QUESTION (HD2201). Is it true that for $a, b \in D\backslash(0)$, $aX+b$ is a prime element in $D[X]$ if and only if $aD \cap bD = abD$. If so, how do you prove it? If not, how do you disprove it?

ANSWER: The statement is true and this is how we can prove it.(See also Remarks 1 and 6.)

Suppose that a and b are such that $aD \cap bD = abD$, then $aX+b$ is a prime, by (a) of Proposition 7.6 of [4]. Conversely, suppose that $aX+b$ is a prime in $D[X]$. Here you may need to recall a few notions related to the v and t operations. The best source is sections 32 and 34 of Gilmer's book on multiplicative ideal theory. I will use them freely. However just to get started I include the following:

Let D be an integral domain with quotient field K and let $F(D)$ denote the set of fractional ideals of D. Denote by A^{-1} the fractional ideal $D :_K A = \{x \in K | xA \subseteq D\}$. The function $A \mapsto A_v = (A^{-1})^{-1}$ on $F(D)$ is called the v-operation on D (or on $F(D)$). Associated to the v-operation is the t-operation on $F(D)$ defined by $A \mapsto A_t = \cup\{H_v |\ H$ ranges over nonzero finitely generated subideals of $A\}$. The v and t-operations are examples of the so called star operations, well explained in sections 32 and 34 of [Multiplicative Ideal Theory, Marcel-Dekker, New York, 1972]. Indeed $A \subseteq A_t \subseteq A_v$. A fractional ideal $A \in F(D)$ is called a v-ideal (resp., a t-ideal) if $A = A_v$ (resp., $A = A_t$). An integral t-ideal maximal among integral t-ideals is a prime ideal called a maximal t-ideal. If A is a nonzero integral ideal with $A_t \neq D$ then A is contained in at least one maximal t-ideal. A prime ideal that is also a t-ideal is called a prime t-ideal. Call $I \in F(FD)$ v-invertible (resp., t-invertible) if $(II^{-1})_v = D$ (resp., $(II^{-1})_t = D$). Indeed an invertible ideal is t-invertible. A prime t-ideal that is also t-invertible was shown to be a maximal t-ideal in Proposition 1.3 of [3].

(1) Given that $(a) : (b)$ is a proper ideal, i.e. $a \nmid_D b$, a prime ideal P minimal over $(a) : (b)$ is called an associated prime of a principal ideal. Let $Ass(D)$ denote the set of associated primes of principal ideals of D.The go to reference for associated primes of principal ideals is [1], where it is shown that (A) $D = \cap_{P \in Ass(D)} D_P$ (Proposition 4). Also that (B) if P is an associated prime of a nonzero polynomial in $D[X]$, then $P \cap D = (0)$ or $P = Q[X]$ where Q is an associated prime of a principal ideal of D. (Corollary 8).

(2) Since $(a) : (b)$ is a v-ideal and hence a t-ideal, P being minimal over $(a) : (b)$ is a t-ideal (See Lemma 6 of[Manuscripta Math. 24 (1978)191-204] for an alternative proof.) It is well known and can be easily established that for a nonzero ideal I of D and for an indeterminate X over D, $(I[X])_v = I_v[X]$. Using this, one can easily establish that if I is a t-ideal, then so is $I[X]$ (see e.g. [2]). Thus if P is an associated prime of a principal ideal then, $P[X]$ is a (prime) t-ideal.

(3) With reference to (A), above, and (3) of Theorem 2.2 of [6],one can show that for a nonzero finitely generated ideal A we have $A_v \neq D$ if and only if A is contained in an associated prime of a principal ideal of D. Thus if A is a nonzero finitely generated ideal of D with $A_v \neq D$, then A is contained in an associated prime of a principal ideal P of D and so $A[x]$ is contained in a prime t-ideal $P[X]$ of $D[X]$.

(4) Let X be an indeterminate over K. Given a polynomial $g \in K[X]$, let A_g

denote the fractional ideal of D generated by the coefficients of g. A prime ideal P of $D[X]$ is called a prime upper to 0 if $P \cap D = (0)$. Thus a prime ideal P of $D[X]$ is a prime upper to 0 if and only if $P = h(X)K[X] \cap D[X]$, for a prime h in $K[X]$. It follows from [3] that P a prime upper to zero of D is a maximal t-ideal if and only if P is t-invertible if and only if P contains a polynomial f such that $(A_f)_v = D$. A domain D is called a UMT domain if every upper to zero of D is t-invertible.

