QUESTION (HD 1902): Given that * is a star operation of finite type. You call a *-finite *-ideal I homogeneous if I is contained in a unique maximal *-ideal in your paper with Dumitrescu in [JPAA, 214 (2010) 2087-2091] and you call I *-rigid if I is a finitely generated ideal that is contained in a unique maximal *-ideal in your Arxiv paper (I): https://arxiv.org/pdf/1712.06725.pdf. Are these the same concepts? Also in your Arxiv paper you call a maximal *-ideal M, potent if M contains a *-rigid ideal and in another Arxiv paper (II): https://arxiv.org/pdf/1802.08353.pdf you call M *-potent if M contains a *-homog ideal. Are they the same?

Answer: It appears that they are, in that they produce the same results.

One simple answer: [4] uses the same definition of $*$-homogeneous as that of $*$-rigid in [**?**], i.e. (I), and reproduces almost word to word the results stated in [3], i.e. (II). (Of course, minus in some cases the proofs crippled by an error in an earlier version of [5].) In any case I have decided to bury a possibility of a controversy. So, here goes my explanation. But first a simple lemma.

Lemma A. Let $*$ be of finite character and let I be such that for some finitely generated ideal ideal J we have $I^* = J^*$ then there is a finitely generated ideal $K \subset I$ with $K^* = I^*$.

For let $J = (a_1, a_2, ..., a_n)$ and note that $* = *_s$ and so $I^* = \cup\{F^* | 0 \neq F \subseteq I$ and F finitely generated$\}$. Now as $J \subseteq I^*$ we have $a_i \in F_i^*$ where F_i are the f.g. subideals of I described above. But then $K = \cup F_i$ is a f.g. ideal contained in I such that $J \subseteq K^*$ and hence $J^* \subseteq K^* \subseteq I^*$.

Note B. Lemma A has already been used in [6, Theorem 1.1], in the proof of $(1) \to (4)$.

Now my definition of a $*$-homog ideal, for a $*$ of finite type, is: A $*$-ideal of finite type that is contained in a unique maximal $*$-ideal M, same as the homogeneous ideal in the JPAA paper you mention.

A more careful definition was forged by the authors of ([**?**] and) (I) that is to appear as [5, Definition 1.1], based on some very sketchy notes of mine the second author, as: Let $*$ be a finite-type star operation on the domain R. Call a finitely generated ideal I of R $*$-rigid if it is contained in exactly one maximal $*$-ideal of R. (A v-ideal of finite type contained in a unique maximal t-ideal was called rigid in [**?**] also.)

My claim: Both definitions should get the same results. For if you take a homogeneous ideal I then I contains a finitely generated $*$-rigid ideal J with $J^* = I$ by Lemma A. Moreover if you take I to be $*$-rigid, then I^* is homogeneous contained in the unique maximal $*$-ideal containing I.

Also the test of the pie is in the eating. Let $*$ be of finite type, I $*$-rigid and $J = I^*$ $M(I)$ the unique maximal containing I. Claim $M(I) = \{x \in D | (x, I)^* \neq D\}$. For $I \subseteq M(I)$ and so $x \in M(I)$ implies $(x, I)^* \neq D$ because $(x, I) \subseteq M(I)$ and $(x, I)^* \neq D$ requires that $(x.I)$ must be contained in the same maximal $*$-ideal of D that contains I. Now note that $(x, I)^* = (x, I^*)^*$ and consequently $M(I) = M(I^*) = M(J)$.

Consider on the other hand that H is homogeneous and let $N(H)$ be the unique maximal $*$-ideal containing H and J a finitely generated ideal such that $H = (J)^*$. Then J is $*$-rigid because J^* and hence J is contained in a unique maximal $*$-ideal.

Now let's go a bit further. I define a $*$-super homog ideal in (II) as: A nonzero integral $*$-ideal I of finite type is called $*$-super homogeneous ($*$-super homog) if (1) if each integral $*$-ideal of finite type containing I is $*$-invertible and (2) For every pair of proper integral $*$-ideals A, B of finite type containing I, $(A+B)^* \neq D$.

This definition works out to be: A $*$ homog ideal I such that every $*$-homog ideal containing I is $*$-invertible.

Now the authors of (I) call $*$-super rigid a finitely generated ideal I such that every finitely generated integral ideal J containing I is $*$-invertible.

Let I be $*$-super homog then there is a f.g. ideal J contained in I such that $J^* = I$. I claim that J is $*$-super rigid. For if H is a finitely generated ideal containing J then H^* contains I and so must be $*$-invertible and that makes H $*$-invertible.

Next let I be a $*$-super rigid ideal. We claim that I^* is $*$-super homog. For if H is a $*$-homog ideal containing I^* then $H = (b_1, b_2, ..., b_n)^*$. But $H \supseteq L = (b_1, ..., b_n) + I$ is finitely generated containing I. Since I is $*$-super rigid, L is $*$-invertible. But as $H = L^*$ we conclude that H is $*$-invertible and so I^* is $*$-super homog.

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