QUESTION (HD 2102) I was reading the paper "On a general theory of factorization in integral domains", by Anderson and Frazier, Rocky Mountain J. Math., 41 (3) (2011), when I came across "In [4], the first author following suggestions of Zafrullah extended these notions to star operations." Now, what more does one want in terms of appreciation? On your part what I have seen lately, is complaints against Professor Anderson and attacks. Isn't it akin to biting the hand that feeds you?

ANSWER: You have raised a very complicated question! I would try to answer it to the best of my ability. First, let's look up the paper that is referred to as "[4]" in the Anderson-Frazier paper [9]. First thing first the title given for "[4]", in [9], is not quite right. In [9], the whole reference given is "Non-atomic factorization in integral domains, in Arithmetic properties of commutative rings and monoids, S. Chapman, ed., 1-21; Lect. Notes Pure Appl. Math. 241, CRC Press Boca-Raton, 2005." The copy of the paper that matches the description, downloaded from ResearchGate, gives the reference, written in hand, to be "Non-atomic unique factorization in integral domains, in Arithmetic properties of commutative rings and modules, S. Chapman, ed., 1-21; Lect. Notes Pure Appl. Math. 241, CRC Press Boca-Raton, 2005.

As a public service and future reference, I include the right reference to "[4]" of [9] as [1]. In [1] too Dan Anderson mentions me. At the beginning of [1]. Dan Anderson mentions my name twice in three sentences: "The goal of this chapter is to survey various generalizations of unique factorization into prime powers in integral domains. This follows the thesis of M. Zafrullah that the $p_i^{\alpha_i}$ are the building blocks (of factorization) in a UFD. The author would like to thank M. Zafrullah for a number of discussions of these topics." (In the second sentence the insertion in brackets is mine to make sense of what the fellow seems to be saying. Personally I can only be seen dead with a sentence like the one he wrote and that too in someone else's handwriting.) Dan is usually very conservative in expressing appreciation, yet these three sentences say a lot. The last sentence says that I have been a source of inspiration for Dan Anderson, over an extended period of time, in matters pertaining to factorization and the second talks about "M. Zafrullah's thesis", without saying where "M. Zafrullah's thesis" took shape. Well that observation was made in my Doctoral thesis submitted to the University of London, in 1974 [31]. Also, some examples in some books by Paul Cohn (circa 1987) show that I was fiddling with factorization long before I came to the US and before the big three papers ([3], [4], [6]) involving factorization were ever written. So, while Dan and David Anderson helped a lot, I was the force behind factorization. Of course that's just a "BTW" mention and your question is not yet answered.

Getting back to your comment that in [9] I am mentioned. Well, let's talk about the mention of "M. Zafrullah" and of "[4]", i.e. [1]. Let me start with the surmise that probably Dan Anderson in his usual conservative mode of giving little or no credit to people like me, had not made a mention that he had in an earlier version of the paper and had tried to publish it at some places and had failed. Then some failures, perhaps instilled some fear of God in his heart and he included the above quoted mention. (My reason for the surmise will become clear as you read on.) As fate had it, Andrea gave a talk at a meeting. The talk was on the socalled τ-factorization, that the paper [9] is about, and when Andrea started with her song and dance about McAdam and Swan did this and that without mentioning what I had done that led to her big theory, I got angry and walked out in the middle of the talk without saying a word. Probably at that point the thought that I could possibly be the referee behind his misery filled Dan with fear of God and he included the above reference that you mention. (Well I wasn't as I am not considered a well known Mathematician and an expert.)

Hopefully you now know all about τ-factorization, as you claim to be reading the paper. But for other readers who look into my rants to gain some knowledge I include a description.

(Copying as faithfully as possible.) They say on page 666 of [9]: We now define the key notions of our theory. Let τ be a symmetric relation on $D^{\#}$. (Here $D^{\#}$ is the set of nonzero non units of D.) For $a \in D^{\#}$ a τ-factorization of a is a factorization $a = \lambda a_1.a_2...a_n$ where $\lambda \in U(D)$ (the set of units of D) and $a_i \neq a_j$ for $(i \neq j)$. In this case we call a_i a τ-factor of a and say that a_i τ-divides a written $a_i \mid_\tau a$. Call $a \in D^{\#}$ τ-irreducible or a τ-atom if $a = \lambda(\lambda^{-1})$ are the only τ-factorizations of a and call D τ-atomic if each element of $D^{\#}$ has a τ-factorization into τ-irreducibles. Finally $a \in D^{\#}$ is τ-prime (respectively $\mid_\tau$-prime) if whenever $a \mid \lambda a_1.a_2...a_n$ (respectively $a \mid_\tau \lambda a_1.a_2...a_n$) where $\lambda a_1.a_2...a_n$ is a τ-factorization of a, then $a \mid a_i$ for some i .

I won't go into the merits and demerits of the theory, as so many folks have written doctoral dissertations; unless the circumstances push me to. So let's get back to your quote: "In [4], the first author following suggestions of Zafrullah extended these notions to star operations." As I have already mentioned that in "4" or [1] Dan thanks me for discussions, there were discussions and often very long discussions, but there was no discussion about how the star operations can be used to mimic the co-maximal factorization thing of McAdam and Swan [29], that was done in section 4 of [1]. But there was an email that I wrote to McAdam that addressed part of what was included in section 4 of [1], with reference to star operations. Before I get to the contents of that email. During the years 2002-2005 I wrote several e-mails to Steve McAdam, mainly to stake my claim that things similar to CFD and UCFD [29] were started in my doctoral dissertation, albeit in the GCD domain setting, to no avail. I was in a habit of attaching pdf version of the technical part. As fate had it, one of the emails, last in my futile rants about CFDs a UCFDs and my work, was actually in tex with the attached file in tex and copied to Dan! I have created a pdf file of the email and put it on my webpage as https://lohar.com/images/researchpdf/Sample c.pdf This email has some other emails that tagged along during printing, thanks to how my email software works. Now just to complete the story let me put the contents of that email in a more readable form below.

"Dear Steve,

I did realize that. The trouble is that I am still wandering in the realm of divisibility and smoothness. I tried to think about replacing primes by maximal ideals but that too would be somewhat smooth. But in trying to get myself straightened in this set up I stumbled onto something that you might like. I am using the language of star operations, if you do not like the stars just disregard them; the results would still make sense.

Let $*$ be a star operation of finite character. The operation d on the set $F(D)$ of nonzero fractional ideals of D, defined by $A^d = A$ is also a star operation of finite character. So disregarding the star operation in what follows will take you directly into the realm of ordinary ideals.

Lemma A. Let A and B be any $*$-comaximal integral ideals. If $C^* \supseteq AB$ then $C^* = (HK)^*$ where $H^* \supseteq A^*$ and $K^* \supseteq B^*$. In particular if A, B, C are principal and $* = d$ then $C = HK$ where $H = (C, A)$ and $K = (C, B)$.

Proof. Note that $((C,A)(C,B))^* = (C^2, CA, CB, AB)^* = (C^2, (C(A,B))^*, AB)^* = (C^2, C^*, AB)^* = (C^2, C, AB)^* = (C, AB)^* = C^*$.

This lemma shows that if A and B are comaximal integral ideals and if C is an invertible integral ideal containing the product AB then (A, C) and (B, C) are both invertible and $C = (A, C)(B, C)$. Now if you assume that A, B and C are all principal and suppose that all of a sudden you decide to work in a domain in which every two generated ideal is principal then in such a domain $c \mid ab$, a, b comaximal would directly imply that $c = rs$ where r divides a and s divides b. Now throw in the restriction that c cannot be expressed as a product of two comaximals then r is a unit or s is a unit. Making c a pseudo prime.

What is amusing is that I can produce the star operation version of this conclusion. This I would do when I can find time and finally a word about Lemma A. It can be stated for any collection of mutually $*-$comaximal set of integral ideals $A_1, A_2, ..., A_n$. That is if $C^* \supseteq A_1A_2...A_n$ then $C^* = (\prod(C + A_i))^*$. I tend to think of it as Multiplicative ideal theory's Chinese remainder theorem.

