$*$-ALMOST SUPER-HOMOGENEOUS IDEALS IN $*$-H-LOCAL DOMAINS

SHIQI XING*, D. D. ANDERSON, AND MUHAMMAD ZAFRULLAH

Abstract. In this paper, we introduce $*$-almost independent rings of Krull type ($*$-almost IRKTs) and $*$-almost generalized Krull domains ($*$-almost GKDs) in the general theory of almost factoriality, neither of which need be integrally closed. This fills a gap left in [10]. We characterize them by $*$-almost super-SH domains, where a domain D is called a $*$-almost super-SH domain if every nonzero proper principal ideal of D is a $*$-product of $*$-almost super-homogeneous ideals. We prove that (1) a domain D is a $*$-almost IRKT if and only if D is a $*$-almost super-SH domain, (2) a domain is a $*$-almost GKD if and only if D is a type 1 $*$-almost super-SH domain, and (3) a domain D is a $*$-almost IRKT and an AGCD-domain if and only if D is a $*$-afg-SH domain. Further, we characterize them by their integral closures. For example, we prove that a domain D is an almost IRKT if and only if $D \subseteq \overline{D}$ is a root extension with D t-linked under $\overline{D}$ and $\overline{D}$ is an IRKT. Examples are given to illustrate the new concepts.

1. Introduction

It is well-known that Krull domains play a central role in the development of multiplicative ideal theory. The concept of a Krull domain has been generalized in many different ways, for example, by independent rings of Krull type, generalized Krull domains and weakly Krull domains. There is an important commonality among the above domains, i.e., they are all $\mathscr{F}$-IFC domains. Recall that a set $\mathscr{F}$ of prime ideals in a domain D is a *defining family* if $D = \bigcap\{D_P \mid P \in \mathscr{F}\}$. Further, $\mathscr{F}$ is *of finite character* (or *locally finite*) if every nonzero nonunit of D belongs to at most finitely many members of $\mathscr{F}$ and $\mathscr{F}$ is *independent* if no two members of $\mathscr{F}$ contain a common nonzero prime ideal. As in [9], a domain D is called an *$\mathscr{F}$-IFC domain* if D has a defining family $\mathscr{F}$ such that $\mathscr{F}$ is independent and of finite character. Now suppose that D is an $\mathscr{F}$-IFC domain. Then D is called a *weakly Krull domain* (*WKD*) in [5] if $X^{(1)}(D) = \mathscr{F}$, where $X^{(1)}(D)$ is the set of height-one prime ideals of D. We can further put conditions on D_P for $P \in \mathscr{F}$. If D is an $\mathscr{F}$-IFC domain and D_P is a valuation domain for each $P \in \mathscr{F}$, then we get the *independent rings of Krull type* (*IRKTs*) of Griffin [23]. If D is a WKD and D_P is a valuation domain for each $P \in \mathscr{F}$, then we get the *generalized Krull domains* (*GKDs*) of Ribenboim [34]. In particular, if D is a WKD and D_P is a

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$*$ Corresponding Author.

DVR for each $P \in \mathscr{F}$, then D is precisely a *Krull domain*. Our original motivation for this paper was to give two classes of generalizations of Krull domains by $\mathscr{F}$-IFC domains.

