QUESTION (HD 2104) The Wikipedia article

https://en.wikipedia.org/wiki/Schreier_domains

describes Schreier domains as integrally closed integral domains in which every nonzero element is primal, i.e., whenever x divides yz, x can be written as x = x_1 x_2 so that x_1 divides y and x_2 divides z. An integral domain is said to be pre-Schreier if every nonzero element is primal. The article also says that the term "pre-Schreier" was introduced by Muhammad Zafrullah. On the other hand the article https://planetmath.org/schreierdomain on planet Math calls pre-Schreier as a synonym of Schreier domains. Can you provide a reason why you introduced this new term?

ANSWER: Let us agree that D is an integral domain with quotient field K. Let's recall that a partially ordered group G is called a Riesz group if G is directed and satisfies the Riesz interpolation property:

given that $x_1, x_2, ..., x_m; y_1, y_2, ..., y_n \in G$ such that $x_i \leq y_j$ for all $i \in [1, m], j \in [1, n]$ there is $z \in G$ such that $x_i \leq z \leq y_j$ for all $(i, j) \in [1, m] \times [1, n]$. Let's also recall that the group of divisibility $G(D) = \{hD | h \in K \backslash \{0\}$ which is partially ordered by reverse containment, i.e., by $hD \leq kD$ if and only if $hD \supseteq kD$. That $G(D)$ is directed can be easily seen.

Cohn [1] called an integrally closed domain D Schreier if each element of D was primal. Noting that at the same time he wanted the group of divisibility to be a Riesz group, whose definition depends only on nonzero elements of K, I decided to stick to nonzero elements and called a domain D a pre-Schreier domain if each nonzero element of D was primal and obviously as zero being primal has no big effect, an integrally closed pre-Schreier domain is still Schreier. The other reason for "introducing" the new term was that Schreier domains have this property that if D is Schreier and X an indeterminate over D, then $D[X]$ is Schreier (Theorem 2.7 of [1], notice no converse is stated here). Yet if D is strictly pre-Schreier then $D[X]$ is not pre-Schreier. On the other hand some authors such as McAdam and Rush [2] seemed to give the impression that nonzero elements being primal was the characterizing property of Schreier domains. These were the circumstances that led me to write [4]. In [4], I collected what was available on domains whose nonzero elements were primal, calling them pre-Schreier, and added some of my thoughts. Of my thoughts, one was an example (Example 4.5 of [4]) of a pre-Schreier domain D that is not integrally closed and hence not Schreier and using it to demonstrate in Remark 4.6 (1) of [4] that the polynomial ring $D[Y]$ over D is not pre-Schreier.

If there was any doubt about pre-Schreier domains having a separate existence, David Rush removed it in [3], by characterizing pre-Schreier domain with the following result. Call $f \in R[X]$ a special quadratic over R if $f(X) = a(X + m/a)(X + n/a)$ with

$a, m, n \in R$ such that a divides mn. Then R is pre-Schreier if and only if every special quadratic $f(X) = a(X + m/a)(X + n/a)$ over R is expressible as a product of linear polynomials from $R[X]$. That is if and only if $a|mn$ and $f(X) = a(X + m/a)(X + n/a)$ implies that $f(X) = (hX + k)(lX + m)$ where h, k, l and m belong to R, see Theorem 1.2 of [3]. Rush [3] also gives a method of constructing pre-Schreier domains using pullbacks. Of course the same method was used in constructing example 4.5 of [4], but hey, some folks might want to listen to David Rush, rather than Muhammad Zafrullah. (Frankly, I'd listen to David Rush carefully, very carefully.) While I am at it, let me mention that a somewhat interesting example was constructed by Jim Coykendall for [5] (Example 2.10), using the pullback method. This example has the interesting property that its quotient field is algebraically closed.

Now coming back to "Muhammad Zafrullah" being mentioned in a Wikipedia article. I have a feeling that it started as a lark. You see I had a run in with some highly educated and very influential folks from a top-notch school. In the course of a heated argument I said "I introduced pre-Schreier domains" and gave my reasons for it and the funny Wikipedia mention is what I got. Of course the fellows were unlucky as David Rush made sure that pre-Schreier domains do have a separate existence. (Also, the appearance of [5] could have cooled the fellows down too.) On the planetmath entry, the name C.J. Woo comes from a culture where all life is sacred, but Muslim babies can be burned alive. You can only forgive such folks for being ignorant. You may find this rant of mine interesting too, in this connection: https://lohar.com/mithelpdesk/hd2004.pdf

References

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[5] D.D. Anderson and M. Zafrullah The Schreier property and Gauss' lemma, Bollettino U. MI, 8 (2007), 43-62.