Proof of the converse. Suppose by way of getting a contradiction that $aX + b$ is a prime and that $(a, b)_v \neq D$. Then by (3) there is an associated prime P of D such that $(a, b) \subseteq P$ and so $(a, b)[X] \subseteq P[X]$. But then $(aX + b)D[X] \subseteq (a, b)[X] \subseteq P[X]$. Now $(aX + b)D[X]$ is a prime upper to zero which is principal, so invertible and so t-invertible. In short $(aX + b)D[X]$ is a maximal t-ideal contained, properly, in another t-ideal, giving us the desired contradiction. Whence $(a, b)_v = D$. Now, using definitions it is easy to check that $(a, b)_v = D$ if and only if $aD \cap bD = abD$.

Remark 1. Moshe Roitman has offered the following simple proof of the fact that if $aX + b$ is a prime then $(a, b)_v = D$. The idea is based on a slick use of the fact that if $aD \cap bD = abD$ (i.e. $(a, b)_v = D$) if and only if $c \in aD \cap bD$ implies that $c = hab$ where $h \in D$. This is how his proof goes. Let $c \in aD \cap bD$ and note that $\frac{c}{a}(aX + b) \subseteq bD[X]$, where obviously, $\frac{c}{a} \in D$. So

$$bf = \tfrac{c}{a}(aX + b) \text{ for some } f \in D[X],....(1).$$

Now, in $D[X]$, we have $(aX + b)|bf$. Because of the degree considerations $aX + b \nmid b$ in $D[X]$ and so $(aX + b)|f$, again in $D[X]$. This gives $f = h(aX + b)$ with h in $D[X]$. Thus we can write (1) as

$$bh(aX + b) = \tfrac{c}{a}(aX + b) \ ... \ (2)$$

Cancelling $aX + b$ from both sides of (2) we get $bh = \frac{c}{a}$. Again, by the degree considerations, we have $h \in D$. But then $c = abh$.

Evan Houston has also offered something similar.

Remark 2. Obviously as $(a, b)_v = D$ if and only if $(b, a)_v = D$, $aX + b$ is a prime in $D[X]$ if and only if $bX + a$ is a prime.

Remark 3. The content of this Q/A may be used to dispel the feeling that if $aX + b$ is a prime then a, and/or b have to be irreducible or prime. The reason is that, as can be easily established (see e.g. Proposition 2.2 of [7]), $(x, r)_v = D$ and $(x, s)_v = D$ if and only if $(x, rs)_v = D$. Setting $x = yz$, where $y, z \in D$ we can claim that $yzX + rs$ is a prime in $D[X]$ (and so is $rsX + yz$). Thus $22X + 35$ and $35X + 22$ are primes in $Z[X]$, the ring of polynomials with integer coefficients. This is because 22 and 35 are coprime in Z, the ring of integers. (Indeed in Z, a, b being coprime means that a, b are comaximal, that is $aZ + bZ = Z$ or $(a, b) = Z$ in Z and that is the same as $(a, b)_v = Z$, in Z).

Remark 4. The fact that $aX + b$ is a prime in $D[X]$ if and only if $(a, b)_v = D$ can be used to show that a UMT domain D is a GCD domain if and only if for each t-invertible t-ideal A of $D[X]$ with $A \cap D \neq (0)$ we have A principal. The proof goes along the following lines.