Our "generalizations" of Krull domains mainly consist of replacing "valuation domain" by "almost valuation domain". Denoting the set of positive integers by $\mathbb{N}$, recall from [8] that a domain D is called an *almost valuation domain* (*AV-domain*) if for $0 \neq a, b \in D$, there is an $n = n(a,b) \in \mathbb{N}$ with $a^n \mid b^n$ or $b^n \mid a^n$. It is clear that every valuation domain is an AV-domain. Now AV-domains are of interest in that by using AV-domains many classical results on valuation domains can be extended to the general theory of almost factoriality introduced in [44]. For example, a domain D is called an *almost Prüfer domain* (*AP-domain*) in [8] if for any $0 \neq a, b \in D$, there is an $n = n(a,b) \in \mathbb{N}$ with (a^n, b^n) invertible. It is shown in [8, Theorem 5.8] that a domain D is an AP-domain if and only if D_P is an AV-domain for each maximal ideal P of D. Also, a domain D is called an *almost Prüfer v-multiplication domain* (*APvMD*) in [30] if for $0 \neq a, b \in D$, there is an $n = n(a,b) \in \mathbb{N}$ with (a^n, b^n) t-invertible. It is shown in [30, Theorem 2.3] that a domain D is an APvMD if and only if D_P is an AV-domain for each maximal t-ideal P of D. Now in the definition of an IRKT, we can use an AV-domain instead of a valuation domain to define an almost independent ring of Krull type. If D is an $\mathscr{F}$-IFC domain and D_P is an AV-domain for any $P \in \mathscr{F}$, then D is said to be an *almost independent ring of Krull type* (*almost IRKT*). Accordingly, if D is a WKD and D_P is an AV-domain for any $P \in \mathscr{F}$, D is said to be an *almost generalized Krull domain* (*almost GKD*). In this paper, we shall create a suitable theory of unique factorization of ideals as in [10] and study them in a slightly more general setting using finite character star-operations.

Briefly, let D be a domain with quotient field K and let $F(D)$ be the set of nonzero fractional ideals of D. A *star-operation* on D is a map $* : F(D) \to F(D)$ such that for all $A, B \in F(D)$ and $0 \neq x \in K$

(1) $(x)_* = (x)$ and $(xA)_* = xA_*$,
(2) $A \subseteq A_*$, $A_* \subseteq B_*$ whenever $A \subseteq B$, and
(3) $(A_*)_* = A_*$.

We note that for $A, B \in F(D)$, $(AB)_* = (A_*B)_* = (A_*B_*)_*$ and call it the $*$-*product*. A fractional ideal A is called a $*$-*fractional ideal* if $A = A_*$ and A is called a fractional ideal of $*$-*finite type* if there exists a finitely generated fractional ideal $B \in F(D)$ such that $A_* = B_*$. A star-operation $*$ is said to be *of finite character* or *of finite type* if $A_* = \bigcup\{B_* \mid 0 \neq B$ is a finitely generated fractional ideal contained in $A\}$ for each $A \in F(D)$. For $A \in F(D)$, define $A^{-1} := \{x \in K \mid xA \subseteq D\}$ and call A $*$-*invertible* if $(AA^{-1})_* = D$. If $*$ is a star-operation on a domain D, then $*$ always induces two finite character star-operations, $*_s$ and $*_w$. Let $A \in F(D)$. Then $A_{*_s} = \bigcup\{B_* \mid 0 \neq B \in F(D)$ f.g. and $B \subseteq A\}$, and $A_{*_w} = \{x \in K \mid xJ \subseteq A$ for some nonzero f.g. ideal J with $J_* = D\}$. We refer the reader to [38, Chapter 7], for more on star-operations. Yet for our purposes we note that given a star-operation $*$ of finite type each nonzero integral $*$-ideal A of D is contained in a maximal integral $*$-ideal M of D.

The classical star-operations are the v-, t-, w-, and d-operations. Let A be a nonzero fractional ideal of D. Then $A_v := (A^{-1})^{-1}$, $A_t := \bigcup\{B_v \mid 0 \neq B \in F(D)$

f.g. and $B \subseteq A\} = A_{v_s}$, $A_w := \{x \in K \mid Jx \subseteq A$ for some $J \in \mathrm{GV}(D)\} = A_{v_w}$, where $\mathrm{GV}(D) = \{J \mid J$ is a nonzero f.g. ideal of D with $J^{-1} = D\}$, and $A_d = A$. In particular, a domain D is called a *DW-domain* in [32] if each nonzero ideal of D is a w-ideal, i.e., $d = w$ over D. It is pointed out in [21, Corollary 3.2] that a quasilocal DW-domain is precisely a t-local domain, where a domain D is called *t-local* in [21] if D is a quasilocal domain with maximal ideal P a t-ideal.

