QUESTION (HD 2103) I am reading your paper "Factorizations in Integral Domains II", and I have some questions regarding S being a splitting multiplicatively closed set (mcs) of R. If S is a mcs generated by primes, S is not necessarily a splitting mcs.

1. If R is Archimedean, is S generated by primes a splitting mcs?

2. Is the Archimedean property a strong hypothesis?

3. If S is generated by only one prime element, when is S a splitting mcs?

ANSWER: A splitting set of D is a multiplicative set S of D such that

(a) S is saturated and

(b) Every nonzero element $d \in D$ can be written as $d = rs$ where $s \in S$ and and $r \in D$ such that $rD \cap tD = rtD$ for all $t \in S$.

Thus we have the following observations.

Lemma A. Let S be a saturated multiplicative (ly closed) set (smcs). Then S is a splitting set if and only if for each $x \in D\backslash(0)$ we have $xD_S \cap D$ principal.

Proof. Let S be a smcs such that for each $x \in D\backslash(0)$ we have $xD_S \cap D$ principal and let $xD_S \cap D = dD$. Obviously as $x \in xD_S \cap D$ we have $d|x$. But then $x = ds$. We claim that $s \in S$. Because $xD_S \cap D = dD$ implies $xD_S = dD$ which forces $xt = dt'$ for some $t, t' \in S$. But then $dst = dt'$ cancelling d from both sides we have $st = t'$. As S is a smcs we conclude that $s \in S$. Next if $xD_S \cap D = dD$, then obviously $dD_S \cap D = dD$. Now let $s \in S$ and consider $dD \cap tD$. As $dtD \subseteq dD \cap tD$, all we need is the reverse containment. Let $z \in dD \cap tD$. Then $z = da = tb$. Multiplying the second equality by s we get $das = tbs$. So $ax = ads = tbs = zs$. This gives $b = d(as/st) = d(a/t)$ where $a \in S$ because $dD_S \cap D = dD$. So $b \in d(a/t)D_S \cap D = dD$ a principal ideal. But then $b = dc$ for some $c \in D$. That gives $bt = dct$ and so $z = dtc \in dtD$. Thus $x = ds$ where $s \in S$ and $dD \cap tD = dtD$ for all $t \in S$. Conversely if for each $x \in D\backslash\{0\}$ $x = ds$ where $s \in S$ and $dD \cap tD = dtD$ for all $t \in S$ we get $xD_S \cap D = dsD_S \cap D = dD_S \cap D$. Obviously $dD \subseteq dD_S \cap D$. For the reverse inclusion let $z \in dD_S \cap D$. Then $z = d(r/s)$, for $r \in D$ and $s \in S$. Then $zs = dr \in dD \cap sD = dsD$ giving $d|z$ or $dD_S \cap D \subseteq dD$.

Proposition B. Let S be a smcs generated by principal primes, then S is a splitting mcs if and only if every non zero non unit of D is (i) divisible by at most a finite number of nonassociated primes from S and (ii) for every prime p in S we have $\cap p^n D = (0)$.

Proof. Let S be an smcs generated by principal primes. Then for each $d \in D\backslash\{0\}$, there are only finitely many primes from S dividing d by (i). Let $T = \{p_1, p_2, ..., p_m\}$ be the set of nonassociated primes from S dividing d. By (ii) $\cap p_i^n D = (0)$ for each prime p in S. So $d = d_1(p_1)^{n_1}$ such that $p_1 \nmid d_1$. Set $d_1 = d/(p_1)^{n_1}$ and note that as $\cap p_2^n D = (0)$ there must be n_2 such that $p_2^{n_2}|d_1$ and $p_2^{n_2+1} \nmid d_1$. Thus $d_1 = d_2 p_2^{n_2}$ giving $d = d_2 p_1^{n_1} p_2^{n_2}$ where $p_1, p_2 \nmid d_2$. Similarly continuing we get $d = d_m p_1^{n_1} p_2^{n_2} ... p_m^{n_m}$ where $p_1, ..., p_m \nmid d_m$. But then no prime from S divides d_m and so $d = xs$ where $x = d_m$ such that $xD \cap tD = xtD$ for all $t \in S$ and $s = p_1^{n_1} p_2^{n_2} ... p_m^{n_m} \in S$. Conversely, suppose that S is a splitting set then for each $x \in D\backslash\{0\}$ we have $x = ds$ where $d \in D, s \in S$ and $dD \cap tD = dtD$ for all $t \in S$. Indeed as $s = p_1^{n_1} p_2^{n_2} ... p_m^{n_m} \in S$, x is divisible by at most a finite number of nonassociated primes from S. Let p be a prime dividing x. Suppose, by way of contradiction, that $\cap\, p^nD \neq (0)$.Then there is $0 \neq x \in \cap\, p^nD$. But then $xD_S \cap D$ must be principal because S is a splitting set. Say $xD_S \cap D = dD$, so that $x = ds$ where $s \in S$ and d is coprime to all primes in S. On the other hand $p^n|x$ for all n and only a finite power of p can divide s. whence some powers of p divide d a contradiction. Thus $\cap\, p^nD = (0)$ for every $p \in S$.

