$A + XB[X]$ construction and the pre-Schreier property

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Let D be an integral domain with quotient field K. Call $x \in D\backslash\{0\}$ primal if for all $a, b \in D\backslash\{0\}$, $x \mid ab$ in D implies x can be factorized in D as $x = rs$ where $r \mid a$ and $s \mid b$. Call $x \in D\backslash\{0\}$ completely primal if every factor of x is primal. Also call D Schreier if D is integrally closed and every nonzero element of D is primal. Cohn introduced these notions in [C] where he also showed that a GCD domain is Schreier and that $D[X]$ is Schreier if and only if D is. One may jettison the integrally closed condition, as did [Z] and essentially [MR], and call D pre-Schreier if every $x \in D\backslash\{0\}$ is primal. Next let $A \subseteq B$ be an extension of integral domains, i.e. A is a subring of B, and let X be an indeterminate over B. Consider the construction $A + XB[X] = \{f \in B[X] : f(0) \in A\}$. It is well known, and easy to check, that $A + XB[X]$ is an integral domain. In [CMZ], just before Theorem1.1, it was indicated that if D is a GCD domain, S a multiplicative set in D and X an indeterminate then the construction $D + XD_S[X]$ is a Schreier domain. Indeed the same explanation works and one can see that if D is Schreier and S a multiplicative set in D then $D + XD_S[X]$ is Schreier. Since then most of the mathematicians working on these topics have been contented with results like the above two results to get integrally closed generalizations of Schreier domains, from integrally closed generalizations of Schreier domains. In case of GCD domains Anderson and El-Abidine showed in Theorem 2.10 of [AE] that the $A + XB[X]$ is a GCD domain if and only if A is a GCD domain and $B = A_S$ where S is a specialized type of multiplicative set called a splitting set. On the other hand Dumitrescu et al [DRSS] addressed and answered the question: When is the $A + XB[X]$ construction a pre-Schreier domain? I write this short note to indicate a slightly different approach providing the following result and some of its consequences, hoping that this approach will lead to a better understanding.

Proposition A. The $A + XB[X]$ construction is pre-Schreier if and only if (a) each $a \in A\backslash\{0\}$ is primal in $A + XB[X]$ and (b) B is a ring between A and the quotient field of A.

I will use a somewhat modified version of the following result from [C]. Cohn called it a Nagata type theorem for Schreier domains.

Theorem B (Theorem 2.6 of [C]). Let R be an integrally closed integral domain and let S be a multiplicative set of R. Then (i) if R is a Schreier domain, then so is R_S, (ii) if R_S is Schreier and S is generated by completely primal elements, then R is a Schreier ring.

Since nowhere in the proof of the above theorem does Cohn use the condition of R being integrally closed, as the reader can check one can assume that Theorem B holds for pre-Schreier domains. I restate this theorem below with just a minor difference, to suit my particular bent, for pre-Schreier domains.

Theorem. C (Cohn). Let D be an integral domain and let S be a multiplicative set of D. Then (i) if D is pre-Schreier, then so is D_S, (ii) if D_S is pre-Schreier and S is a saturated multiplicative set with every element of S primal in D, then D is pre-Schreier.

Note that when S is saturated, "every element primal" implies that every factor of each member of S is completely primal. So we are dealing with primal elements which are completely primal. Paul was tacitly using the facts that the product of two completely primal elements is completely primal ([C], Lemma 2.5) and that if T is the saturation of the multiplicative set S of D then $D_S = D_T$. I could reproduce Cohn's proof verbatim but that would be a waste of time, besides in a recent paper [DK] Dumitrescu and Khalid have redone it for a generalization of pre-Schreier domains (see Theorem 4.3 of [DK]) and necessary introduction below. Next recall that a nonzero nonunit element x in D is called irreducible if whenever we write $x = rs$ where $r,s \in D$ either r is a unit or s is. An irreducible element is also called an atom and an integral domain D is called atomic if every nonzero element of D is expressible as a finite product of atoms of D.