Let $*$ be a finite character star-operation on a domain D. Denote the set of maximal $*$-ideals by $*\text{-}\max(D)$. Then $D = \bigcap\{D_P \mid P \in *\text{-}\max(D)\}$ by [38, Theorem 7.2.11]. Hence $*\text{-}\max(D)$ is a defining family on D. As in [10], a $*\text{-}\max(D)$-IFC domain is said to be *$*$-h-local*. Indeed, following [22, page 136], a $*$-h-local domain can be also called $h_{\mathcal{P}}$-local where $\mathcal{P} = *\text{-}\max(D)$. In particular, a d-h-local domain is precisely an *h-local domain* of Matlis [31], and a t-h-local domain is precisely an *$h_{\mathscr{M}}$-local domain* [20], which was introduced as a *weakly Matlis domain* [9]. In an effort to uniformize the terminology concerning $*$-h-local domains, WKDs, IRKTs, GKDs, and Krull domains are redefined in [10] as $*$-WKDs, $*$-IRKTs, $*$-GKDs, and $*$-Krull domain, respectively. More precisely, a domain D is called a *$*$-WKD* if D is $*$-h-local and $*\text{-}\max(D) = X^{(1)}(D)$; a domain D is called a *$*$-IRKT* if D is $*$-h-local and D_P is a valuation domain for each $P \in *\text{-}\max(D)$; a domain D is called a *$*$-GKD* if D is a $*$-WKD and D_P is a valuation domain for each $P \in X^{(1)}(D)$; a domain is called a *$*$-Krull domain* if D is a $*$-WKD and D_P is a DVR for each $P \in *\text{-}\max(D)$. It is easy to check that a t-WKD (resp., t-IRKT, t-GKD and t-Krull domain) is just a WKD (resp., IRKT, GKD and Krull domain) while a d-WKD (resp., d-IRKT, d-GKD and d-Krull domain) is a one-dimensional finite character domain (resp., finite character Prüfer domain, one-dimensional finite character Prüfer domain and a Dedekind domain), all being served by a single notation "$*$". Using $*$-homogeneous ideals, the second and third authors have given some nice characterizations for these domains. Let $*$ be a finite character star-operation on a domain D. Recall from [10] that a nonzero ideal A of D is called *$*$-homogeneous* if A is finitely generated and A is contained in a unique maximal $*$-ideal. The unique maximal $*$-ideal containing A is often denoted by $M(A)$. It is shown in [10, Theorem 4] that a domain D is a $*$-h-local domain if and only if D is a $*$-SH domain, where D is called a *$*$-semi-homogeneous domain* (*$*$-SH domain*) if every proper nonzero principal ideal of D is a $*$-product of $*$-homogeneous ideals. Furthermore, if A is $*$-homogeneous and $M(A) = \sqrt{A_*}$, then A is called a *type 1 $*$-homogeneous* ideal; if A is $*$-homogeneous and $A_* = (M(A)^n)_*$ for some $n \geq 1$, then A is called a *type 2 $*$-homogeneous ideal*. Accordingly, a domain D is called a *type 1 $*$-SH domain* if every proper nonzero principal ideal of D is a finite $*$-product of type 1 $*$-homogeneous ideals of D; a domain is called a *type 2 $*$-SH domain* if every proper nonzero principal ideal of D is a finite $*$-product of type 2 $*$-homogeneous ideals of D. It is shown in [10, Theorem 7] that a domain D is a $*$-WKD if and only if D is a type 1 $*$-SH domain. It is shown in [10, Theorem 8] that a domain D is a $*$-Krull domain if and only if D is a type 2 $*$-SH domain. Also, a $*$-homogeneous ideal A of D is called *$*$-super-homogeneous* if each $*$-homogeneous ideal containing A is $*$-invertible and a domain D is called a *$*$-super-SH domain* if every proper nonzero principal ideal of D is a finite $*$-product of $*$-super-homogeneous ideals of D. It is proved [10, Theorem 10] that a domain D is a $*$-IRKT if and only if D is a $*$-super-SH domain. In analogy with $*$-super-homogeneous ideals, we introduce $*$-almost super-homogeneous ideals and $*$-almost super-SH domains in Section 2. Here a *$*$-almost super-homogeneous ideal* A is a $*$-invertible P-$*$-homogeneous ideal with the additional condition that given $b_1, \dots, b_s \in P$ with $A^r \subseteq (b_1, \dots, b_s)_*$ for some $r \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ with $(b_1^n, \dots, b_s^n)$ $*$-invertible. Accordingly, a domain D is called a *$*$-almost super-SH domain* if every nonzero proper principal ideal of D is a $*$-product of $*$-almost super-homogeneous ideals. This fills a gap left in [10]. The $*$-almost factorial-SH domains introduced in [10] are integrally closed while the $*$-almost super-SH domains introduced in this paper need not be integrally closed. In Section 3, we study $*$-almost IRKTs and we prove in Theorem 3.2 that a domain D is a $*$-almost IRKT if and only if D is a $*$-almost super-SH domain. We also consider the relationship between a domain D and its integral closure $\overline{D}$. To do this, we first use the w_R-operation to study the integral closure of a t-h-local domain, where the w_R-operation is a star-operation of finite character induced by the w-operation [36]. We prove in Theorem 3.6 that if $R \subseteq T$ is a root extension of domains with R t-linked over T, then R is a t-h-local domain (i.e, w-h-local domain) if and only if T is a w_R-h-local domain. Further, we obtain in Theorem 3.10 that if $R \subseteq \overline{R}$ is a root extension with R t-linked under $\overline{R}$, then R is a t-h-local domain (resp., WKD) if and only if $\overline{R}$ is a t-h-local domain (resp., WKD). And then we characterize a series of domains (such as almost IRKTs, an almost GKDs, and so on) by their integral closures. For example, we show in Theorem 3.12 that D is an almost IRKT if and only if $\overline{D}$ is an IRKT and $D \subseteq \overline{D}$ is a root extension with D t-linked under $\overline{D}$. In Section 4, we study $*$-almost GKDs and we prove in Theorem 4.3 that a domain D is a $*$-almost GKD if and only if D is a type 1 $*$-almost super-SH domain. Furthermore, in Section 5, we study $*$-almost factorial general-SH domains ($*$-afg-SH domains) and we prove in Theorem 5.7 that a domain D is a $*$-afg-SH domain if and only if D is an AGCD-domain and a $*$-almost IRKT, where a domain D is called an *almost GCD domain* (*AGCD-domain*) in [44] if for $0 \neq a, b \in D$, there exists an $n = n(a, b) \in \mathbb{N}$ with $(a^n, b^n)_v$ principal. Throughout this paper we also give various examples to illustrate these new concepts. We now proceed to state and prove our main results.