Corollary C. Let S be multiplicatively generated by a finite set of nonassociated prime elements $T = \{p_1, ..., p_m\}$ such that for all $p \in T$, we have $\cap p^nD = (0)$. Then S is a splitting set.

Note D. In the absence of the restriction that for all $p \in T$ we have $\cap$ $p^nD = (0)$ a finite set $T = \{p_1, ..., p_m\}$ of primes of a domain D does not generate, multiplicatively, a splitting set.

Examples E. (a) Take a valuation domain (V, M) of rank ≥ 2 with $M = pV$ and consider $S = \{p^n\}_{n=0}^{n=\infty}$. Then the saturation of S is not a splitting multiplicative set as no element x of $\cap\, p^nD$ can be written as $x = dp^m$ where d is coprime to every power of p.

(b) Let $D = Z_{(2)} + XR[[X]]$ where Z is the ring of integers and R the field of real numbers. Then D is a quasilocal ring with maximal ideal $2D$, $S = \{\pm 2^n|n$ a nonnegative integer$\}$ is a saturated multiplicative set, but not a splitting set.

Next, (c) D is Archimedean if $\cap x^nD = (0)$ for all x in D. So, in an Archimedean domain D, a saturated multiplicative set generated only by a finite number of principal primes is a splitting set, by Corollary C.

Next (d) A completely integrally closed domain D is Archimedean (Corollary 5, of Gilmer and Heinzer's [J. Aust. Math. Soc. 6 (1966), 351-361] and the ring of entire functions is completely integrally closed.

Finally (e) Given that D is a GCD domain, a splitting set in D is what is termed as an lcm splitting multiplicative set, has the extra property that for each $s \in S$ and $d \in D$ we have $sD \cap dD$ principal. Using Theorem 2.10 of D.F Anderson and Noure-el-Abidine's paper [J. Pure Appl. Algebra 159 (2001) 15–24] (or Corollary 1.5 of [J. Pure Appl. Algebra 50(1988), 93-107]) we show that for a GCD domain D, $D + XD_S[X]$ is a GCD domain if and only if S is a splitting multiplicative set of D.

Claim: a saturated multiplicative set, in an Archimedean domain D, generated by an infinite set of nonassociated principal primes may not be a splitting set. To establish our claim we need to recall some information on the ring of entire functions. A function that is analytic in the entire finite plane is called an entire function. It is not too hard to establish that the set of all entire functions E is an integral domain, with elements that are nowhere zero serving as units. Olaf Helmer [Duke Math. J. 6 (1940), 345-356] showed that every finitely generated ideal A in E is principal, thus showing that E is a Bezout domain and hence a GCD domain. (See also Exercise 18 p 147 of [Multiplicative Ideal Theory, Marcel Dekker, New York, 1972].) Next, a zero $(z - \alpha)$ of an entire function determines a height one principal prime p of E, the set of zeros, including the multiplicities of zeros, of an entire function is a discrete set, while the multiplicity of a zero is a positive integer (see Theorems 3-6 of [Duke Math. J. 6 (1940), 345-356]). As a consequence of Theorem 6 of [Duke Math. J. 6 (1940), 345-356] we conclude that every nontrivial (that is neither zero nor everywhere nonzero) entire function can be written as a countable product $\varepsilon\Pi p_i^{n_i}$of finite powers of nonassociated height one primes of E, where ε is a unit. Using this much information one can show that E is completely integrally closed.

Example. Let E be the ring of entire functions and let S be the multiplicative set of E generated by all the principal primes of D. Then S is not a splitting set, because if X is an indeterminate over E_S then the ring $E+XE_S[X]$ is not a GCD domain.

The above example is illustrated in Example 2.6 of my paper "$D+XD_S[X]$ construction from GCD domains" [J. Pure Appl. Algebra 50(1988), 93-107] and in a slightly different manner in Example 4.7 of my survey " Various facets of rings between $D[X]$ and $K[X]$" [Comm. Agebra 31 (5) (2003), 2497–2540]. Come to think of it, Example 4.7 of the survey may be easier to follow and last few lines of Example 4.7 suffice to show why $E+XE_S[X]$ is not a GCD domain for E the ring of entire functions and S the saturated set generated by principal primes.

To show that $E+XE_S[X]$ is not a GCD domain, all we need do is take α to be an infinite product $\Pi p_i^{n_i}$ and suppose that $d=GCD(\Pi p_i^{n_i},X)$. Then $d|X$ and $d\in E$ because $d|\alpha$. So $d\in S$. But then d is a finite product of powers of height one primes, say the first r factors in $\Pi p_i^{n_i}$. Thus $1=GCD(\frac{\Pi p_i^{n_i}}{\Pi_{i=1}^r p_i^{n_i}},\frac{X}{\Pi_{i=1}^r p_i^{n_i}})$. But then $p_{r+1}|\frac{\Pi p_i^{n_i}}{\Pi_{i=1}^r p_i^{n_i}}$ and $p_{r+1}|\frac{X}{\Pi_{i=1}^r p_i^{n_i}}$, because every principal prime divides X. Thus $p_{r+1}|1$ which contradicts the fact that p_{r+1} is a non unit.

To complete the answer set. The above example shows that the Archimedean and completely integrally closed properties aren't strong enough.