Proposition D. Any primal atom in D is a prime and so a pre-Schreier atomic domain is a UFD.

Proof. Note that if a is a primal atom and for $b,c \in D\backslash\{0\}$ $a \mid bc$ in D, i.e., $bc = ax$ for some $x \in D$, then $a = rs$ where $r \mid b$ and $s \mid c$. But as a is an atom, one of r,s say r is a unit. So $cD \subseteq sD = aD$. Similarly if s were a unit we would have $bD \subseteq aD$. This translates to: if $a \mid bc$ then $a \mid b$ or $a \mid c$. The remainder is easy as D is a UFD if, and only if, D is atomic and every atom in D is a prime.

To prove proposition A we need another set of results.

Proposition E. If in $A + XB[X]$, the ring A is a field then $A + XB[X]$ is pre-Schreier if and only if $B = A$.

Proof. Suppose that A is a field and $A = B$ then $A + XB[X] = A[X]$ which is a PID and hence Schreier. Conversely note that as A is a field, by Proposition 1.1 of [BIK], $A + XB[X]$ satisfies ACCP (ascending chain condition on principal ideals) and so is atomic. Now by Proposition D above, being also a pre-Schreier domain makes $A + XB[X]$ a UFD. But then note that if $B \neq A$ then the atom X is not a prime; for if $b \in B\backslash A$ then $X \mid (bX)^2 (= (b^2X)(X))$, yet $X \nmid bX$.

Proof of Proposition A. Suppose the construction $A + XB[X] = D$ is pre-Schreier then every nonzero element of D is primal and so is every nonzero element of A in D. This take care of part (a). Now for part (b) Let $S = A\backslash 0$. Since D is pre-Schreier, D_S is pre-Schreier and $D_S = A_S + XB_S[X]$ and, A_S is a field, we use Proposition E to conclude that $A_S = B_S$. Conversely suppose that the conditions (a) and (b) hold and let $S = A\backslash\{0\}$ and $L = qf(A) = A_S = B_S = qf(B)$ and note that $(A + XB[X])_S = A_S + XB_S[X] = L[X]$ a pre-Schreier domain, forcing $A + XB[X]$ to be pre-Schreier via Cohn's Nagata-type theorem.

Corollary E. Let A be an integral domain and X an indeterminate over A. Then $A[X]$ is pre-Schreier if and only if every nonzero element of A is primal in $A[X]$.

It was shown in Proposition 6 of [ADZ], that if a,b are nonzero elements of a domain A such that b is primal in $A[X]$. If $\frac{a}{b}$ is integral over A then $\frac{a}{b} \in A$. From this it follows that if every nonzero element of A is primal in $A[X]$ then A is integrally closed and pre-Schreier, i.e., A is Schreier. This argument led to Corollary 7 of [ADZ] which says that for a domain A, $A[X]$ is a Schreier domain if and only if every nonzero element of A is primal in $A[X]$. Thus Corollary E can be restated as follows.

Corollary F. Let A be an integral domain and X an indeterminate over A. Then $A[X]$ is pre-Schreier if and only if every nonzero element of A is primal in $A[X]$ if and only if A is Schreier.

Both corollaries E and F raise the question: Can we construct a domain $A + XB[X]$ such that $A + XB[X]$ is pre-Schreier and not Schreier? One answer is simple.

Corollary G. If A is pre-Schreier with quotient field L then $A + XL[X]$ is pre-Schreier and is not a Schreier domain if and only if A is not a Schreier domain.