2. $*$-ALMOST SUPER-HOMOGENEOUS IDEALS

In this section we introduce $*$-almost super-homogeneous ideals and $*$-almost super-SH domains. Suppose that A is a $*$-homogeneous ideal of a domain D. If P is the unique maximal $*$-ideal containing A, then A is said to be *P-$*$-homogeneous*. If both A and B are P-$*$-homogeneous, we say that A is *similar* to B, denoted by $A \sim B$. Now we start with the following definition.

Definition 2.1. Let $*$ be a finite character star-operation on a domain D.

(1) An ideal A of D is called *$*$-almost super-homogeneous* if there exists a maximal $*$-ideal P of D such that A is a $*$-invertible P-$*$-homogeneous ideal and given $b_1, \dots, b_s \in P$ with $A^r \subseteq (b_1, \dots, b_s)_*$ for some $r \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ with $(b_1^n, \dots, b_s^n)$ $*$-invertible.
(2) D is called a *$*$-almost super-SH domain* if every nonzero proper principal ideal of D is a $*$-product of $*$-almost super-homogeneous ideals.

Remark 2.2. In Definition 2.1 (1), the n depends on $b_1, \ldots, b_s$. It may also be noted that every proper nonzero principal ideal in an AV-domain is indeed $*$-almost super-homogeneous.

Next we investigate the properties of $*$-almost super-homogeneous ideals. We need the following lemmas.

