QUESTION (HD 2004) While reading "Unique factorization property of non-unique factorization domains II", by G.W. Chang and Andreas Reinhart [CR, JPAA 224 (12) (2020), 106430], I found the following sentences "Clearly, GCD-domains are Schreier domains. Schreier domains were introduced by Cohn [6], and later, in [14], Zafrullah introduced the notion of pre-Schreier domains. (Pre-)Schreier domains are rather "nice" integral domains." Is there something wrong with pre-Schreier domains?

ANSWER: Well I found the quotes on nice odd too. The concepts seem to be sound. But it's hard to tell what they meant, mother tongues of both of the authors are different from English and they do not have enough experience with it (English). They probably tried to qualify the quotes on nice in the next paragraph. But what surprises me is the fact that in that whole section, section 2 of [CR], the authors are dealing with the Schreier property. So a mention of pre-Schreier domains in that disjointed fashion, is rather odd and the referee should have caught it, if the referee was not sleeping or if the referee was not a part of the game. But the referee didn't.

This paper has some very odd features. For instance the authors keep talking about "weakly Matlis GCD domains, being UVFD". Now a weakly Matlis domain D is a locally finite intersection of localizations at maximal t-ideals such that no two maximal t-ideals contain a nonzero prime ideal. Next a GCD weakly Matlis domain is an independent ring of Krull type right off the bat and according to Corollary 3.8 of Anderson, Mott and Zafrullah [AMZ, Boll. Unione Mat. Ital. Ser. 8 2-B (1999) 341–352] a GCD independent ring of Krull type is just a semirigid GCD domains, a concept studied by me long ago in [Z semi, Manuscripta Math. 17(1975), 55-66]. (I wonder if the purpose of writing the paper was to confuse the readers about the history of who started the study of the property of unique factorization in non-unique factorization domains. I hope not, the paper, [CR] has some interesting features too.)

Briefly a nonzero nonunit r of an integral domain D is called a rigid element if for all $x, y \in D$, $x, y|r$ in D implies that $x|y$ or $y|x$. I showed in [Z semi] that if a nonzero nonunit x in a GCD domain is expressible as a product of finitely many rigid elements then x is uniquely expressible as a product of finitely many (mutually coprime) rigid elements. I called a domain D semirigid GCD domain if D is a GCD domain in which every nonzero nonunit is expressible as a product of finitely many rigid elements. I also showed that a semirigid GCD domain was an independent ring of Krull type as in Griffin's [Gr, J. Reine Angew. Math. 229, 1–27 (1968)]. It was shown later that a GCD independent ring of Krull type is a semirigid GCD domain, Theorem B of [Z rig, J. Natur. Sci. and Math. 17 (1977), 7–14]. Later the result, that D is a semirigid GCD domain if and only if D is a GCD independent ring of Krull type, was included in a compact form in Corollary 3.8 of [AMZ, Boll. Unione Mat. Ital. Ser. 8 2-B (1999) 341–352] and in Theorem 3.8 of [Dan, nonat Properties of Commutative Rings and Modules (ed. S. Chapman) SRC Press Boca Raton, (2005), 1-21]. (Looks like Dan liked this result very much!)

Then on page 2 the authors talk about PVMDs and reference B.G. Kang [K, J. Algebra 123 (1989) 151–170], and actually miss no chance of referencing

Kang, but when it comes to class groups they just give the definition and say that a GCD domain is a PVMD D with $Cl_t(D) = 0$, no reference here. (It was shown in Corollary 1.5 of Bouvier and Zafrulla's [BZ, Bull. Soc. Math. Grece 29(1988) 45-59] that a PVMD with $Cl_t(D) = (0)$ is a GCD domain.) The referee, obviously didn't check. Perhaps it was all happening according to the referee's wishes.

Now let's look at Corollary 2.3 of this paper. It reads:

Let D be a VFD.

(1) $Cl_t(D) = \{0\}$.

(2) Every atom of D is a prime element.

(3) If D is a t-finite conductor domain, i.e., the intersection of each two principal ideals of D is t-finite, then D is a GCD-domain. Here t-finite is simply v-finite, but the authors are out to confuse the reader into believing that they have something new.

