QUESTION (HD 2105) While reading your paper [J. Pure Appl. Algebra 212 (2008), 376–393] I got stuck at Lemma 3.7. Really, how do you propose to show that if an ideal A of $R = D + XD\_S[X]$ is such that $A/\backslash S$ is non empty then $A = (A/\backslash D)R$? Also, was it necessary to use $A/\backslash S$ is non empty in the proof of $(JR)_v = (J_v R)_v$?

ANSWER: Let A be an ideal of $D^{(S)} = D + XD_S[X]$. Then, $A \cap S \neq \phi$ or $A \cap S = \phi$.

If $A \cap S \neq \phi$ then $A \supseteq XD_S[X]$, because there is $s \in A$. So for each $f \in A$ we have $f(0) \in A \cap D$ for $f = a_0 + Xg(X), Xg(X) \in A$. So $A \subseteq A \cap D) + XD_S[X]$. Obviously as $(A \cap D)$, $XD_S[X] \subseteq A$ we have $A = (A \cap D) + XD_S[X]$.

Next we already have $A \cap D \subseteq A$.So $A \supseteq (A \cap D)(D + XD_S[X])$. On the other hand $D^{(S)} \supseteq D, XD_S[X]$. So $(A \cap D)D^{(S)} \supseteq (A \cap D)D, (A \cap D)XD_S[X]$. Therefore $(A \cap D)D^{(S)} \supseteq (A \cap D)D + (A \cap D)XD_S[X] = (A \cap D) + (A \cap D)XD[X]_S = (A \cap D) + (A \cap D)XD[X]_S = (A \cap D) + XD[X]_S = (A \cap D) + XD_S[X] = A$. Thus we have $A = (A \cap D)D^{(S)} = (A \cap D) + XD_S[X]$.

This gives us the following statement.

Lemma AB. Suppose that S is a multiplicative set of D and X an indeterminate over D_S. If A is an ideal of $D^{(S)}$ such that $A \cap S \neq \phi$, then $A = (A \cap D)(D + XD[X]_S) = (A \cap D) + XD[X]_S$.

I'd be using star operations freely in what follows. A reader who needs to know about star operations may consult sections 32 and 34 of [4].

It was shown in [5] that if A is a nonzero finitely generated ideal of D then $(AD^{(S)})^{-1} = A^{-1}\, D^{(S)}$ [5, Lemma 3.1]. Later it was decided in [1] that as $D^{(S)}$ is flat over D, we have not only [5, Lemma 3.1] but also $(AD^{(S)})_v = (A_v\, D^{(S)})_v$. It appears that when we proved Lemma 3.7 of [3] we missed the results of [1] and wrote the following.

Lemma BB. ([3],Lemma 3.7). Let D be a domain, S a multiplicative subset of D and X an indeterminate over D. Let I be an ideal

of $D^{(S)}$such that $I \cap S \neq \phi$. Then $I = JD^{(S)} = J + XD_S[X]$ for some ideal J of D with $J \cap S \neq \phi$. Moreover, $I_t = J_t + XD_S[X]$.

In the proof we showed that $(JD^{(S)})_v = (J_v D^{(S)})$ using the fact that $J \cap S \neq \phi$ which was totally unnecessary, in view of [1]. Yet, if we look closely, we have the following result hidden in the proof of [3, Lemma 3.7].

Theorem CB. Let $D \subseteq R$ be an extension of domains and let A be a nonzero ideal of D such that $(AR)^{-1} = A^{-1}R$. Then $(A_{v_D}R)_{v_R} = (AR)_{v_R}$, where v_D (resp., v_R) denotes the v-operation on D (resp., the v-operation on R). Moreover if $D \subseteq R$ is such that for every finitely generated nonzero ideal F of D we have $(F^{-1}R) = (FR)^{-1}$, then $(I_{t_D}R)_{t_R} = (IR)_{t_R}$ for every nonzero ideal I of D.

Proof. Note that $A_{v_D}A^{-1} \subseteq D$ and so $(A_{v_D}R)(A^{-1}R) \subseteq R$. But then $(A_{v_D}R) \subseteq (A^{-1}R)^{-1} = ((AR)^{-1})^{-1} = (AR)_{v_R}$. Thus $(A_{v_D}R) \subseteq (AR)_{v_R}$ which gives $(A_{v_D}R)_{v_R} \subseteq (AR)_{v_R}$. Note, for the reverse containment, that $A \subseteq A_{v_D}$ and so $AR \subseteq A_{v_D}R$, leading to $(AR)_{v_R} \subseteq (A_{v_D}R)_{v_R}$. For the moreover part note that $(IR)_{t_R} = \cup \mathcal{F}_{v_R}$ where $\mathcal{F}$ ranges over finitely generated sub-ideals of IR. Indeed each of $\mathcal{F}$ is contained in FR for some finitely generated sub-ideal F of I and for each nonzero finitely generated sub-ideal F of I, FR is a finitely generated sub-ideal of IR. Thus $(IR)_{t_R} \subseteq \cup(FR)_{v_R}$, where F ranges over nonzero finitely generated sub-ideals of I. But, since $FR \subseteq IR$ we have $(FR)_{t_R} \subseteq (IR)_{t_R}$. As FR is finitely generated $(FR)_{t_R} = ((FR)_{v_R}$ we have $(IR)_{t_R} \subseteq \cup(FR)_{v_R} \subseteq (IR)_{t_R}$. So $(IR)_{t_R} = \cup(FR)_{v_R}$ where F ranges over nonzero finitely generated sub-ideals of I. Next, by the formula $(FR)_{v_R} = ((F_{v_D}R)_{v_R}, F_{v_D} \subseteq F_{v_D}R \subseteq (FR)_{v_R}$ for each finitely generated nonzero sub-ideal F of I. Thus $I_{t_D} = \cup F_{v_D} \subseteq \cup(FR)_{v_R} = (IR)_{t_R}$, which gives $I_{t_D}R \subseteq (IR)_{t_R}$ and eventually $(I_{t_D}R)_{t_R} \subseteq (IR)_{t_R}$. But we have the reverse containment also, since $IR \subseteq I_{t_D}R$ and that gives $(IR)_{t_R} \subseteq (I_{t_D}R)_{t_R}$.

Corollary DB. Let $D \subseteq R$ be an extension of domains such that R is flat over D. Then for any nonzero ideal A of D we have $(AR)_{t_R} = (A_{t_D}R)_{t_R}$.

References

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