Lemma 2.3. *Let $A = (a_1, \ldots, a_k)$ be an ideal of a domain D. Then $A^{nk} \subseteq (a_1^n, \ldots, a_k^n) \subseteq A^n$ for any $n \in \mathbb{N}$.*

Proof. It is clear that $(a_1^n, \ldots, a_k^n) \subseteq A^n$. Let $a_1^{l_1} \cdots a_k^{l_k} \in A^{nk}$, $\sum_{i=1}^k l_k = nk$. Then there is some $l_j \geq n$. Otherwise, as $l_i < n$ for all i, we have $\sum_{i=1}^k l_k < kn$. Which is a contradiction. Hence $a_1^{l_1} \cdots a_k^{l_k} \in (a_1^n, \ldots, a_k^n)$. It follows that $A^{nk} \subseteq (a_1^n, \ldots, a_k^n)$. So $A^{nk} \subseteq (a_1^n, \ldots, a_k^n) \subseteq A^n$. □

Lemma 2.4. *Let $*$ be a finite character star-operation on a domain D and $\{a_\alpha\} \subseteq D \setminus \{0\}$. If $(\{a_\alpha\})_*$ is $*$-invertible, then $(\{a_\alpha^n\})_* = ((\{a_\alpha\})^n)_*$ for any $n \in \mathbb{N}$.*

Proof. See [27, Lemma 2.2]. □

Proposition 2.5. *Let $*$ be a finite character star-operation on a domain D and A a P-$*$-almost super-homogeneous ideal of D.*

(1) *If $(b_1, \ldots, b_s)$ is a P-$*$-homogeneous ideal of D, then $(A^n + (b_1^n, \ldots, b_s^n))_* = (A^n)_*$ or $(A^n + (b_1^n, \ldots, b_s^n))_* = (b_1^n, \ldots, b_s^n)_*$ for some $n \in \mathbb{N}$.*
(2) *If B is a P-$*$-almost super-homogeneous ideal of D, then there exists an $n \in \mathbb{N}$ such that $B^n \subseteq (A^n)_*$ or $A^n \subseteq (B^n)_*$.*
(3) *If B is a P-$*$-almost super-homogeneous ideal of D, then so is AB.*
(4) *A^n is $*$-almost super-homogeneous for any positive integer n.*

Proof. (1) Let $A = (a_1, \ldots, a_k)$. Then $A \subseteq (a_1, \ldots, a_k, b_1, \ldots, b_s)_* \subseteq P$. Hence for some $n \in \mathbb{N}$, $(a_1^n, \ldots, a_k^n, b_1^n, \ldots, b_s^n)$ is $*$-invertible. So $(((a_1^n, \ldots, a_k^n) + (b_1^n, \ldots, b_s^n))(a_1^n, \ldots, a_k^n, b_1^n, \ldots, b_s^n)^{-1})_* = ((a_1^n \ldots, a_k^n, b_1^n, \ldots, b_s^n)(a_1^n, \ldots, a_k^n, b_1^n, \ldots, b_s^n)^{-1})_* = D$. It follows that $(a_1^n, \ldots, a_k^n)(a_1^n \ldots, a_k^n, b_1^n, \ldots, b_s^n)^{-1} \nsubseteq P$ or $(b_1^n, \ldots, b_s^n)(a_1^n \ldots, a_k^n, b_1^n, \ldots, b_s^n)^{-1} \nsubseteq P$. We claim that $(a_1^n, \ldots, a_k^n)(a_1^n, \ldots, a_k^n, b_1^n, \ldots, b_s^n)^{-1}$ and $(b_1^n, \ldots, b_s^n)(a_1^n, \ldots, a_k^n, b_1^n, \ldots, b_s^n)^{-1}$ can not be contained in any maximal $*$-ideal other than P. In fact, if $(b_1^n, \ldots, b_s^n)(a_1^n, \ldots, a_k^n, b_1^n, \ldots, b_s^n)^{-1} \subseteq Q$ for some $Q \in *\text{-}\max(D)$, then $(b_1^n, \ldots, b_s^n)_* \subseteq (Q(a_1^n, \ldots, a_k^n, b_1^n, \ldots, b_s^n))_* \subseteq Q_* = Q$. Hence $b_1, \ldots, b_s \in Q$. So $Q = P$ since $(b_1, \ldots, b_s)$ is P-homogeneous. Similarly we can show that $(a_1^n, \ldots, a_k^n)(a_1^n, \ldots, a_k^n, b_1^n, \ldots, b_s^n)^{-1}$ can not be contained in any maximal $*$-ideal other than P. Thus $((a_1^n, \ldots, a_k^n)(a_1^n \ldots, a_k^n, b_1^n, \ldots, b_s^n)^{-1})_* = D$ or $((b_1^n, \ldots, b_s^n)(a_1^n, \ldots, a_k^n, b_1^n, \ldots, b_s^n)^{-1})_* = D$. In the first case, we have $(A^n)_* = (a_1^n, \ldots, a_k^n)_* = (a_1^n \ldots, a_k^n, b_1^n, \ldots, b_s^n)_*$. So $(A^n)_* = (A^n + (b_1^n, \ldots, b_s^n))_*$. In the second case, we have $(b_1^n, \ldots, b_s^n)_* = (a_1^n, \ldots, a_k^n, b_1^n, \ldots, b_s^n)_*$. So $(b_1^n, \ldots, b_s^n)_* = (A^n + (b_1^n, \ldots, b_s^n))_*$.