For the proof they send the reader on to a wild goose chase of facts that are in the literature, yet need proof. For instance it is true that if D is a Schreier domain then $Cl_t(D) = (0)$, but it was proved some place. And if you shove references to Kang's "results", most of which are picked off from authors who were not looking, into your reader's face every time you create a situation to mention them, then you should provide a reference for this fact too; of course this is Proposition 1.4 of [BZ]. Next an atom is a prime in a Schreier domain is a result of Cohn [Proc. Camb. Philos. Soc. 64 (1968) 251–264]. The last one is actually that takes the cake! They tell the reader to use Proposition 2 and Corollary 6 of my paper with Dumitrescu [DZ, Commun. Algebra 39 (2011) 808–818] for the proof. But then why not use Theorem 3.6 of [Z pres,Commun. Algebra 15 (1987) 1895–1920]? It has a part that would be enough! Besides, while the result in [DZ] is correct, telling the reader to use it is like saying: Go figure what a t-Schreier domain is. Then you may see the simple result that can be proved in one go. So, why did the authors choose this route? The answer here is partly provided by the quotes on nice that you mention in your question. The references and the tone of the paper have been set so that the paper only goes to a certain set of folks and if the authors mention one good reference to my paper, there would be a chance that the paper might end up at my table. (I let Tiberiu write that t-Schreier domain paper [DZ] thinking I was helping. While he kept asking me questions about Riesz groups he was removing what I had modeled my original results, on t-Schreier domains, after. And I had modeled my results after the paper on pre-Schreier domains. Way to go Tiberiu! See if that hurts me.)

Now let's read Corollary 1.9 of this paper. It says: Let D be an integral domain that is not a field. Then D is a VFD with $t\text{-}dim(D) = 1$ if and only if D is a weakly factorial GCD-domain. Let me ask: Isn't it the case that accoding to Theorem 10 of [AAZ, Bollettino U. M. I. (7) 9-A (1995), 401-413], D is a weakly factorial GCD domain if and only if D is a generalized Krull domain (GKD) that is also a GCD domain if and only if D is a Generalized UFD (GUFD)?

Briefly, a GUFD is defined as below. Call a nonzero nonunit q of D a prime quantum if q satisfies the following conditions. Q_1. For every nonunit $r|q$ there is a natural number n such that $q|r^n$,

Q_2. For every natural number n, $r, s|q^n \Rightarrow r|s$ or $s|r$,

Q_3. For every natural number n, each element t with $t|q^n$ has the property: $t|ab \Rightarrow t = rs$ in D where $r|a$ and $s|b$, for all $a, b \in D$.

An integral domain whose nonzero non units are expressible as products of prime quanta was called a generalized unique factorization domain (GUFD) in the first chapter of my doctoral dissertation in 1974, [Z diss], where it was shown that (a) Given two non-coprime prime quanta, one divides the other, (b) a finite product of prime quanta is uniquely expressible as a finite product of mutually coprime prime quanta, (c) if q is a prime quantum then the ideal (q) is primary to the prime ideal $Q(q) = \{x|GCD(x, q) \neq 1\}$ and (d) D is a GUFD if and only if D is a generalized Krull domain (GKD) that is also a GCD domain, if and only if every nonzero ideal of D contains a prime quantum.

Now a GKD D is a locally finite intersection $D = \cap_{P \in X^1(D)} D_P$ of localizations at height one prime ideals such that D_P is a valuation domain for each $P \in X^1(D)$. This notion was later generalized into a weakly Krull domain (WKD) as a locally finite intersection at height one primes, [AMZ fini, Boll. Un. Mat. Ital. B (7) 6 (1992), 613-630].

I could not publish my research under my name for one reason or another. So I requested Dan and David Anderson to join in as coauthors. The kind and helpful gentlemen graciously joined in and we published [AAZ guf], which was an expanded version of my original work. In fact it was a much improved version, in that it also included a mention of weakly factorial domains and some of my later work. Recall that D is a weakly factorial domain if every nonzero non unit of D is expressible as a product of primary elements. Weakly factorial domains were studied, in 1988, by Dan Anderson and Lou Mahaney in [AM wfd]. Indeed, as the authors indicate, via Theorem of [AZ wf, Proc. Amer. Math. Soc. 109 (4) (1990) 907-913], a WKD D with $Cl_t(D) = (0)$ is a weakly factorial domain. Thus a pre-Schreier WKD is a WFD.