Now using this for $* = d$ and the tacit assumption that, in D, every two generated invertible ideal is principal, it is easy to see that in such a D for $x \mid a_1a_2...a_n$, where a_i are mutually comaximal we have $x = r_1r_2...r_n$ such that $r_i \mid a_i$. Now, r_i are mutually comaximal yet, even if D is a CFD, r_i do not have to be pseudo irreducible. Actually, each r_i would have to be a product of mutually comaximal pseudo irreducibles. Boy that is hard (a factor having worse factorization than the factored!) and now I know why I could not find time to go back to my unique representation domains. I see your UCFD's as a generalized d-version of URD's and I know the nooks and crannies of what I created. (By the way, in another paper I showed that a finite intersection of valuations of a field is a URD, but your result that a semilocal domain is a UCFD is far superior.)

You are deciding to branch out, well it is your choice. I can give you my experience, I tried to go into differential equations and then to coding theory. Did not seem to work out, in that every time I seemed to make some progress in branching out I would start having showers of new ideas. Now this semester I decided to read some extra Statistics, while teaching an introductory course on Statistics and I am inundated with ideas in ideal theory! I might keep on trying though and that is what you can do too.

I am sending a copy of this letter to Dan. He can often see some sense in my madness. Hopefully with Dan's help, I would like to produce at least a "t-version" of your paper, and of course we would like to cite your monumental work and of course I would try to keep you informed of what we produce.

Sincerely,

Muhammad"

(Apparently, instead of helping me Dan decided to help himself!)

Oh yes this email was written on 11-25-2002. It appears, from my email software, that I wrote several emails to Steve after that and never mentioned CFD's and UCFD's except possibly for praising them. I did not also follow it up with Dan Anderson either, possibly because I had realized that generally the property of being comaximal or $*$-comaximal requires something else so that a product of elements can be grouped into a product of comaximal (resp., $*$-comaximal). Because for regrouping a product as a product of comaximal (resp., $*$-comaximal) elements you need to first look at the product in terms of non-comaximal (resp., non-$*$-comaximal) elements. As a parting gift here is an example (if you know about $*$-operations, at least as much as described in sections 32 and 34 of [23]. Here only $*$-operations of finite character will be used): Define a relation ρ on D by $x\rho y \Leftrightarrow$ there is a maximal $*$-ideal M such that both x and y belong to M. The relation is obviously symmetric but may not be transitive. Now introduce elements called say $*$-pure (or $*$-homogeneous) if they, each, belong to a unique maximal $*$-ideal and throw in the condition that every nonzero nonunit x of D is a finite product of $*$-pure elements. Now, for M a maximal $*$-ideal, let P_M denote the set of all $*$-pure elements belonging to M. Restricted to P_M, ρ becomes an equivalence relation. Now if $x = x_1...x_n$, and if $M_1, ...M_j$ are all the distinct maximal $*$-ideals containing x, regroup as in the proof of Proposition 1 of [38] to write $x = p_{M_1}p_{M_2}...p_{M_j}$ where each p_{M_i} is a product of factors of x that belong to M_i. Now with arguments similar to the ones used in Lemma 2.1 of [38] you can show that each of p_{M_i} is a $*$-pure element and as for $i \neq j$, p_{M_i} and p_{M_j} are $*$-comaximal, because they belong to distinct maximal $*$-ideals, x can be expressed (uniquely) as a product of mutually $*$-comaximal $*$-pure elements (as done in Theorem 3.1 of [10]). We can call such a domain a $*$-pure $*$-UCFD. Here if $* = d$ we have a UCFD each of whose pseudo atoms is a nonzero nonunit that belongs to a unique maximal ideal. Similarly for $* = t$ we have a semi-t-pure domain or a HofD of Chang [17].

Theorem A0. A semi $*$-pure domain is a $*$-pure $*$-UCFD, which when $* = t$, is a HoFD of Chang [17] and a semi- t-pure domain of [10].

Next let $Inv_*(D)$ be the set of $*$-invertible $*$-ideals and let $P(D)$ be the set of nonzero principal fractional ideals of D. $Inv_*(D)$ is a group under $*$-multiplication and $P(D)$ is a subgroup of $Inv_*(D)$. The quotient group $Inv_*(D)/P(D) = Cl_*(D)$ is called the $*$-class group you can read all about it in David Anderson's paper [13]. If $* = d$ we have the usual ideal class group of D or $Pic(D)$ and for $* = t$ we have the t-class group or $Cl_t(D)$ of D. (The t-class group was introduced in [15] and further studied in [16] much earlier .)

Now let D be a $*$-pure $*$-UCFD and let h be a $*$-pure element belonging a maximal $*$-ideal M of D. Then, as done in [10] $hD = \cap_{Q\in t\text{-}Max(D)} hD_Q = \cap_{Q\neq M} D_Q \cap hD_M = D \cap hD_M$ we conclude that for any $*$-pure element h belonging to a $*$-maximal ideal M we have $hD = D \cap hD_M$. Now let A be a $*$-invertible $*$-ideal of our $*$-pure $*$-UCFD D and let $M_1, M_2, ..., M_r$ be all the distinct maximal $*$-ideals containing A. Then A is $*$-locally principal. That is AD_M is principal for each maximal $*$-ideal. Of course AD_M is a nonunit only for $M = M_i$ $(i = 1, ..., r)$. Consider $AD_{M_i} = \alpha_i D_{M_i} = \frac{h}{k} D_{M_i} = \frac{\rho_{1i}\rho_2 \cdots \rho_m}{\tau_1\tau_2 \ldots \tau_n} D_{M_i}$, where $\rho_{1i}\rho_2...\rho_m$ is the $*$-pure $*$-comaximal factorization of h and $\tau_1\tau_2...\tau_n$ is the $*$-pure $*$-comaximal factorization of k. Since we are dealing with a $*$-pure $*$-UCFD and since we can assume that $k \notin M_i$, only one of the rhos , say $\rho_i = h_i$, is a nonunit in M_i we have $AD_{M_i} = h_i D_{M_i}$ where h_i is a $*$-pure element belonging to M_i. Now as A is a $*$-invertible $*$-ideal we have $A = \cap_{M\in t\text{-}Max(D)} AD_M = \cap h_1 D_{M_1} \cap ...h_r D_{M_r} \cap (\cap_{Q\in t\text{-}Max(D)\setminus\{M_1,...,M_r\}} D_Q$ or $A = \cap(D \cap h_i D_{M_i})$ or $A = \cap h_i D = \Pi h_i D$ because h_i are mutually $*$-comaximal. Thus A is principal and as a consequence we have the following result.

Theorem A. Let D be a $*$-pure $*$-UCFD. Then $Cl_*(D) = (0)$.

Call a domain $*$-h-local if D is of finite $*$-character, i.e every nonzero non unit of D belongs to at most a finite number of maximal $*$-ideals, and no two maximal $*$-ideals of D contain a nonzero prime ideal. Observe also that D is a $*$-h-local domain if and only if for each nonzero non unit x of D we have a unique expression $xD = (I_1...I_n)^*$ where each of I_i belongs to a unique maximal $*$-ideal [11, Theorem 6]. Noting that each of I_i is $*$-invertible, couple this information with $Cl_*(D) = (0)$ to conclude that if D is a $*$-h-local domain with $Cl_*(D) = (0)$ then D is a $*$-pure $*$-UCFD. These observations give us the following result.

Theorem B. A $*$-pure $*$-UCFD D is a $*$-h-local domain with $Cl_*(D) = (0)$ and a $*$-h-local domain D with $Cl_*(D) = 0$ is a $*$-pure $*$-UCFD.

Of course D is of finite $*$-character because every nonzero non unit of a $*$-pure $*$-UCFD is expressible as a finite product of finitely many mutually $*$-comaximal $*$-pure elements. For the other part let M and N be two maximal $*$-ideals, let P be nonzero prime ideal contained in $M \cap N$ and let x be a nonzero element in P. Because D is a $*$-pure $*$-UCFD, x is a product of $*$-pure elements and as no $*$-pure element can be in P we are left with a unit in P and the desired contradiction.