(2) Suppose that $B = (b_1, \ldots, b_s)$. Now by (1), for some $n \in \mathbb{N}$, either $(b_1^n \ldots, b_s^n)_* \subseteq (A^n + (b_1^n, \ldots, b_s^n))_* = (A^n)_*$ or $(A^n)_* \subseteq (A^n + (b_1^n, \ldots, b_s^n))_* = (b_1^n, \ldots, b_s^n)_*$. Since B is $*$-invertible, $(B^n)_* = (b_1^n, \ldots, b_s^n)_*$ by Lemma 2.4. So $B^n \subseteq (A^n)_*$ or $A^n \subseteq (B^n)_*$.

(3) By [10, Proposition 2], it follows that AB is P-$*$-homogeneous and certainly AB is $*$-invertible. Let $C = (c_1, \ldots, c_l)$ be a P-$*$-homogeneous ideal with $(AB)^r \subseteq$

C_* for some $r \in \mathbb{N}$. By (3) we have $A^n \subseteq (B^n)_*$ or $B^n \subseteq (A^n)_*$ for some $n \in \mathbb{N}$. Without loss of generality, suppose that $A^n \subseteq (B^n)_*$. Then by Lemma 2.3, $A^{2nlr} \subseteq (A^{2nlr})_* \subseteq ((AB)^{nlr})_* \subseteq ((c_1, \ldots, c_l)^{nl})_* \subseteq (c_1^n, \ldots, c_l^n)_* \subseteq ((c_1, \ldots, c_l)^n)_* \subseteq P$. Since A is P-$*$-almost super-homogeneous, there exists some $m \in \mathbb{N}$ with $(c_1^{mn}, \ldots, c_l^{mn})$ $*$-invertible. Hence AB is P-$*$-almost super-homogeneous.

(4) This follows from (3). □

Corollary 2.6. *Let $*$ be a finite character star-operation on a domain D and A a P-$*$-almost super-homogeneous ideal of D. If B is $*$-invertible and $A^r \subseteq B_* \neq D$ for some $r \in \mathbb{N}$, then B is P-$*$-almost super-homogeneous.*

Proof. By Proposition 2.5 (4), A^r is P-$*$-almost super-homogeneous. Let $B^l \subseteq (c_1, \ldots, c_k)_* \subseteq P$ for some $l \in \mathbb{N}$. Then $A^{rl} \subseteq (B^l)_* \subseteq (c_1, \ldots, c_k)_*$. Hence there exists some $n \in \mathbb{N}$ with $(c_1^n, \ldots, c_k^n)$ $*$-invertible. So B is also P-$*$-almost super-homogeneous. □

Proposition 2.7. *Let $*$ be a finite character star-operation on a domain D. If A is a $*$-super-homogeneous ideal of D, then A is a $*$-almost super-homogeneous ideal of D.*

Proof. Suppose that $A^r \subseteq (c_1, \ldots, c_k)_* \subseteq P$ for some $r \in \mathbb{N}$. Now as A^r is $*$-super-homogeneous and similar to A by [25, Theorem 1.11], $(c_1, \ldots, c_k)$ is $*$-invertible. Hence A is $M(A)$-$*$-almost super-homogeneous. □

Corollary 2.8. *If D is a $*$-super-SH domain, then D is a $*$-almost super-SH domain.*

Proof. This follows from Proposition 2.7. □

We now give a uniqueness result for $*$-products of $*$-almost homogeneous ideals.