Now let's get back to the business at hand and let's show that if x is a valuation element in a VFD D with $t\text{-}dim(D) = 1$ then x is actually a prime quantum. Indeed as, x^n is a valuation element and hence a rigid element we conclude that x satisfies Q_2. Next, as a VFD is claimed to be Schreier, x satisfies Q_3 too. Since x is a valuation element, $P = \sqrt{(x)}$ is a prime ideal and t-dim$(D) = 1$ we conclude that x is P-primary. But then, for every nonzero nonunit $h|x$ we have $h \in P$, which forces $x|h^n$ for some natural number n. So a VFD D with t-dim$(D) = 1$ is a GUFD and hence a weakly factorial GCD domain. The converse follows from (6) of Theorem 10 of [AAZ guf].

If that was all that needed done, why do all those definitions and explanations? My reason is that you may need far less than a VFD to pull off results like Corollary 1.9. For now let's call a nonzero nonunit x compact if x satisfies Q_2-Q_3 and call a domain D semi compact if every nonzero nonunit of D is a finite product of compact elements.

Proposition A. A semi compact domain D is a GUFD if and only if t-dim$(D) = 1$.

Proof. Let x be a compact element. Then x is rigid and completely primal in particular. Now rigid is homogeneous in the terminology of [AMZ uni] and by Theorem 2.3 of [AMZ uni], x belongs to a unique maximal t-ideal M, which must be of height one. But then $\sqrt{(x)} = M$ and so for every nonunit factor h of x there is n such that $x|h^n$. Now this means that if t-dim$(D) = 1$, every compact element satisfies Q_1 too. Thus a semi compact domain with t-dim$(D) = 1$ is a GUFD. For the converse note that a GUFD is of t-dim one.

Let's prove one more "result".

Proposition B. A semi compact domain D is a semirigid GCD domain if and only if D is a v-finite conductor domain if and only if D is a UVFD.

Proof. Use (4) of Theorem 3.6 of [Z pres].

Now, of course Proposition 4.6 of that paper can also be done for semi compact domains. I can go on and on about the oddities of that paper, but what's the point? Let me try to answer your question fully and to the best of my abilities.

As was my custom, in the beginning, I circulated the first version of my paper and that was well received in the sense that it was cited by several authors. But when I tried to publish it, it was rejected with one negative comment after another. After five or six rejections, I sat down to see what was essentially wrong with my paper. It seemed to me that there could be one point that the geniuses might not be clear about but were too afraid of being "wrong". So I included an example to clear the point and the paper got accepted. This is not the first time that this paper has been "put down", there was a time that it was dragged all over the internet. But hey, written in my usual clear style, this paper has some useful new results and new ideas, along with the coverage of old material. As a result, it gets cited. Hence the attempts to sideline it.

One of the peculiarities of the pre-Schreier domains is that if D is pre-Schreier and X is an indeterminate over D, then the polynomial ring $D[X]$ may not be pre-Schreier; unless D is Schreier (see [Z pres]). As a reviewer for the Math Reviews I had once to review a paper that purpoted to prove that if D was pre-Schreier, then so was $D[X]$. I pointed out the mistake in the paper, in my review of it. That gentleman may not have forgotten that, it seems. If push comes to shove, I'd start naming names.

Poke fun or not, but you have got to admit that the property of pre-Schreier domains: $((\cap(a_i))(\cap(b_j)) = \cap_{ij}(a_ib_j)$ for all $a_1, ..., a_n, b_1, ...b_m \in D$, that was isolated in [Z pres], has been used at more places than you can shake a stick at, and occasionally without reference to [Z pres]. For instance an early preprint gets mentioned in [Houston J. Math. 7 (1981), 1-10]. Then Kang just defines the $*$-property and starts proving his big replicas of known results in [K, J. Algebra 124 (1989) 284-299]. (If you don't believe me, read Theorem 1.10 of my paper [Z g-ded, Mathematika (1986) (33)285–295] and see how it's spread out in [K]. Especially almost all of Theorem 5.1 of [K] can be picked off from Theorem 1.10 of [Z g-ded], by replacing "G-Dedekind" with "pseudo Dedekind". I have related the whole sad story in hd2006) Now, the reason why I have brought the $*$-property in this "answer" is that in ([CR]) too I have noticed the same trick of changing terminology to sideline Muhammad Zafrullah, like teacher like student! My question is what has Muhammad Zaftrullah done to be treated like this? Are they doing it because they can? If they do what they can, I can shove their history in their shameless faces.