QUESTION (HD2205) (1) In the definition of t-spitting set, why (A,s)_t=D are equivalent to A_t \bigcap sD=sA_t? (2) Let Q be the field of rational numbers. How to prove that Q[X^{1/n}] is a PID? Is C[X^{1/n}] also a PID for the field of complex numbers C?

ANSWER: (1). Let me repeat the definition from [1]: Let D be an integral domain with quotient field K and let S be a multiplicatively closed subset of D. We say that $d \in D\backslash\{0\}$ is t-split by S if $(d) = (AB)_t$ for integral ideals A and B of D, where $A_t \cap sD = sA_t$ (or equivalently, $(A_t, s)_t = D$.)

Note that $(A_t, s)_t = D$ if and only if A and s share no maximal t-ideals. Now as A and s share no maximal t-ideals $(A \cap sD)_w = (As)_w$, because it holds t-locally. This gives $A_w \cap sD = A_w sD$. Since A is t-invertible $A_w = A_t$ and so $A_t \cap sD = A_t sD$. (Alternatively, note that $(A_t, s)_t(A_t \cap sD) \subseteq sDA_t$. Applying the t-operation throughout and noting that $(A_t, s)_t = D$ we conclude that $(A_t \cap sD) \subseteq sDA_t$. Since $sDA_t \subseteq (A_t \cap sD)$ always holds we have $(A_t \cap sD) = sDA_t$.)

Conversely, let $A_t \cap sD = A_t sD$. So $A_t \cap sD$ is t-invertible, because A is t-invertible. Since A is t-invertible we have $A_t = (z_1) \cap ... \cap (z_m)$ for some $z_i \in K\backslash\{0\}$, see (3) of Remark 3.2 of [2] Thus $((A_t \cap (s)) = ((z_1) \cap ... \cap (z_m) \cap (s))$ and $((A_t \cap (s))^{-1} = ((z_1) \cap ... \cap (z_m) \cap (s))^{-1} = ((1/z_1, ..., 1/z_m, 1/s)^{-1} = (A_t^{-1}, s^{-1})_v$. Thus $((A_t \cap (s))^{-1} = (A_t^{-1}, s^{-1})_v$. Substituting for $(A_t \cap (s)$ $(= sA_t)$ in the previous equation we have $s^{-1}A^{-1} = (A_t^{-1}, s^{-1})_v$. Multiplying the last equation by sA, throughout, and applying the t-operation we get $D = (sAA^{-1}, A)_t = (s(AA^{-1})_t, A)_t = (s, A)_t$. (Alternatively, note that $(A_t^{-1}, s^{-1})^{-1} = A_t \cap sD$ and so $(A_t \cap sD)^{-1} = (A_t^{-1}, s^{-1})_v = s^{-1}A_t^{-1}$ (because $A_t \cap sD = A_t sD$). Now multiply the equation $(A_t^{-1}, s^{-1})_v = s^{-1}A_t^{-1}$ by sA_t and apply the t-operation to get $D = (A_t, s)_v$.)

ANSWER: (2). For any field F the expression $F[X^{1/n}]$ means a ring of polynomials taking $X^{1/n}$ as an indeterminate. That is why $F[X^{1/n}]$ is a PID. (A general element of $F[X^{1/n}]$ is $f = \sum_{i=0}^{m} f_i X^{i/n} = f_0 + f_1 X^{\frac{1}{n}} + f_2 X^{\frac{2}{n}} + ... + f_i X^{\frac{1}{n}} + ... + f_m X^{\frac{m}{n}}$.)

References

[1] D.D. Anderson, D.F. Anderson an d M. Zafrullah, The ring $D + XD_S[X]$ and t-splitting sets, Commutative algebra. Arab. J. Sci. Eng. Sect. C Theme Issues 26 (2001), no. 1, 3–16.

[2] M. Zafrullah, Revisiting G-Dedekind domains, Canad. Math. Bull. 2022, pp. 1–15 http://dx.doi.org/10.4153/S000843952100103X