Theorem 2.9. *Let $*$ be a finite character star-operation on a domain D and let $A_1, \ldots, A_n$ be $*$-almost super-homogeneous ideals of D. Then the $*$-product $(A_1 \cdots A_n)_*$ can be expressed uniquely, up to order, as a product of pairwise $*$-comaximal $*$-almost super-homogeneous ideals.*

Proof. Write $A = (A_1 \cdots A_n)_*$. Let $M(A_{i_1}), \ldots, M(A_{i_s})$ be the distinct maximal $*$-ideals among $M(A_1), \ldots, M(A_n)$. Set $B_k := \prod\{A_j \mid A_j \sim A_{i_k}\}$ $(k = 1, \ldots, s)$. Then by Proposition 2.5 (3), $B_1, \ldots, B_s$ are $*$-almost super-homogeneous ideals of D that are pairwise $*$-comaximal and $A = (B_1 \cdots B_s)_*$. The uniqueness follows from [10, Theorem 3].

□

3. $*$-almost IRKTs

In this section we introduce $*$-almost IRKTs. Using AV-domains instead of valuation domains, we define a $*$-almost IKRT as follows.

Definition 3.1. A domain D is called a *$*$-almost independent ring of Krull type* (*$*$-almost IRKT*) if D is a $*$-h-local domain and D_P is an AV-domain for each $P \in *\text{-}\max(D)$.

It is clear that a domain D is a $*$-almost IRKT if and only if D is an almost IRKT and $\mathscr{F} = *\text{-}\max(D)$, where a domain D is called an *almost IRKT* if D is an $\mathscr{F}$-IFC domain and D_P is an AV-domain for each $P \in \mathscr{F}$. It is easy to see that a d-almost IRKT is just a finite character, independent AP-domain. At the other extreme, a t-almost IRKT is just an almost IRKT [7, Lemma 2.1].

Recall from [27] that a domain D is called an *almost Prüfer $*$ multiplicatiion domain* (*AP$*$MD*) if for $0 \neq a, b \in D$, there exists an $n = n(a, b) \in \mathbb{N}$ with (a^n, b^n) $*_s$-invertible. Let $*$ be a finite character star-operation on a domain D. Then D is an AP$*$MD if and only if D_P is an AV-domain for each $P \in *\text{-}\max(D)$ [27, Theorem 2.4]. Hence it is clear that D is a $*$-almost IRKT if and only if D is a $*$-h-local domain and an AP$*$MD. Next we prove that D is a $*$-almost IRKT if and only if D is a $*$-almost super-SH domain.

Theorem 3.2. *Let D be a domain and $*$ a finite character star-operation on D. Then the following statements are equivalent for D.*

(1) *D is a $*$-almost IRKT.*
(2) *D is a $*$-h-local domain and an AP$*$MD.*
(3) *D is a $*$-h-local domain and every $*$-invertible $*$-homogeneous ideal of D is $*$-almost super-homogeneous.*
(4) *D is a $*$-almost super-SH domain.*

Proof. (1) $\Leftrightarrow$ (2) This follows from [27, Theorem 2.4].

(1) $\Rightarrow$ (3) Suppose that A is a $*$-invertible P-$*$-homogeneous ideal of D. Let $B = (b_1, \ldots, b_k)$ be P-$*$-homogeneous with $A^r \subseteq B_*$ for some positive integer r. Since D_P is an AV-domain, D_P is an AB-domain by [8, Theorem 5.6]. Hence by [8, Lemma 4.3] there exists some $n \in \mathbb{N}$ such that $(b_1^n, \ldots, b_k^n)D_P = (b_1^n/1, \ldots, b_k^n/1)$ is a principal ideal of D_P. Let $Q \in *\text{-}\max(D)$ with $Q \neq P$. Then $B \nsubseteq Q$ since B is P-$*$-homogeneous. Take $b_j \in B \setminus Q$. Then, $b_j^n \notin Q$. Hence $(b_1^n, \ldots, b_k^n) \nsubseteq Q$ and so $(b_1^n, \ldots, b_k^n)D_Q = D_Q$. It follows that $(b_1^n, \ldots, b_k^n)$ is a locally principal ideal of D. So $(b_1^n, \ldots, b_k^n)$ is $*$-invertible by [38, Theorem 7.2.15]. Consequently, A is $*$-almost super-homogeneous.