Next, call D a $*$-prufer domain if every nonzero finitely generated ideal is $*$-invertible. These domains were born as P$*$MDs in [26]. These are advanced PVMDs in that a result about $*$-Prufer domains is a result about PVMDs (i.e. t-Prufer domains) and a result about Prufer domains (i.e. d-Prufer domains) at the same time. Now the local characterization of $*$-Prufer domains is the same as that of PVMDs and Prufer domains, i.e., D is a $*$-Prufer domain if and only if D_M is a valuation domain for each maximal $*$-ideal M of D. Finally, following [11] let's call D a $*$-Bezout domain if for every nonzero finitely generated ideal A of D we have A^* principal. Indeed as $Cl_*(D) = (0)$ means that every $*$-invertible $*$-ideal is principal we have the following result.

Theorem C. An integral domain D is $*$-Bezout if and only if D is $*$-Prufer and $Cl_*(D) = (0)$.

Theorem C can be put to an immediate use in the form of the following result.

Theorem D. A $*$-pure $*$-UCFD D that is also $*$-Prufer is $*$-Bezout.

Theorem D is new in that it states at least two results in one go.

Corollary E. A Prufer d-pure d-UCFD D is a Bezout domain.

Corollary F. A PVMD t-pure t-UCFD is a weakly Matlis GCD domain.

Of course Corollary F can be extracted from Chang's paper [17]. but as he probably mentions, it is a consequence of Proposition 2 of [15] that says something like "A PVMD D with $Cl_t(D) = (0)$ is a GCD domain.) So, no big deal. Of course I have not gone through all the trouble of typing this stuff with aching limbs to say just that. So there is more, and possibly more useful stuff, to follow.

Let's start with recalling some terminology. Let me quote from [10]. Dan Anderson wrote, as he usually took on the big task of writing and I usually wandered off to "greener pastures" or other ideas: "Another property of p^n is that if $x, y|p^n$, then $x|y$ or $y|x$. With this in mind, following P. M. Cohn, the third author [13] defined a nonzero nonunit $h \in R$ to be rigid if $x, y|h\ddot{} \Rightarrow x|y$ or $y|x$." (The fellow seems to be enamored with "Zafrullah's Thesis!) Here "[13]" is [38].

The first statement that I want to make is the following (after noting that a "star operation of finite character' is to mean one of the operations d, t or at most w).

Lemma G. Let x be a $*$-pure element belonging to a maximal $*$-ideal M, in an integral domain D. Then every non-unit factor h of x is in M. (In other words every non unit factor of a $*$-pure element is $*$-pure.)

The proof is straightforward because if h is a nonunit factor of x, then h has to be in some maximal $*$-ideal and x belongs to whichever maximal $*$-ideal h belongs to, while x can belong only to M.

Call two nonzero elements x, y of D comparable if $xD \subseteq yD$ or $yD \subseteq xD$ (i.e. $x|y$ or $y|x$). Also, following Cohn [20] let's call a nonzero non unit r of D rigid if for all $x, y|r \Rightarrow x, y$ are comparable, as mentioned above. Also call two nonzero elements x, y of D $*$-comaximal if $(x, y)^* = D$. It is easy to see that x and y are $*$-comaximal if and only if x and y do not share any maximal $*$-ideals. (Indeed if x and y share a maximal $*$-ideal M, then $(x, y) \subseteq M$ and hence $(x, y)^* \subseteq M$.)

Lemma H. Let D be a $*$-Bezout domain .Then the following hold. (1) Let x and y be two $*$-pure elements in D, then x and y are $*$-comaximal or comparable. (2) Every $*$-pure element x in D is rigid. (3) Every rigid element of D is $*$-pure. (4). Let r be a rigid element in D. Then the set $P(r) = \{x \in D|(x, r)^* \neq D\}$ is a maximal $*$-ideal of D. (5) If D is a d-Bezout (resp., t-Bezout) and S a multiplicative set in D we have D_S a d-Bezout (resp., t- Bezout) domain. (6) $D_{P(r)}$ is a valuation domain.

Proof. (1) Suppose x, y are $*$-pure and $(x, y)^* \neq D$. Then both x and y belong to a unique maximal $*$-ideal M, being $*$-pure. Since D is $*$-Bezout, we have $(x,y)^* = dD$ for some $d \in D\backslash\{0\}$. This gives $(x/d, y/d)^* = D$. If $dD \neq xD$ or yD then x/d and y/d are both non units and hence both in M by Lemma G, because x and y are both $*$-pure they belonging to M. But this is impossible. Whence either of x/d or y/d is a unit, forcing the conclusion that $x|y$ or $y|x$.

(2) Let x be a $*$-pure element in the $*$-Bezout D. Since, by Lemma G, every pair of non unit factors of a $*$-pure element are $*$-pure belonging to the same maximal $*$-ideal, they must be comparable by (1). But then, as all pairs of factors of x are comparable, making x a rigid element. (3). Note that a rigid element in a general integral domain can pass as a pre-homogeneous element of [38] and as a $*$-Bezout domain is a GCD domain and hence a PSP domain at least, a (t-) homogenous element and following the argument of [38, Proposition 6] in more general terms, one can show that in a $*$-Bezout domain a rigid element is $*$-pure. (4). Noting that $(x,r)^* \neq D$ and that D is $*$-Bezout we conclude that for all $x \in P(r), r$ or a factor of r divides x. Now using (1) and (2) and the arguments similar to those in [33] we conclude that $x_1, x_2 \in P(r)$ implies that $x_1 + x_2 \in P(r)$ and for all $y \in D$ and $x \in P(r)$ we have $yx \in P(r)$ and so $P(r)$ is an ideal. Next because r is $*$-pure r belongs to a maximal $*$-ideal M. But then $P(r) \subseteq M$. On the other hand, by the definition of $P(r)$ any $y \in M\backslash P(r)$ would have to be such that $(y,r)^* = D$ which being impossible because M is a maximal $*$-ideal, we have $P(r) = M$. Whence the conclusion that $P(r)$ is a maximal $*$-ideal. (5) Well known as d-Bezout is Bezout and a t-Bezout domain is a GCD domain. (6) Note that, because D is a $*$-Bezout domain, for all $x, y \in P(r)$ we have $(x,y)^* = dD \subseteq P(r)$ which extends to $(x,y)D_{P(r)}$ if $* = d$ and to $(x,y)_t D_{P(r)} = dD_{P(r)}$ or $(x,y)_v D_{P(r)} = dD_{P(r)}$ because (x,y) is finitely generated. Now taking the v-image of both sides of $(x,y)_v D_{P(r)} = dD_{P(r)}$ we get via Lemma 4 of [34] $((x,y)D_{P(r)})_v = dD_{P(r)}$. Thus for every pair a, b in $P(r)D_{P(r)}$ we have $(a,b)_{v_1} = ((x,y)D_{P(r)})_{v_1} = dD_{P(r)}$, where v_1 is the v-operation in $D_{P(r)}$. This leads to the conclusion that for all nonzero $a, b \in P(r)D_{P(r)}$, a, b have a GCD that is in $P(r)D_{P(r)}$ and this fact couple with the fact that $D_{P(r)}$ is a GCD domain can be used to show that $D_{P(r)}$ is a valuation domain.

In [33, Theorem 2] it was shown that in a GCD (i.e. t-Bezout) domain a finite product of rigid elements is uniquely expressible as a finite product of mutually coprime elements. Using similar arguments and the above lemmas one can prove the following result.

Theorem I. Let D be a $*$-Bezout domain (for $* - d$ or t). Then a finite product of rigid elements of D can be written as a finite product of mutually $*$-comaximal rigid elements, uniquely, up to associates.

Proof. Since for any nonzero ideal A and for any star operation $*$ of finite character, we have $(A^*)_t = A_t$, a $*$-Bezout domain is a t-Bezout domain or a GCD domain to start with. So we can modify the proof of [33, Theorem 2] to fit our statement.

Let's call a $*$-Bezout domain D a semirigid $*$-Bezout domain if every every nonzero nonunit of D is expressible as a finite product of rigid elements. Mimicking the proof of Theorem 5 of [33] one can prove the following statement.

