. Suppose $R = D + XK[X]$ is divisorial. Then D is divisorial can be shown in the same manner as we showed that D is a TV domain, in Theorem G. As, being divisorial, $D + XK[X]$ is a TV domain D is divisorial. Next being divisorial $D + XK[X]$ is t-independent, i.e., no two maximal t-ideals share a nonzero prime ideal, [H], D must be local. (You may use (2) of Remark G1 to conclude that D is local.) Conversely, let D be a local super TV divisorial domain and consider, as in the corresponding part of the proof of Theorem G, a typical nonzero ideal A of $D + XK[X]$, that is $A = f(X)(F + XK[X])$ where $f(X) \in K[X]$ and F is a D-submodule of K such that $f(0)F \subseteq D$. If F is such that $F^{-1} \neq 0$ then F is a fractional ideal of D and so is divisorial because D is divisorial. If on the other hand $F^{-1} = 0$ then $F_v = K$ requiring that $F = K$, by the condition.

Finally let's delve into the w-operation. For $A \in F(D)$, define $A_w = \{x \in K | xH \subseteq A$ for some finitely generated ideal H of D such that $H^{-1} = D$. Indeed $A_w \in F(D)$ for each $A \in F(D)$. It was shown by Wang and McCasland [WM], that the map $A \mapsto A_w$ is a star operation. Now a domain D is called a w-divisorial domain if every nonzero w-ideal of D is a v-ideal. The w-divisorial domains were studied by Said El Baghdadi and Stefania Gabelli in [BG] where it was shown that D is a w-divisorial domain if and only if (1) D_M is a divisorial domain for each maximal t-ideal M of D (2) D is of finite t-characrer and (3) D is t-independent.

Corollary L. The following are equivalent for a w-divisorial domain D with quotient field K.

(1) $D + XK[X]$ is divisorial,

(2) $D + XK[X]$ is w-divisorial

(3) D is a local conducive divisorial domain.

Proof. (1) $\Rightarrow$ (2) because divisorial is w-divisorial. (2) $\Rightarrow$ (3) w-divisorial being t-independent requires that D be t-local. But that makes every maximal ideal of $D + XK[X]$ a maximal t-ideal and $D + XK[X]$ a DW domain, a domain in which every nonzero ideal is a w-ideal [M]. Also, as $D + XK[X]$ is w-divisorial, every nonzero ideal of $D + XK[X]$ is divisorial. That also makes D divisorial. But looking at a typical nonzero ideal $A = f(X)(F + XK[X])$, which we now know to be divisorial we conclude that every D-submodule F of K is either K or a fractional ideal, whence D is conducive. (3) $\Rightarrow$ (1) D is divisorial and conducive and that makes D a super TV domain. As D is local in addition, Propsition K applies.

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