Proof. Note that if $A + XL[X]$ is pre-Schreier then every nonzero element of A is automatically primal in $A + XL[X]$. The proof of the converse consists in showing that every nonzero element of A is primal in $A + XL[X]$. For this let $a \in A\backslash\{0\}$ and let $a \mid fg$ where $f, g \in A + XL[X]$. Write $f = f_0 + \sum_{i=1}^{n} f_i X^i$ and $g = g_0 + \sum_{j=1}^{r} g_j X^j$. Suppose for a start that both f_0, g_0 are nonzero. Then as $f = f_0(1 + \sum_{i=1}^{n} \frac{f_i}{f_0} X^i)$ and $g = g_0(1 + \sum_{j=1}^{r} \frac{g_j}{g_0} X^j)$ $a \mid fg \Leftrightarrow a \mid f_0 g_0$ in A by the degree considerations. But then $a = rs$ where $r \mid f_0$ and $s \mid g_0$, which leads to $r \mid f$ and $s \mid g$. In case any of $f_0,\ g_0$ is zero, say $f_0 = 0,$ we have $f = \sum_{i=1}^{n} f_i X^i = (X)h(X)$ where $h(X) \in L[X]$. Since the coefficients of X come from L, $a \mid Xh(X)$. So, we can write $a = rs$ where $r = a$ and $s = 1$. Thus in all cases a can be shown to be primal in $A + XL[X]$. Since a is a typical nonzero element of A and since $L = qf(A)$ the requirements of Proposition A are met and $A + XL[X]$ is pre-Schreier. Indeed it can be easily shown (see the appendix) that if A is an integral domain and $L = qf(A)$, $A + XL[X]$ is integrally closed if and only if A is.

Remark H. If A is pre-Schreier and M is a field that properly contains $L = qf(A)$ as a subfield, we have every element of A primal in $A + XM[X]$ but we cannot expect $A + XM[X]$ to be pre-Schreier. For if $S = A\backslash\{0\}$, $(A + XM[X])_S = L + XM[X]$ is not pre-Schreier by Proposition E.

We can construct some more general examples than the ones given via Corollary G.

Proposition K. Let A be a pre-Schreier domain and let S be a saturated multiplicative set in A such that A_S is a Schreier domain. If X is an indeterminate then $A + XA_S[X]$ is a pre-Schreier domain.

Proof. Let S be a saturated multiplicative set in A. Then as A is pre-Schreier, S consists of elements that are primal and hence completely primal elements of A. We show that every element of S is primal in $A + XA_S[X]$. It is easy to see that a typical element of $A + XA_S[X]$ can be written as $a + Xf(X)$ where $f(X) \in A_S[X]$. Now let $s \in S$ and suppose that $s \mid FG$ where $F = a + Xf(X)$ and $G = b + Xg(X)$ are in $A + XA_S[X]$. We can write $FG = ab + Xh(X)$. When both a and b are nonzero $s \mid FG$ implies that $s \mid ab$ in A; indeed as $s \mid FG$ and $s \mid Xh(X)$ because $s \in S$ we have $s \mid FG - Xh(X)$. Now as s is primal in A we can write $s = uv$ where $u \mid a$ and $v \mid b$. because S is saturated $u, v \in S$, and because every element of S divides X we conclude that $u \mid a + Xf(X) = F$, and $v \mid b + Xg(X)$. Next suppose that either of a, b say a is zero then, because $s \mid Xf(X)$ we can write $u = s$ and $v = 1$. From this it is easy to see that in each possible form of $F, G \in A + XA_S$ and for each $s \in S$, $s \mid FG$ implies that $s = uv$ where $u \mid F$ and $v \mid G$. Thus every element of the saturated set S is primal and hence completely primal in $A + XA_S[X]$. Now note that $(A + XA_S[X])_S = A_S[X]$ is Schreier because A_S is Schreier. So by Theorem C, $A + XA_S[X]$ is pre-Schreier.

Remark L. If S is a multiplicative set in A then $A_S = A_{\overline{S}}$ where $\overline{S}$ is the saturation of S in A, so $A + XA_S[X] = A + XA_{\overline{S}}[X]$ there is no harm in stating Proposition K as: "Let A be a pre-Schreier domain and let S be a multiplicative set in A such that A_S is a Schreier domain. If X is an indeterminate then $A + XA_S[X]$ is a pre-Schreier domain." But in the proof we must mention that because $A_S = A_{\overline{S}}$ it is better to assume that S is saturated. Otherwise we will need to establish that every element of S is indeed completely primal, which is not hard but can be an extra bother.

