QUESTION (HD 2001) "Fuchs in [On primal ideals, Proc. Amer. Math. Soc. 1 (1950), 1-6.] showed that every irreducible ideal is primal. I need an example to show the converse is not true." (This question was asked by Farimah Farokhpay of Shahid Chamran University, Ahvaz, Iran.)

ANSWER: The theorem (Theorem 1 of the above mentioned paper) says: Every irreducible ideal is primal. The converse of this theorem would be: Every primal ideal is irreducible. So you want an example of a primal ideal that is not irreducible. But it so happens that Fuchs did provide, in that paper, an example of such an ideal. Before I get to that, let me recall the definition of a primal ideal. Given an ideal I in a ring R, an element $x \in R$ is said to be prime to I if for $y \in R$ we have $xy \in I$ only if $y \in I$. We say that x is not prime to I if x is such that for some $y \in R \backslash I$ we have $xy \in I$. An ideal I of a ring R is primal if the adjoint, the set of elements not prime to I, is an ideal which is necessarily a prime ideal. In other words an ideal I of a ring R is primal if and only if the zero divisors of R/I form an ideal P/I. With this "introduction" here is the answer.

On page 2 of the paper you mention there is an example of a primal ideal that's not primary. The ideal that I refer to can be described as: $A = (X^2, XY)$ in the ring $R = Q[X, Y]$ where Q is the field of rational numbers. This ideal A is primal with adjoin the prime ideal (X, Y). But A is not primary because $XY \in A = (X^2, XY)$, $X \notin A$ and no power of Y is in A.

Now reason as follows. An irreducible ideal in a Noetherian ring must be primary and A is not primary. So, the ideal (X^2, XY) in the ring R is not irreducible. [In fact $(X^2, XY) = (X) \cap (X^2, Y)$].

(For more examples of and more information on primal ideals you may also look up https://lohar.com/mithelpdesk/hd1103.pdf

and https://lohar.com/mithelpdesk/hd1104.pdf)

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Note (1/24/2020): I showed an earlier version of this answer to Bruce Olberding and he noted that in the ring $R = K[X, Y]$, where K is a field and X, Y independent indeterminate over K, the ideal $A = (X^2, XY, Y^2)$ is primary and hence primal (primal generalizes primary), but not irreducible. (Indeed as (X, Y) is a maximal ideal in R and $(X, Y)^2 = (X^2, XY, Y^2)$ is primary. Now as $(X^2, XY, Y^2) = (X^2, Y) \cap (X, Y^2)$ the ideal is not irreducible.)

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