Let D be a GCD domain. Then every upper to zero over D is principal, hence t-invertible. So D is a UMT domain. Also $D[X]$ being a GCD domain, every t-invertible t-ideal of $D[X]$ is principal and so is the t-invertible t-ideal A with $A \cap D \neq (0)$. Conversely, let D be a UMT domain and suppose that for a t-invertible t-ideal A of $D[X]$ with $A \cap D \neq (0)$ we have A principal. Now let $a, b \in D \backslash \{0\}$ and consider $(aX + b)$. Because $aX + b$ is non-constant $aX + b$ belongs to an upper to zero, say P. Thus $(aX + b) \subseteq P$. Since P is t-invertible we have $(aX + b)P^{-1} \subseteq D[X]$. So there is an ideal A of $D[X]$ such that $(aX + b)P^{-1} = A$. Multiplying both sides by P and taking the t-image we get $(aX + b) = (AP)_t = (A_tP_t)_t$ we can assume that A and P are both t-nvertible t-ideals. Since $(aX + b)K[X] = (AP)_tK[X] = PK[X]$ we conclude that $A \cap D \neq (0)$. By the condition A is a principal ideal of $D[X]$. That is $A = hD[X]$. But then $(aX + b) = (AP)_t$ becomes $(aX + b) = (hP)_t = hP$, as P is a t-ideal. This leads to the conclusion that $P = \frac{1}{h}(aX + b)$. Since degree of h is zero, we have $P = (\frac{a}{h}X + \frac{b}{h})$ a principal ideal. But then $(\frac{a}{h}, \frac{b}{h})_v = D$ and this leads to $(a, b)_v = hD$. Since a, b were picked arbitrarily we claim that, in this situation, $(a, b)_v$ is principal for each pair of nonzero elements of D and that is a necessary and sufficient condition for D to be a GCD domain.

This leaves the QUESTION: Can the UMT domain restriction be removed from the above result?

Remark 5. I am thankful to Serpil Saydam for pointing out the reference in Samuel's notes.

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Remark 6. Evan Houston has completely nailed the question and in much more general terms. Here's his note.

Many of the ideas in this note come from [5].

Theorem. Let $f \in D[x]$, f non-constant, and let $I = fK[x] \cap D[x]$. Then $I = fD[x] \Leftrightarrow (A_f)^{-1} = D$ (equivalently, $(A_f)_v = D$).

Proof. Suppose that $I = fD[x]$. Let $u \in (A_f)^{-1}$. Then $uf \in fK[x] \cap D[x] = fD[x] = I$, whence $u \in D$. Hence $(A_f)^{-1} = D$. Conversely, suppose that $(A_f)_v = D$. Let $g \in I$. Then $g = fk$ for some $k \in K[x]$. By the content formula, there is a positive integer m with $(A_f)^{m+1}(A_k) = (A_f)^m(A_{fk}) \subseteq D$. We then have

$$(A_k) \subseteq (A_k)_v = ((A_f)^m(A_g))_v \subseteq D.$$

This means that $k \in D[x]$, because $(A_f)_v = D$ and $g \in D[X]$. But then $g \in fD[X]$. Therefore, $I = fD[x]$. ∎

Corollary 1. Let $f \in D[x]$, f non-constant, and let $I = fK[x] \cap D[x]$. Then I is principal $\Leftrightarrow (A_f)_v$ is principal.

Proof. Assume that $I = gD[x]$. We have $f = gh$ for some $h \in D[x]$ and $g = fk$ for some $k \in K[x]$. It follows that $h \in D$ and that $I = gK[x] \cap D[x]$. By the theorem, we have $(A_g)_v = D$, whence $(A_f)_v = h(A_f)_v = Dh$.

Conversely, suppose that $(A_f)_v = aD$. Then $a^{-1}f \in D[x]$, and $I = a^{-1}fK[x] \cap D[x]$ with $(A_{a^{-1}f})_v = D$. Therefore $I = a^{-1}fD[x]$ by the theorem. ■

Corollary 2. Let $f \in D[x]$ with f irreducible in $K[x]$. Then $fD[x]$ is prime $\Leftrightarrow (A_f)^{-1} = D$. In particular, if $f(x) = ax + b$ with $a \neq 0$, then f is prime in $D[x] \Leftrightarrow (a, b)^{-1} = D$.

Proof. Let $P = fK[x] \cap D[x]$. Then P is an upper to (0), that is, P is prime and $P \cap D = (0)$, but $P \neq (0)$. In particular, P has height one (which follows from the fact that $K[x]$ is a quotient ring of $D[x]$). Note that $fD[x] \subseteq P$. If $fD[x]$ is prime then we must have $P = fD[x]$, and hence $(A_f)^{-1} = D$ by the theorem. Conversely, if $(A_f)^{-1} = D$, then, again by the theorem, $P = fD[x]$. ■

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