Theorem J. Let D be a semirigid $*$-Bezout domain. Then D has a family

$\Phi = \{P_\alpha\}\alpha \in I$ of prime ideal such that:

IRKT1. D_{P_α} is a valuation domain for each $\alpha \in I$, IRKT2. Each nonzero non unit x of D belongs to at most a finite number of P_α,

IRKT3. For distinct $P_\alpha, P_\alpha \in \Phi$, $P_\alpha \cap P_\beta$ does not contain a nonzero prime ideal of D and IRKT4. $D = \cap_{\alpha \in I} D_{P_\alpha}$.

The ring described in Theorem H, was called an independent ring of Krull type by Griffin [25]. It can be shown that each member Φ is a maximal t-ideal and every maximal t-ideal of D is in Φ. Further, if Φ consists of all the maximal ideals of D then D is a Prufer domain. Later, by dropping IRKT1, the ring with a family Φ consisting of maximal t-ideals of D satisfying IRKT2-IRKT4 was called a weakly Matlis domain in [12]. An integral domain D with Φ consisting of all maximal $*$-ideals satisfying IRKT2-IRKT4 was called a $*$-h-local domain in [11]. (If we drop IRKT3 we get a ring of Krull type which is PVMD of finite t-character. That is a PVMD each of whose nonzero nonunit belongs to at most a finite number of maximal t-ideals.) Now let us see how we can show that a $*$-Bezout $*$-h-local domain D is semirigid. One way of doing that is to note that if M is a maximal $*$-ideal in a $*$-Bezout domain that is also a $*$-h-local domain, $M = P(r)$ for a rigid element r. For this let $x \in M \backslash \{0\}$. As D is $*$-h-local, x belongs to at most a finite number of maximal $*$-ideals $M_1, ..., M_n$ in addition to M. Choose, by prime avoidance, a $y \in M \backslash \cup_{i=1}^{n} M_i$ and consider $(x,y)^*$. Because D is $*$-Bezout and because $x, y \in M$ we have $(x,y)^* = rD \subseteq M$. But then r belongs to a unique maximal $*$-ideal and hence is a $*$-pure element and r is rigid by (2) of Lemma F. Now (as another name for a $*$-h-local domain is a $*$-SH domain) by Theorem 6 of [11] we conclude that xD can be expressed uniquely as $xD = (I_1...I_n)^*$ where $I_i = xD_{M_i} \cap D$ a $*$-pure ideal belonging to the maximal $*$-ideal M_i. Obviously as D is $*$-Bezout and each of I_i a $*$-invertible $*$-ideal we have $I_i = r_i D$, forcing $xD = r_1...r_n D$ or $x = \epsilon r_1...r_n$ where r_i are rigid and ϵ a unit. Thus we have the following statement.

Theorem K. A $*$-Bezout $*$-h-local domain is a $*$-Bezout Semirigid domain.

Weakly Matlis GCD domains have figured prominently in recent literature, see e.g. [17] and [18], without giving a clue to what the blazes they actually are. To thwart production of "new research" with "to the best of the author's knowledge" I include the following corollary.

Corollary L. For an integral domain D the followiong are equivalent.

(1) D is a weakly Matlis GCD domain,

(2) D is a Semirigid GCD domain.

(3) D is a GCD-IRKT,

(4) D is a t-h-local GCD domain,

(5) D is a t-$Max(D)$-$I\mathcal{FC}$ GCD domain,

(6) D is a semirigid domain in which the product of every pair of non-v-coprime rigid elements is rigid

(7) D is a PVMD HofD of Chang [17], i.e. D is a semi-$\vec{t}$-pure PVMD of [10].

(8) D is a PVMD t-pure t-UCFD.

Proof. (1) $\Rightarrow$ (2) is Theorem K for $* = t$. (2) $\Rightarrow$ (3) is Theorem 5 of [33] (3) $\Rightarrow$ (1) because IRKT meets the requirements of being a weakly Matlis domain (this result was proved somewhat laboriously in [32, Theorem A]. (In other words these results were proved way before modern day factorization became the talk of the town.) Next (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) by definitions. (2) $\Leftrightarrow$ (6) was shown in Corollary 1 of [38]. (1) $\Leftrightarrow$ (7) because a HoFD is semi t-pure and using the fact that a semi t-pure domain D is weakly Matlis with $Cl_t(D) = 0$ and that a PVMD with $Cl_t(D) = (0)$ is a GCD domain we end up with a weakly Matlis GCD domain and of course a weakly Matlis GCD domain is a weakly Matlis PVMD, with trivial t-class group. (1) $\Rightarrow$ (8) A weakly Matlis GCD domain D is a PVMD because D is a GCD domain and $Cl_t(D) = (0)$ because D is a GCD dom,ain. Next D is t-pure t-UCFD because D is a t-h-local domain with $Cl_t(D) = (0)$ and Theorem B applies (8) $\Rightarrow$ (1) A t-pure t-UCFD is a t-h-local domain with $Cl_t(D) = (0)$, by Theorem B. Couple it with D being a PVMD to get a t-h-local GCD domain. But a t-h-local domain is a weakly Matlis domain.

Let me point out that a lot of the results mentioned in the above corollary are well known if one cares to read. In other words a weakly Matlis GCD domain is a Semirigid GCD domain is old information and refers to [10] and anyone who denies it is lying through his/her teeth. But let's push this aside and get to do something new. I plan to show how to use factorization of rigid elements in GCD domains to get new examples of semirigid GCD domains and of $*$-UCFDs.

(Some of these examples are at least as old as 11-25-2002 as shown by my e-mail to Steve McAdam, a pdf version of which can be found at

https://lohar.com/images/researchpdf/Sample%20d.pdf

My first example of a UCFD or d-UCFD was a simpler form of the following.

Example M. Let (D, M) be a quasi local domain, K its quotient field, L a field extension of K and X an indeterminate over K. Then (1) $R = D+XL[X] = \{f \in K[X] | f(0) \in D\}$ is a (d-) UCFD with pseudo atoms either principal primes of height one or elements that belong only to the maximal ideal $M + XL[X]$ and no other maximal ideal of $D + XL[X]$. (2) If M is the t-ideal of D then $M + XL[X]$ is a t-ideal of R and R is a t-h-local domain. (3) D is a GCD domain if and only if $D + XK[X]$ is a GCD domain, (4) D is a PVMD if and only if $D + XK[X]$ is a PVMD, however (5) R is a PVMD t-h-local if and only if D is a valuation domain (if and only if R is a GCD t-h-local domain).

Illustration. If you assimilate the comments between Corollary 17 and its proof in [8] you are ready to understand the illustration.

Before we start let's recall from [8] that generally the prime (maximal) ideals of R are of the form $P + XL[X]$ where P is a prime (maximal) ideal of D and principal primes of the form $(1 + Xf(X))R$ and the prime (maximal) t-ideals of R are of the form $P + XL[X]$ where P is a prime (maximal) t-ideal of D and principal primes of the form $(1 + Xf(X))R$.

(1). A typical element of R is given by $a+Xf(X)$ where $a \in D$ and $f \in L[X]$. Two cases arise: (a) $a \neq 0$ and (b) $a = 0$. The case (a) is straightforward in that if $a \neq 0$ then $a + Xf(X) = a(1 + Xf(X)/a)$. Here a and $1 + Xf(X)/a$ are co-maximal and $1+Xf(X)/a$ is a finite product of irreducible elements of the form $1+Xg(X)$ which are height one primes in R and hence maximal ideals, making $1+Xf(X)/a$ a product of powers of primes that are mutually co-maximal. Thus in this case $a+Xf(X) = a(1+Xf(X)/a) = ap_1^{n_1}...p_r^{n_r}$ is a product of mutually co-maximal non-units, each belonging to a distinct and unique maximal ideal.

