QUESTION (HD 2007) You write in your paper, [DZ, Comm. Algebra 39 (2011) 808–818] the following about Proposition 12: "The following result extends [28, Proposition 2.1]". Now, the above mentioned proposition is: "Let D be a t-Schreier domain and $x_1, ..., x_n \in D \backslash \{0\}$ such that $(x_1, ..., x_n)_v \neq D$. Then there exists a t-invertible t-ideal H such that $(x_1, ..., x_n) \subseteq H \neq D$." On the other hand the result [28, Proposition 2.1] is about sums of mutually disjoint homogeneous elements. Could you explain the connection? Similarly, I do not see any connection of [28, Proposition 2.1] with Proposition 13 of [DZ]. Is there an explanation?

ANSWER: I think it's a typographical error, at least in the first case. Obviously, this explanation is not enough. So, I'd try below to find the actual links in both cases and some explanation. But first let's get the necessary information together. The reference [28] is [MRZ, J. Group Theory (11) (2008), 23–41]. We have Proposition 12 of [DZ] in your question, so let's also have

Proposition 13 of [DZ]: Let D t-Schreier domain and A a proper t-invertible ideal of D. Then A is homogeneous if and only if A is contained in a unique maximal t-ideal.

Now obviously as [28, Proposition 2.1] is about sums of mutually disjoint elements of a Riesz group, neither of the above propositions extends it. The closest link to [28] that Proposition 12 has is that (1) of Proposition 1.1 of [28] says for two elements x, y of a Riesz group G: $x \wedge y \neq 0$ if and only if there exists a $t \in G$ with $0 < t \leq x, y$

Here "$x \wedge y \neq 0$" stands for "x and y are non-disjoint". This translates to $xD \cap yD \neq xyD$, which is equivalent to saying that $(x, y)_v \neq D$. So (1) of Proposition 1.1 of [28] translates to: For x, y in a pre-Schreier domain D we have $(x, y)_v \neq D$ if and only if there is a $t \in D$ such that $t|x, y$. (Because the group of divisibility of a pre-Schreier domain is a Riesz group.) Now I had proved in Lemma 2.1 of [Z wb, J. Pure Appl. Algebra 65(1990) 199-207] that if D is Schreier and $x_1, x_2, ..., x_n \in D \backslash [0\}$, $(x_1, x_2, ..., x_n)_v = D$ if and only if x_i have no non unit common factor. Proposition 12 of [DZ] is obviously a t-Schreier rendering of this result ([Zwb, Lemma 2.1]). The statement that I was really looking for came about as a property of Riesz groups in Proposition 3.1 of [YZ, Arab. J. Sci. Eng. 36 (2011), 1047–1061]

In a Riesz group G the following property holds. (pR): If $0 < x_1, x_2, ..., x_n \in G$ with $\mathcal{L}(x_1, x_2, ..., x_n) \neq \mathcal{L}(0)$, then there exists $r \in G$ such that $0 < r \leq x_1, x_2, ..., x_n$.

Here pR stands for "pre-Riesz" and $\mathcal{L}(x_1, x_2, ..., x_n)$, (resp., $\mathcal{L}(0)$) is the set of lower bounds of $x_1, x_2, ..., x_n \in G$ (resp., of 0) in G. Because a t-Schreier domain D is defined in [DZ] as a domain such that $Inv_t(D)$ is a Riesz group, the above statement translates to the following proposition.

Proposition A. If $I_1, I_2, ..., I_n$ are integral t-invertible t-ideals of a t-Schreier domain D such that $(I_1, I_2, ..., I_n)_v \neq D$ there is an integral t-invertible t-ideal I such that $I_i \subseteq I$, for each i.

Now you can see that Proposition 12 of [DZ] is a special case of Proposition A. (Indeed, Proposition A could have been stated, with the same proof as that of [DZ, Proposition 12]. But that would have meant mentioning some more of my work.)

For Proposition 13 the closest connection I could find in [28] is (2) of proposition 1.1. I say it is the closes because it characterizes a homogeneous element as an element x of G such that for all $h, k \in (0, x]$ there is t with $0 < t \leq h, k$. Of course there is no maximal t-ideal there. Now there is a result that is close to Proposition 13 can be found, hidden away in a comment in paragraph 2 of page 343, in [AMZ, Boll. Unione Mat. Ital. Ser. 8 2-B (1999) 341–352]. Of course that comment is about how to show that an element h is t-pure.

Let's say, for the sake of brevity, that an element h is t-pure if h belongs to a unique maximal t-ideal. The above mentioned comment is reproduced below for comparison.

For a nonzero non unit h, put $P(h) = \{x \in R | (c, h)_v \neq R\}$. So $P(h) = \cup\{M \in t\text{-}Max(R) | h \in M\}$. Thus $P(h)$ is an ideal, necessarily a maximal t-ideal, if and only if h is t-pure. Also note that $P(h)$ is an ideal if and only if $P(h)$ is closed under addition, that is, if $(x, h)_v \neq R$ and $(y, h)_v \neq R$, then $(x + y, h)_v \neq R$. I believe [DZ, Proposition 13] is indeed an extended and improved version of the above comment. Also worth a mention is the fact that t-pure was for elements what is "t-homogeneous" for t-ideals of finite type (or for t-invertible t-ideals).