Corollary M. Let A be a Schreier domain, S a multiplicative set in A and let X be an indeterminate. Then $A + XA_S[X]$ is Schreier.

Proof. That $A + XA_S[X]$ is pre-Schreier follows from Proposition K, and for $A + XA_S[X]$ being integrally closed note that $A + XB[X]$ is integrally closed if and only if B is integrally closed and A is integrally closed in B. This result was stated without proof in [AAZ], as a part of Theorem 2.7, but we shall include the proof in the appendix. Now as A_S is integrally closed and A is integrally closed and hence integrally closed in A_S we conclude that $A + XA_S[X]$ is an integrally closed pre-Schreier domain.

Corollary N. Let A be a GCD domain and let S be a multiplicative set in A then $A + XA_S[X]$ is a Schreier domain.

Corollary N was hinted at, as mentioned above, in [CMZ] by saying, "Since each of the properties, integral closure and Schreier, is preserved under polynomial ring extensions and direct limits we see that $T^{(S)}$ is integrally closed or Schreier if D is integrally closed or Schreier." Here $T^{(S)} = D + XD_S[X]$. For those who who are not quite well-versed in the first order properties or "local properties", as Cohn calls them, the above results and their proofs will provide a better understanding. In fact some of the above results such as Proposition K cannot, apparently be proved using the reasoning provided in [CMZ].

This leaves us with two frequently asked questions: (i) Is there an example of a pre-Schreier domain that is not Schreier? (ii) Why shouldn't the $A + XA_S[X]$ construction from a GCD domain A be a GCD domain and (iii) what is the simplest exampe of a Schreier domain as an $A + XA_S[X]$ construction from a GCD domain?

For (i) there is an example in [MR] of a pre-Schreier domain that is not Schreier and there is an example in [Z] (Example 4.5). There is also an example in [AZ] (Example 2.10) of a pre-Schreier domain but that is more or less in the spirit of the example in [Z]. For (ii) and (iii) the reader may look up http://www.lohar.com/mithelpdesk/hd1209.pdf

Both the examples in (i) are one dimensional quasi-local and somewhat less complicated. In section 3 of [R] David Rush shows how to construct such examples from non-discrete valuation domains. We can use Proposition K to create examples that are of higher dimension and non-quasi-local. In [ACHZ] it was shown, using Cohn'a Nagata-type theorem for pre-Schreier domains of the type $A = Q + R[\{Y^{\alpha} \colon \alpha \in Q^{+}\backslash\{0\}\}]$ where Q^{+} denotes the set of positive rational numbers. If we let $S = \{Y^{nb} : b$ is a single element of $Q^{+}\backslash\{0\}\}_1^{\infty}$ and let T the saturation of S in A then $A_S = A_T = R[\{X^{\alpha} \colon \alpha \in Q^{+}\backslash\{0\}\}]_T$ is a Schreier domain, actually a one dimensional Bezout domain. If X is an indeterminate the ring $A + XA_S[X]$ is a pre-Schreier domain of dimension greater than 2 and is certainly non-quasi-local. So using Proposition K we can construct examples of pre-Schreier domains of dimension greater than any pre-assigned number.

The question remains: If $A + XB[X]$ is pre-Schreier what can be B? This question has been answered in [DRSS] in a somewhat compact manner. We give below an answer using divisibility techniques.

Suppose that $A + XB[X]$ is pre-Schreier then X is primal in $A + XB[X]$. Note that for $b \in B$,

