QUESTION (HD2204). Can you give me a direct proof of the fact that an integral domain D is a PVMD if and only if every t-linked overring of D is integrally closed? The proof given in Comm. Algebra 17(1989) 2835-2852, seems a little involved.

ANSWER: Sure, but let me first tell other readers to look into [1], for the star operation lingo etc., if they want to see what is going on. You are of course talking about Theorem 2.10 of [1] and there seems to be nothing wrong with it, but perhaps you got confused by the indirectness of the proof, but hey everyone has a right to get confused, once in a while. Let's start with the following lemma.

Lemma A. Let D be an integral domain and let P be a prime t-ideal of D. Then for every $u \in K\backslash\{0\}$ the ring $D_P[u]$ is t-linked over D.

Proof. Let $T = D[u]$ and let P be a prime t-ideal of D. Then by Propotion 2.9 of [1], $T_{D\backslash P} = D[u]_{D\backslash P}$ is t-linked over D. But $D[u]_{D\backslash P} = D_P[u]$. This follows because if we set $S = D\backslash P$, then S is a multiplicative set of $D[u]$ and $D \subseteq D[u]$. So $D_S \subseteq D[u]_S$ and as u belongs to the RHS we have $D_S[u] \subseteq D[u]_S$. On the other hand $x \in D[u]_S$ implies that $x = f(u)/s$ for some $s \in S$ and $f(u) \in D[u]$. But $f(u) \in D_S[u]$ and s is a unit in $D_S[u]$, forcing $x = f(u)/s$ in $D_S[u]$. Thus $D_S[u] = D[u]_S$ or substituting for $S = D\backslash P$ we have

$D_P[u] = D[u]_{D\backslash P}$.

This Lemma was given to me by Evan Houston. Now recall that usually by an overring of D we mean a ring R between D and its quotient field K.

Theorem B. An integral domain D is a PVMD if and only if every t-linked overring of D is integrally closed.

Proof. Let D be a PVMD and let T be any t-linked overring of D. Then by Proposition 2.13 of [1] $T = \cap T_{D\backslash P}$ where P varies over prime t-ideals of D. Now for each prime t-ideal P of D, $T_{D\backslash P}$ is an overring of D_P, a valuation ring, and so $T_{D\backslash P} = D_\wp$ for some prime ideal $\wp \subseteq P$. Since $D_\wp$, being an overring of D_P is a valuation domain, $\wp$ must be a prime t-ideal [5]. Thus T is what was termed as a subintersection in [4] of the PVMD D (where it was also shown that a subintersection of a PVMD is a PVMD and hence integrally closed). Being an intersection of valuation domains T is integrally closed. Conversely suppose that every t-linked overring of D is integrally closed. Let $u \in K\backslash\{0\}$ and let P be a maximal t-ideal of D. Then, by Lemma A $D_P[u^2]$ is t-linked over D and so by the condition $D_P[u^2]$ is integrally closed. But then u being integral over $D_P[u^2]$, we conclude that $u \in D_P[u^2]$. Then there are elements v_0 , . . . , $v_n \in D_P$ such that $u = v_0 + v_1u^2 + ... + v_nu^{2n}$...(I). If we multiply the previous equation ((I)) by v_0^{2n-1}/u^{2n}, we obtain $(v_0/u)^{2n} - (v_0/u)^{2n-1} + v_1v_0(v_0/u)^{2n-2} + ... + v_nv_0^{2n-1} = 0$. Thus v_0/u is integral over D_P, hence $v_0/u \in D_P$. If v_0/u is a unit in D_P, then $u \in D_P$. If v_0/u is not a unit in D_P, then $1 - (v_0/u)$ is a unit in D_P. If we multiply the equation expressing u in terms of powers of u^2, that is (I), throughout by $1/u^{2n}$, we get $(1 - v_0/u)(1/u)^{2n-1} - v_1(1/u)^{2n-2} - ... - v_n = 0$. Since $(1 - v_0/u)$ is a unit in D_P we conclude that $1/a$ is integral over D_P and so is in D_P. Now as for any $u \in K\backslash\{0\}$ we have $u \in D_P$ or $u^{-1} \in D_P$, D_P is a valuation domain. Since P was an arbitrarily chosen maximal t-ideal, we conclude that D is a PVMD. This is because D is a PVMD if and only if D_P is a valuation domain for every maximal t-ideal of D [2].

Remark C. The proof of the converse of Theorem B has been lifted from the proof of Theorem 6.13 of [3].

One take away from the above exercise is the following statements.

Corollary D. An integral domain D is a PVMD if and only if every simple overring of every t-linked overring of D is integrally closed.

Here, by a simple overring of a domain D we mean a ring of the form $D[u]$, where $u \in K \backslash \{0\}$.

The proof is simple as the statement includes what we do in the proof. Similarly one can state the following corollary.

Corollary E. An integral domain D is a Prufer domain, if and only if, every simple overring of every overring of D is integrally closed.

Again it is easy to prove Corollary E. Yet if someone says that there is no characterization of Prufer domains involving simple overrings you can offer this one. In any case there is a saying in Persian that goes as: Dashtah awyad bekar[redacted] (if you) keep something, it might come in handy (one day). Hence the recording of the two corollaries. (This nice adage was turned around by some funny folks in Pakistan, where dashtah also means a keep or a mistress. So they translated the adage as: Dashtah comes with a car, i.e., if you have a car then you can have a dashtah. This has an easy rendering in English as: a keep comes with a car, once you have a car you can keep a keep. But sadly keeps and cars have lost their meaning in the English speaking world. Now your keep, if you have one, can come in her car to pick you up.)

References

[1] D. Dobbs, E. Houston, T. Lucas. E. Houston and M. Zafrullah, t-linked overrings and Prüfer v-multiplication domains, Comm. Algebra 17(1989) 2835-2852.

[2] M. Griffin, Some results on v-multiplication rings. Can. J. Math. 19, 710–722 (1967).

[3] M.D. Larsen and McCarthy, Multiplicative Theory of Ideals, Pure and applied mathematics (Academic Press) volume 43, 1971.

[4] J. L. Mott and M. Zafrullah, On Priifer v-multiplication domains, Manuscripta Math. 35 (1981), 1-26.

[5] M. Zafrullah,"Finite conductor domains" Manuscripta Math. 24(1978) 191-203.