Of these $a \in M + XL[X]$ and is a "pseudo atom" in that no two non unit factors of a are co-maximal. On the other hand, each of $p_i^{n_i}$ is obviously a pseudo atom. In case (b) we can write $a + Xf(X) = X^r g(X)$, where $g(X) \in L[X]$ such that $g(0) = s \neq 0$. But then $a + Xf(X) = X^r g(X) = X^r/sg'(X) = X^r/s(1 + Xh(X))$. Thus we have $a + Xf(X) = X^r/s(1 + Xh(X)) = (X^r/s)p_1^{n_1}...p_t^{n_t}$, where $X^r/s \in M + XL[X]$ and so is a pseudo atom in that no two non unit factors of X^r/s are co-maximal and each of $p_i^{n_i}$ is a pseudo atom, in the lingo of McAdam and Swan [29]. So if D is a quasi local domain of any type $D + XL[X]$ is a UCFD. Below I mention some of the oddest quasi local domains that give rise to UCFDs $R = D + XL[X]$. (I) The first that comes to mind is a regular local ring D of dimension $n > 1$. (II) D can be any Noetherian local domain or a non-Noetherian quasi local domain. In short you get examples of UCFDs. We can indeed have (III) $D = K$ a field, taking K to be quasi local with $M = (0)$. In this case $R = K + L[X]$ which can be shown to be atomic (See e.g. [4, Theorem 2.9]) This can serve as an example of a UCFD that is atomic but its pseudo atoms are different from its atoms. Now these rings $D + XL[X]$ are generally not integrally closed when $K \neq L$, unless D is integrally closed in L [4, Theorem 2.7] and as we have established, are UCFDs when D is quasi local. Oh and generally, if D is not a field, $D + XL[X]$ is not atomic, as it allows $X = d^m(X/d^n)$ for all nonzero non unit $d \in D$.

The story doesn't end here. With the same description we can have t-h-local domains, if we start with a t-local domain, i.e a quasi local domain (D, M) with M a t-ideal. The resulting $D + XL[X]$ will be UCFD, which is also a t-h-local domain. (The only maximal ideal $M + XL[X]$ that intersects D is a t-ideal and the prime ideals that intersect D trivially are all principal height one maximal and hence t-ideals.). Now there are all sorts of t-local domains (see e.g. [22]). So if you are interested in constructing t-h-local domains that are not integrally closed or integrally closed of an odd kind) and are t-pure t-UCFDs you may use $D + XL[X]$ with D a t-local domain with specific properties ($D + XL[X]$ is integrally closed if D is integrally closed in L etc.) Of course you can take D to be an almost valuation domain (AV domain), D that allows for each pair $x, y \in D\backslash\{0\}$ a natural number $n = n(x, y)$ such that $x^n | y^n$ or $y^n | x^n$. Let's also recall that a saturated multiplicative subset S of D is said to be a splitting set of D if for each $x \in D\backslash\{0\}$ we have $x = dr$ where $r \in S$ and $dD \cap sD = dsD$ for all $s \in S$. A splitting set is said to be an lcm splitting set if for each $s \in S$ and $d \in D\backslash\{0\}$ we have $dD \cap sD$ principal.

Indeed as the set S_1generated by primes of the form $1 + Xf(X)$ of $D + XL[X]$ is a multiplicative set of $D + XL[X]$. Let S be the saturation of S_1 in $D + XL[X]$. Then by the description of elements of $D + XL[X]$, S is an lcm splitting set of $D + XL[X]$. But then, using Theorem 4.2 of [5] we have $Cl_t(D + XL[X]) \cong Cl_t(D + XL[X])_S = Cl_t(D + XL[X]_{(X)})$. But if D is a t-local domain, so is $D + XL[X]_{(X)}$, forcing $Cl_t(D + XL[X]_{(X)}) = (0)$ and giving $Cl_t(D + XL[X]) = (0)$. These observations lead to the following conclusion.

Proposition N. Let D be a t-local domain with quotient field K, let L be an extension field of K and let X be an indeterminate over L. Then (a) $D + XL[X]$ is t-h-local, (b) $Cl_t(D + XL[X]) = (0)$ and consequently $D + XL[X]$ is a t-pure

t-UCFD.

Corollary P. Let D be a t-local domain with quotient field K, let L be an extension field of K and let X be an indeterminate over L. Then $D+XL[X]$ is a PVMD if and only if D is a valuation domain and $K=L$. Also the following are equivalent.

(1) $D+XL[X]$ is a PVMD

(2) $D+XL[X]$ is a semirigid GCD domain

(3) D is a Prufer domain

(4) $D+XL[X]$ is a Bezout semirigid domain.

Looking at the above examples one gets the feeling that if we want to construct a semirigid GCD domain or an independent ring of Krull type, we are stuck with the above situation. But it doesn't have to be that way. Using rigid elements of a special kind we can construct all sorts of interesting examples. Following [11] call an element $r\in D$ t-f-rigid (t-factorial rigid), if rD is a t-homogeneous ideal such that every proper t-homogeneous ideal containing r is principal. (rD is $\star$-homogeneous $=$ r belongs to a unique maximal t-ideal $=$ r is t-pure.)

One way of producing examples was used in now classical, almost forgotten, [33] as follows.

Example 2 of [33] says: Let V be a valuation and X be an indeterminate over V. Then $V[X]$ is a semirigid GCD domain. Of course using the same reasoning one can prove the following result and its converse, using rigid elements.

Proposition Q. Let X be an indeterminate over an integral domain D. Then D is a semirigid GCD domain if and only if $D[X]$ is.

Proof. Let D be a semirigid GCD domain. Then $D[X]$ is a GCD domain. To see that $D[X]$ is semirigid take a typical element $f(X)\in D[X]$. Then $f(X)=df'(X)$ where d is the GCD of coefficients of f and f' the primitive polynomial f/d. Now note that for a rigid element $r\in D$, $h(X)|r$ in $D[X]$ implies that $r=h(X)k(X)$, forcing the degrees $h(X)$ and $k(X)$ to be zero and any factors of r in D. Based on this we conclude that rigid elements of D are rigid in D. Now d being an element of D is a product of rigid elements, say $d=r_1...r_m$ and f' being a primitive polynomial in the GCD domain $D[X]$ is expressible as a finite product of primes i.e $f'=p_1^{l_1}...p_n^{l_n}$. Thus $f=r_1...r_mp_1^{l_1}...p_n^{l_n}$ is a product of rigid elements. The converse is obvious in that if $D[X]$ is a semirigid GCD domain then D is known to be a GCD domain and the rigid factorization of each $d\in D$ can be read off from the rigid factorization of $d\in D[X]$.

An alternative proof can be effected by showing that D is an independent ring of Krull type (IRKT) if and only if $D[X]$ is. This can be done by noting that since, according to Lemmas 7 and 8 of [35] every maximal t-ideal M of $D[X]$ is either an upper to zero ($M\cap D=(0)$) or of the form $M=P[X]$ where P is a prime t-ideal and it is easy to see that P is a maximal t-ideal, we conclude that every nonzero non unit of $D[X]$ belongs to at most a finite number of maximal t-ideals of $D[X]$. (A nonzero element of $D[X]$ can belong to only a finite number of uppers to zero because $K[X]$ is a PID and belongs to only a finite number of maximal t-ideals of the form $P[X]$, because D is a ring of Krull type.) Now to show that $D[X]$ is an independent ring of Krull type let M,N

be two maximal t-ideals of $D[X]$ and suppose that P is a nonzero prime ideal contained in $M \cap N$. Now P cannot be an upper to zero because an upper to zero is a maximal t-ideal as D is a PVMD and Lemma 7 of [35] applies. So if there is a common prime P, to M and N then $P \cap D = p \neq 0$. Thus we have shown that if D is an IRKT then so is $D[X]$. Conversely suppose that $D[X]$ is an IRKT, by Theorem 7.1 of [30], D is a ring of finite (t-) character. Now let P and Q be two maximal t-ideals of D. If there is a nonzero prime ideal $m \subseteq P \cap Q$. Since P and Q are valued primes in the terminology of [30] and so are $P[X]$ and $Q[X]$ because $D[X]_{P[X]} = D_P(X)$. But then $P[X]$ and $Q[X]$ are prime t-ideals. If $\mathcal{P} \supseteq P[X]$ and $\mathcal{Q} \supseteq Q[X]$ are maximal t-ideals of $D[X]$, then, by Proposition 1.1 of [28], $\mathcal{P} = P[X]$ and $\mathcal{Q} = Q[X]$. But then $m[X] \subseteq P[X] \cap Q[X]$ contradicting the fact that $D[X]$ is an IRKT. Thus we have the following result.

Proposition R. Let X be an indeterminate over an integral domain D. Then (1) D is an IRKT if and only if $D[X]$ is and (2) D is a semirigid GCD domain if and only if $D[X]$ is.