$X \mid (bX)^2$. Since X is primal we must be able to write $X = fu$ where $f \mid bX$ and $u \mid bX$, both f, u in $A + XB[X]$. Of f and u one is of degree one, say f is then $f = rX + s$ then u is of degree 0 and hence $u \in A$. Thus we have $X = (rX + s)u$. Comparing coefficients we get $s = 0$ and $X = ruX$. This forces $ru = 1$. Since $r \in B$ and u is a unit in B. Now we have $rX \mid bX$ and $u \mid bX$ we have $bX = h(rX)$ and $bX = gu$. From these we get $b = hr$ where $h \in A$ and $r = 1/u$. So $b = h/u$ where u is an element of A that is a unit in B. Let $S = \{s \in A : s$ is a unit in $B\}$, then $B \subseteq A_S$ by the above observstion. But as already $A_S \subseteq B$ we conclude that $B = A_S$. To this point all we have done is reproduce parts of Lemma 2.3 and Corollary 2.4 of [DRSS]. Our addition to these observations is: And $B = A_S$ is a Schreier domain. The reason for this remark is if $A + XA_S[X]$ is pre-Schreier then so is $(A + XA_S[X])_S = A_S[X]$. But $A_S[X]$ pre-Schreier requires that $A_S[X]$ and A_S is Schreier. Thus we have the following statement.

Proposition P. If $A + XB[X]$ is a Schreier domain then B is a Schreier domain such that $B = A_S$ for a multiplicative set S in A.

Possible areas of investigation: At present the $A + XB[X]$ constructions which started with the the study of the $D + XD_S[X]$ constructions of [CMZ] and which were studied in [AAZ] have been generalized to constructions of the type $D + E[\Gamma^*] = \{f \in E[\Gamma] : f(0) \in D\}$ where Γ is a nonzero torsion free grading monoid with $\Gamma \cap -\Gamma = (0)$ and $\Gamma^* = \Gamma \backslash \{0\}$. Jung Wook Lim in [L] studies the usual stuff: When is $D + E[\Gamma^*]$, Prufer, GCD? One can ask: When is $D + E[\Gamma^*]$ a pre-Schreier domain? There is another construction, the so called amalgamated duplication of a ring R which covers both $A + XB[X]$ and $A + XB[[X]]$ constructions and also generalizes Nagata idealization of a module. One can ask: When is the amalgamated duplication of a pre-Schreier domain a pre-Schreier ring? (I am leaving a possibility of rings with zero divisors.) For this type of constructions the reader may look up [DFF] and references there.

Then there is a multitude of generalizations of pre-Schreier domains such as the t-Schreier domains, quasi-Schreier domains, almost Schreier domains, almost quasi-Schreier domains etc. Of these the almost Schreier domains can be defined without burdening the reader with too many definitions. Call an integral domain D almost Schreier (AS-) domain if for each triple $x, y, z \in D \backslash \{0\}$ $x \mid yz$ implies that there is a positive integer m such that $x^m = rs$ where $r \mid y^m$ and $s \mid z^m$. One can ask: When is $A + XB[X]$ almost Schreier? The same with other generalizations.

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Appendix. Proof of "$A+XB[X]$ is integrally closed if and only if B is integrally closed and A is integrally closed in B".

Note that if $A+XB[X]$ is integrally closed and if $S=\{X^n : n\geq 1\}$ then $(A+XB[X])_S = B[X,X^{-1}]$ is integrally closed which requires that B is integrally closed. Next suppose that A is not integrally closed in B. Then there is $b\in B\backslash A$ such that $b^n+a_{n-1}b^{n-1}+\ldots+a_1b+a_0=0$ but then as $b\in qf(A+XB[X])$ and $a_i\in A+XB[X]$ we have b integral over $A+XB[X]$, contradicting the assumption that $A+XB[X]$ is integrally closed. Now suppose that B is integrally closed and A is integrally closed in B. Then as $A+XB[X]\subseteq B[X]$ where $B[X]$ is integrally closed and $qf(A+XB[X]) = qf(B[X])$, it is enough to verify that $A+XB[X]$ is integrally closed in $B[X]$. For this let $f=\sum_{i=0}^{n} a_iX^i \in B[X]$, be integral over $A+XB[X]$. Now we can write $f=a_0+Xg(X)$ where $Xg(X)$ is already in A. This means that $f-Xg(X)=a_0$ is integral over $A+XB[X]$ which , in view of degree considerations, boils down to a_0 being integral over A. But as A is integrally closed in B and $a_0\in B$ we conclude that $a_0\in A$ and so $f\in A+XB[X]$.