Proof. (1) has already been proven and for (2) note that D is semirigid GCD domain if and only if D is a GCD IRKT.

Here's yet another way. To facilitate the proof we need the following lemma.

Lemma S0. Let D be a PVMD, X an indeterminate over D and let $S = \{f \in D[X] | (A_f)_v = D\}$. (Here A_f denotes the content of f, i.e. the ideal generated by coefficients of f.). Let P be a prime (resp., maximal) ideal of $D[X]_S$. Then $P \cap D[X] = p = p'[X] = (p \cap D)[X]$ where p' is a prime (resp. maximal) t-ideal of D. Conversely if p' is a prime (resp., maximal) t-ideal of D, then $p'[X]D[X]_S$ is a prime (resp., maximal) ideal of $D[X]_S$.

Proof. If P is a prime (resp., maximal) ideal of $D[X]_S$, then, since $D[X]_S$ is Bezout, $(D[X]_S)_P$ is a valuation domain. As P is a prime ideal of $D[X]_S$, P corresponds to a prime ideal p of $D[X]$ such that $p \cap S = \phi$. Thus $(D[X]_S)_P = D[X]_p$ and by Corollary 4.2 of [30] p is a prime t-ideal of $D[X]$. Now $p \cap D = p' \neq (0)$. For if $p \cap D = (0)$, then p would not miss S by the proof of Lemma 7 of [35]. But then $p = p'[X] = (p \cap D)[X]$ where p' is a prime t-ideal of D. Now let $\mathcal{P}$ be a maximal t-ideal containing p. Then $\mathcal{P} \cap \mathcal{S} = \phi$ for otherwise $\mathcal{P}$ would not be a t-ideal. But then $\mathcal{P}D[X]_V \supseteq pD[X]_p$, by the order preserving correspondence. Whence $\mathcal{P} = p = p'[X]$. Again if $\wp$ is a maximal t-ideal of D containing p' then since $\wp[X]$ must be disjoint from S to be t-ideal we must have $\mathcal{P} = p = p'[X] = \wp[X]$ forcing p' to be a maximal t-ideal. Conversely if p' is a prime t-ideal of D, then $p'[X]$ is a t-ideal for if $p'[X]$ were not a t-ideal then by [35, Lemma 9] $p'[X] \cap S \neq$ forcing p' to be a non t-ideal. Now that $p'[X]$ is disjoint from S, we conclude that $p'[X]D[X]_S$ is a prime ideal of $D[X]_S$. Finally, if p' is a maximal t-ideal of D. then $p'[X]$ is a maximal t-ideal. For if $\mathcal{P} \supseteq p'[X]$ were a maximal t-ideal then by Proposition 1.1 of [28] we have $\mathcal{P} = (\mathcal{P} \cap D)[X]$. But $(\mathcal{P} \cap D) = p'$ because p' is a maximal t-ideal.

Proposition S. Let D be an integrally closed integral domain, X an indeterminate over D and let $S = \{f \in D[X] | (A_f)_v = D\}$. (Here A_f denotes the content of f, i.e. the ideal generated by coefficients of f.) Then D is an IRKT if and only if $D[X]_S$ is a semirigid Bezout domain.

Proof. Let D be an IRKT. Then in particular D is a ring of Krull type and hence a PVMD. So, according to results on page 720 of [24], $D[X]$ and $D[X]_S$ are rings of Krull type and according to [35, Corollary 13] $D[X]_S$ is Bezout, because D is a PVMD. Moreover, $D[X]$ is an IRKT by Proposition R. To show that $D[X]_S$ is semirigid let M and N be two maximal ideals of $R = D[X]_S$ such that there is a nonzero prime ideal $Q \subseteq M \cap N$. By Lemma S0 $M = m'[X]$, $N = n'[X]$ and $Q = q'[X]$ where m', n' are maximal t-ideals and q' is a nonzero prime t-ideal contained in $m' \cap n'$, which is impossible because $D[X]$ is an IRKT. Thus $D[X]_S$ is an IRKT. But an IRKT, Bezout is a GCD IRKT and so is a semirigid a Bezout domain. Conversely let $D[X]_S$ be a semirigid Bezout domain. Then D is a PVMD to start with. Suppose that there is a nonzero non unit x in D such that x belongs to an infinite set of distinct maximal t-ideals $\{p_\alpha\}$ of D. Then x belongs to infinitely many maximal t-ideals $\{p_\alpha[X]\}$ of $D[X]$, Since each of $p_\alpha[X]$ is a t-ideal, $p_\alpha[X] \cap S = \phi$ and so x belongs to infinitely many maximal ideals $p_\alpha[X]D[X]_S$ of $D[X]_S$ which is impossible because $D[X]_S$ is of finite character. Thus D is of finite t-character. Similarly if m, n are two distinct maximal t-ideals of D containing a nonzero prime ideal p. Then p is a prime t-ideal to start with and $p[X]D[X]_S$ is contained in the intersection of maximal ideals $m[X]D[X]_S \cap n[X]D[X]_S$ a contradiction to the fact that $D[X]_S$ is a Bezout IRKT. Thus D is an IRKT.

There are other ways of constructing semirigid GCD domains. Call $x \in D\backslash\{0\}$ primal if $x|ab$ implies $x = rs$ where $r|a$ and $r|b$. Also call x completely primal if every factor of x is primal. An integrally closed integral domain D was called Schreier by Cohn [19] if every nonzero element of D was primal. It was shown in [19] that a GCD domain is Schreier. (Thus every nonzero element of a GCD domain is completely primal. A nonzero non unit element q of a domain D is said to be a prime quantum if q meets the following conditions:

q_1. For all natural numbers n q^n is rigid'

q_2. q is completely primal,

q_3. For each non unit factor h of q there is a natural number n such that $q|h^n$.

It was shown in [31] that, in D, any pair of non-coprime prime quanta were comparable and thus $x \in D$ was a product of finitely many prime quanta then x was uniquely expressible as a finite product of mutually coprime prime quanta. A domain D was called a generalized UFD (GUFD) if every nonzero non unit of D was expressible as a finite product of prime quanta. It was also shown in [31] that a GUFD was a GCD generalized Krull domain. Here a generalized Krull domain (GKD) is a domain D such that (1) $D = \cap_{P \in X^1(D)} D_P$ where the intersection is locally finite, $X^1(D)$ is the set of height one primes and (2) D_P is a valuation domain for each $P \in X^1(D)$. If we drop (2) from the definition of GKD we get a weakly Krull domain (WKD) while a WKD D with $Cl_t(D) = (0)$ is known as a weakly factorial domain (WFD). It is easy to show that a GCD-WFD, a GCD WKD and a GCD-GKD are the same, each is a GUFD. Moreover, when D is a GCD domain a prime quantum is just a rigid element satisfying q_3.

All my efforts at publishing my results about GUFDs were thwarted with asinine comments about punctuations. In any case, results from the first chapter of [31], with much more, were published as [7], thanks to David and Dan. In any case, coming back to business at hand, given that D is a GCD domain, S a saturated multiplicative set in D and X an indeterminate over D_S the ring $D^{(S)} = D + XD_S[X] = \{f \in D_S[X] | \; f(0) \in D\}$ is a GCD domain if and only if S is a splitting multiplicative set of D [36, Corollary 1.5]. Theorems C and D of [32] state the following.

Theorem T. Let D be a GUFD, S a multiplicative set of D and X an indeterminate over D_S. Then (1) $D + XD_S[X]$ is a GCD domain and (2) $D + XD_S[X]$ is a GCD ring of finite t-character if and only if S meets at most a finite number of height one primes of D.

The theory has moved way beyond GCD domains.

Theorem U. (cf. [8, Theorem 2.4])The following statements are equivalent for $D^{(S)} = D + XD_S[X]$.

(1) $D^{(S)}$ is a ring of Krull type.

(2) D is a ring of Krull type, S is a t-splitting set, and the set of maximal t-ideals of D that intersect S is finite.

Here a multiplicatively closed subset S of D is a t-splitting set if for each $d \in D\backslash\{0\}$, $(d) = (AB)_t$ for some integral ideals A and B of D,

where $(A; s)_t = D$ for all $s \in S$ and $B_t \cap S \neq \phi$.

Corollary V (cf. [8, Corollary 2.6]). $D^{(S)} = D + XD_S[X]$ is an independent ring of Krull type if and only if D is an independent ring of Krull type, S is a t-splitting set, and $|\{P \in t - Max(D)|P \cap S \neq \phi\}| \leq 1$.

But finding a multiplicative set that is a t-splitting set may not be easy. So we must look for some easier way to find a splitting set.

Noting that a generalized Krull domain is an independent ring of Krull type let's first have a simple result like Corollary V.

Corollary W. Let D be a GKD. $D^{(S)} = D + XD_S[X]$ is an independent ring of Krull type if and only if S is a t-splitting set, and

$|\{P \in t - Max(D)|P \cap S \neq \phi\}| \leq 1$.

This leads to the following result.

Corollary X. Suppose that D is a GKD with at least one prime quantum q and let S be the saturation of $\{q^i\}_{i=1}^{\infty}$. If S is a splitting set of D, then $D^{(S)} = D + XD_S[X]$ is an IRKT. Thus given that D is a GUFD $D^{(S)} = D + XD_S[X]$ is a semirigid GCD domain if and only if S is multiplicatively generated by a prime quantum.

Proof. The first part is a straightforward corollary to Corollary W. The second part uses the fact that every saturated multiplicative set in a GUFD is a splitting set and Corollary W.

I guess I must stop here, as going beyond this point would be giving away a lot.

Finally, yes I have had a lot of help. The gentlemen who really helped me were Robert Gilmer and Joe Mott. Both of them acted as mentors for me long before I started thinking of coming to the US. Now get ready for a long story. I came to the University of North Carolina as a visiting instructor or something. Until the end of the contract, at UNC Charlotte, there "was hope" that the contract will be renewed. But apparently that was a false hope. A Hindu friend of mine named Visuanathan had warned me about the pace at which I was going, saying I had written too many good papers with them too soon and so I'd be going away very soon. I did not quite understand as that kind of pace had been business as usual for some time. I did write some good papers with them, including some from the t-linked overring sequence e.g. [21], one on t-invertibility [27] and one on TV domains [28] but not a lot in my opinion. (Course, they contributed a lot, but I brought in the fresh blood, as Joe Quinn the then head of department of Mathematics at UNC Charlotte, put it.) In any case, I was at the brink of being railroaded to leave the US, to go God knows where as I could not go back to Pakistan for fear of my life and I could not go back to Britain. At that point Joe Mott, God bless him, came forward and got me a job at FSU, Tallahassee, Florida. An old college mate and at that time a Mathematician working for the US Govt., the late Dr. Basharat Jameel, helped with the visa etc. and I had a foothold.

Apparently, my bad luck traveled with me to Tallahassee, within days there was hue and cry that not only was my accent hard, but also my terminology was incorrect (I used "a into b" to mean "a times b", in the old Indian/Arabic fashion). (The reason why I believe there was a plan in place to treat me as a Chinese Railroad worker is that those fellows, at UNC Charlotte, never told me that I had that terminology problem.) In any case my classes were observed and those who should have taken notes on how to introduced new topics were "advising" me on how to teach. I was teaching several classes, two of which consisted of non-Math students. I knew that no teacher can really satisfy disinterested students and did not mind being "observed" to pacify them. But I was also teaching a course on trigonometry for engineering students who seemed to be quite satisfied with my teaching. When three "Observers" showed up to observe and to give me pointers on how to teach at that trigonometry class my patience gave out and I walked out of the class, saying that I had had enough of being taught on how to teach. My students also spoke out and actually cornered my "observers" for harassing me, saying I was the best teacher they had had in years. The whole thing sort of died down when, after the first exam, it turned out that the sections that I was teaching had done remarkably better than others. Sometimes, brooding over "What did I do wrong" I get the sick feeling that there were some other forces at work. Because, once there was some peace, a woman named Becky something, who was linked with organization of the courses, decided to chase Joe Mott, apparently to "avenge defeat". In response, Joe got a job offer from another school and gave notice to FSU. Consequently FSU had to make an offer of better conditions to keep him.

All seemed well and there was hope for a peaceful next semester at FSU when the then head of department of mathematics (McWilliams, I think) summoned me to his office and asked "who are Ahmadis?" I was taken aback, as I never took my religious faith to my classes. In any case, I told him that there is a Muslim community called the Ahmadiyya Community in Islam, of which I am a member and members of this community call themselves Ahmadis and that I was in the US because my community was under a lot of fire in Pakistan. In response he said that one of my Pakistani students had applied to be moved to some other section and mentioned as reasons that I was an Ahmadi and as she had been voicing her concerns about my "accent" she was afraid of reprisals from me. I had not distingushed her as a Pakistani girl from the crowd of the noisy lot who thought the Math course was a means of lowering their GPA and so they had the right to avail any chance of letting out some steam. In any case I told McWilliams that as an An Ahmadi, I was supposed to help any and all. Equipped with the young lady's name, I discreetly investigated. I was told with something like a suppressed smile that she was a niece of a Pakistani Professor. It was as mundane an information as could be, in the US environment. But it set me thinking: Oh my God! Here we go again. (I left Libya because I was sick of the intrigues of my Indian and Pakistani Muslim brothers trying to tell my Libyan students that I was an Ahmadi and hence non-Muslim etc., after they were rebuffed by the administration with: Anyone who works as intelligently and as diligently as Muhammad is a good Muslim. Something similar happened at UMIST Manchester. I have some facility with generators and relations and so a Pakistani graduate student was attached with me at UMIST. I noticed that when he could not provide an argument to join one statement with the next he'd say, "it follows that". I was taught to check such "it follows thats"! I pushed him to provide the argument or ask for (my) help. Instead, he went to his supervisor (Roger Bryant) and literally wept in front of him saying that because I was an Ahmadi, I was giving him a hard time, thus diminishing my already dismal chances of staying on at UMIST.)

In short McWilliam's attituded and the threat of Pakistani intrigue scared me stiff. I mentioned my dissatisfaction with FSU to Dan and he got me a job at the University of Iowa. At University of Iowa I had the most comfortable time teaching and doing research, thanks to Dan Anderson. My friends' help went in to get me a tenure track Associate Professorship at Winthrop College, Rock Hill, SC. But of course, my ill luck did not let me be, even there and if I had not managed to run away, some mysterious epilepsy-like) illnesses would have left me useless for life.

https://loharcom.wordpress.com/2020/09/20/my-vagabond-days/

After my kidney transplant I got a part-time job at Bowie State University, Md, teaching a couple of courses and was trying to make both ends meet with handouts from my son and an occasional stint at a nearby Seven Eleven, when Dan Anderson (God bless him for that!) came to the rescue and got me a job at the University of Iowa, providing me with a very much needed turning point and within a couple of years I was at a position which it would have taken me a decade to get to, under ordinary circumstances. The fact that I was thankful to Dan is apparent from the fact that I never, really, objected to Dan using ideas developed with me to help his students write their doctoral dissertations. On the other hand, and let me mention that, Dan had an invitation from Inha University, Incheon, South Korea and took me and Jim Coykendall along to give talks. There he started "Me and my brother" rant about [3]. Jokingly I said, "Hey I am still alive!" At the end of his talk he comes to me and says, "I should have left you working at the seven eleven!" And I was like, "Thank you for helping me out but what kind of gracious talk is that?" And frankly, he did not help me out for nothing. I was working on the w-operation and all the results that I had, including t-$Max(D) = w$-$Max(D)$ went into S. Cook's thesis and his paper with her. All the coffin thief gave me in return was the comment, "The authors thank M. Zafrullah for piquing our interest in the w-operation, and for several helpful suggestions." (I say "coffin thief" because I had had a major operation and was barely out of the woods and needed all the help I could.) I did not care about it then and I do not care about it now. God gives me ideas and I share them and am usually thankful to those who acknowledge the shares.

Finally when I sprang back and started producing papers like [37], when fate struck again. I was invited to spend a month at the Department of Mathematics at Beijing University, by Professor Yichuan Yang. Giving a talk at Chengdu University in China (Professor Fanggui Wang had borrowed me for fifteen days) I had a stroke-like event. Thankfully, I did not lose control and completed the talk and later, my one month in China. Of course my visit to China was all paid for but as I had overspent and as I needed money for doing my share in some family events, I needed some money. I mentioned my problems to Dan and he again came in with help! During my stay at Iowa I wrote my only "travel-log": https://lohar.com/images/researchpdf/exploits of a geriatric in China.pdf . After teaching for two semesters at the University of Iowa I was really drained. But apparently I had not learnt my lessons and to honor a prior commitment took up a job at the Boise State Unbiversity. There I had a stroke in the middle of the semester and the team of healthworkers and psychiatrists declared me unfit for teaching. Researchers in my area seemed very helpful, some even dedicating papers to me. But then, thanks to the mean an despicable streak in me, I sprang back and started writing decent papers. I still had no intention of bitching about what was going on, but then things like funny Korean webpages (see e.g. https://lohar.com/mithelpdesk/hd2008.pdf) and then appeared a paper in JPAA (see e.g. https://lohar.com/mithelpdesk/hd2004.pdf) that broke the (proverbial) camel's back. I got to the roots of the problem and started hacking at them. In spite of all that I still love Dan as my little brother.

In fact if I look at it carefully, and let me walk you through the confused mess of my love hate jungle full of weeds, I have nothing but love for my coauthors in multiplicative ideal theory. Let me explain. Very early on I realized that my solo papers would be fodder for folks such as some South Koreans like B.G. Kang ang G.W. Chang who probably have been listening to Buddhist monks who are known to have thrown Muslim babies into fire in Assam. They probably like to swipe results from Muslim authors as a religious duty. So I chose to raise the slogan of "there's protection in numbers" and often begged my coauthors to join in. (Joe often objected to that and so did Dan, but somehow I convinced them.) They obliged and I am thankful for the protection they provided by joining in and by actually adding a lot more to the literature as their contribution. Through this rant I assure all of my coauthors that I won't do anything to hurt them. But I must tell the world about how my life was stolen. I hope this answers the part to do with "biting the hand that feeds.

Having said all that, let me describe working with Dan and David. I describe them and some of my American coauthors as Mathematical juggernauts, all laiden with books and knowledge. I often had to make a statement of a theorem, they would provide a better proof than I had in mind and much more. Just as with a big truck that stands still by the road side and as you turn the ignition key, it roars into life and with its power steering system needing very little force to take you wherever you want. A lot of my ideas would not see the light of day, if I had not been perched on the shouylders of these giants. My complaints are not against my giants, but against the mean streak in some of them.

References

[1] D.D. Anderson, Non-atomic unique factorization in integral domains, in Arithmetic properties of commutative rings and monoids, S.Chapman, ed., 1-21; Lect. Notes Pure Appl. Math. 241, CRC Press Boca-Raton, 2005.

[2] D.D. Anderson, D.F. Anderson, Factorization in integral domains, IV, Comm. Algebra 38 (2010) 4501-4513.

[3] D.D. Anderson, D.F. Anderson and M. Zafrullah, Factorization in integral domains, J. Pure Appl. Algebra 69 (1990), 1-19.

[4] D.D. Anderson, D.F. Anderson and M. Zafrullah, Rings between $D[X]$ and $K[X]$, Houston J. Math. 17(1)(1991), 109-129.

[5] D.D. Anderson, D.F. Anderson and M. Zafrullah,"Splitting the t-class group" J. Pure Appl. Algebra 74(1991) 17-37.

[6] D.D. Anderson, D.F. Anderson and M. Zafrullah, Factorization in integral domains II, J. Algebra 152(1)(1992), 78-93.

[7] D.D. Anderson, D.F. Anderson and M. Zafrullah, A generalization of unique factorization, Bollettino U. M. I. (7) 9-A (1995), 401-413.

[8] D.D. Anderson, G.W. Chang and M. Zafrullah, Nagata-like theorems for integral domains of finite character and finite t-character, J. Algebra Appl., 14(8) (2015).

[9] D.D. Anderson and Andrea Frazier, On a general theory of factorization in integral domains, Rocky Mountain J. Math., 41 (3) (2011), 663-705.

[10] D.D. Anderson, J. Mott and M. Zafrullah, Unique factorization in non atomic domains, Bollettino U. M. I. 6 (1992), 613 630.

[11] On -Semi$*$-homogeneous Integral Domains. In: Badawi A., Coykendall J. (eds) Advances in Commutative Algebra. Trends in Mathematics. Birkhauser, Singapore, (2019).

[12] D. D. Anderson and M. Zafrullah, Independent locally-finite intersections of localizations, Houston J. Math. 25 (1999), 433-452.

[13] D.F. Anderson, A general theory of class groups, Comm. Algebra 16(1988), 805-847.

[14] D.F. Anderson and Driss N. El Abidine, Factorization in integral domains, III, J. Pure Appl. Algebra, 135 (1999), 107-127.

[15] A. Bouvier, "Le groupe des classes d'un anneau integre" 107 eme Congres National des Societe Savantes, Brest, Fasc. IV(1982) 85-92.

[16] A. Bouvier and M. Zafrullah, On some class groups of an integral domain, Bull. Soc. Math. Grece 29(1988) 45-59.

[17] G.W. Chang, Unique factorization property of non-unique factorization domains, J. Algebra Appl. (2020).

[18] G.W. Chang and Andreas Reinhart, Unique factorization property of non-unique factorization domains II, J. Pure Appl. Algebra 224 (12) (2020), 1-18.

[19] P.M. Cohn, Bezout rings and their subrings, Proc. Camb. Philos. Soc. 64 (1968) 251–264.

[20] P.M. Cohn, Unique factorization domains, Amer. Math. Monthly, 80 (1)(1973) 1-18.

[21] D. Dobbs, E. Houston, T. Lucas. E. Houston and M. Zafrullah, t-linked overrings and Prüfer v-multiplication domains, Comm. Algebra 17(1989) 2835-2852.

[22] M. Fontana and M. Zafrullah, On t-local domains and valuation domains, In: Badawi A., Coykendall J. (eds) Advances in Commutative Algebra. Trends in Mathematics. Birkhauser, Singapore, (2019).

[23] R. Gilmer, Multiplicative Ideal Theory, Marcel-Dekker, New York, 1972.

[24] M. Griffin, Some results on v-multiplication rings, Canad. J. Math.19(1967) 710-722.

[25] M. Griffin, Rings of Krull type. J. Reine Angew. Math. 229, 1–27 (1968).

[26] E. Houston, S. Malik and J. Mott, Characterizations of $*$-multiplication domains, Canad. Math. Bull. 27(1)(1984) 48-52.

[27] E. Houston and M. Zafrullah,"Integral domains in which every t-ideal is divisorial" Michigan Math. J. 35(1988) 291-300.

[28] E. Houston and M. Zafrullah, On t-invertibility II, Comm. Algebra 17(8)(1989) 1955-1969.

[29] S. McAdam and R.G. Swan, Unique comaximal factoration, J. Algebra, 276 (2004), 180-192.

[30] J. Mott and M. Zafrullah, On Prüfer v-multiplication domains, Manuscripta Math. 35(1981)1-26.

[31] M. Zafrullah, Unique factorization and related topics, thesis submitted to the University of London in 1974.

[32] M. Zafrullah, Rigid elements in GCD domains, J. Natur. Sci. and Math. **17** (1977), 7–14.

[33] M. Zafrullah, Semirigid GCD domains, Manuscripta Math. 17(1975), 55-66.

[34] M. Zafrullah, Finite conductor domains, Manuscripta Math. 24(1978) 191-203.

[35] M. Zafrullah, Some polynomial characterizations of Prüfer v-multiplication domains, J. Pure Appl. Algebra 32(1984) 231-237.

[36] M. Zafrullah, The $D + XD_S[X]$ construction from GCD-domains, J. Pure Appl. Algebra 50(1988), 93-107.

[37] M. Zafrullah, t-Invertibility and Bazzoni-like statements, J. Pure Appl. Algebra, 214(5) (2009) 654–657.

[38] M. Zafrullah, Semirigid GCD domains II, https://arxiv.org/pdf/2012.10339.pdf