(3) $\Rightarrow$ (4) Since D is a $*$-h-local domain, it follows from [10, Theorem 5] that D is a $*$-SH domain. Let xD be a proper nonzero principal ideal of D. Then $xD = (A_1 \cdots A_k)_*$ where each A_i is $*$-homogeneous. Since xD is $*$-invertible, each A_i is $*$-invertible. Hence each A_i is $*$-almost super-homogeneous. It follows that D is a $*$-almost super-SH domain.

(4) $\Rightarrow$ (1) Suppose that D is a $*$-almost super-SH domain. Then D is a $*$-SH domain. Hence D is a $*$-h-local domain by [10, Theorem 5]. We only need to prove that D_P is an AV-domain for each $P \in *\text{-}\max(D)$. For a given $P \in *\text{-}\max(D)$, take $0 \neq x \in P$. Then by Theorem 2.9 $xD = (A_1 \cdots A_k)_*$ where the A_i are mutually $*$-comaximal $*$-almost super-homogeneous ideals of D. Hence there exists some A_i such that $A_i \subseteq P$. If $j \neq i$ and $A_j \subseteq P$, then $(A_i + A_j)_* \subseteq P$. But A_i and A_j are $*$-comaximal, so $D = (A_i + A_j)_* \subseteq P$, which is a contradiction. So there is only one A_i such that $A_i \subseteq P = M(A_i)$. Since A_i is $*$-almost super-homogeneous, A_i is $*$-invertible. Hence $(A_i)_*D_P = A_iD_P$ by [38, Corollary 7.2.16]. It follows that $(A_i)_* = ((A_i)_*)_{*_w} = \bigcap\{(A_i)_*D_Q \mid Q \in *\text{-}\max(D)\} = A_iD_P \bigcap D$. So $xD_P \bigcap D = (A_1 \cdots A_k)_*D_P \bigcap D = (A_1 \cdots A_k)D_P \bigcap D = A_iD_P \bigcap D = (A_i)_*$. For

convenience we write $xD_P \bigcap D = A_*$, where A is P-$*$-almost super-homogeneous. Let $0 \neq y \in P$. Then similarly we get that $yD_P \bigcap D = B_*$, where B is also P-$*$-almost super-homogeneous. Thus $A^n \subseteq (B^n)_*$ or $B^n \subseteq (A^n)_*$ for some $n \in N$ by Proposition 2.5(2). Now we claim that $x^n D_P \subseteq y^n D_P$ or $y^n D_P \subseteq x^n D_P$. In fact, if $A^n \subseteq (B^n)_*$, then $((xD_P \bigcap D)^n)_* = (A^n)_* \subseteq (B^n)_* = ((yD_P \bigcap D)^n)_*$. Since B is $*$-invertible, $(B^n)_* = ((yD_P \bigcap D)^n)_*$ is $*$-invertible. Hence $(yD_P \bigcap D)^n D_P = ((yD_P \bigcap D)^n)_* D_P$ by [38, Corollary 7.2.16]. So we have $x^n D_P = (xD_P)^n = ((xD_P \bigcap D)D_P)^n = (xD_P \bigcap D)^n D_P = ((xD_P \bigcap D)^n)_* D_P \subseteq ((yD_P \bigcap D)^n)_* D_P = (yD_P \bigcap D)^n D_P = y^n D_P$. Similarly we can prove that if $B^n \subseteq (A^n)_*$, then $y^n D_P \subseteq x^n D_P$. Therefore for each $P \in *\text{-}\max(D)$, D_P is an AV-domain. □

Next, we point out that a $*$-almost super-SH domain of type 2 is precisely a $*$-Krull domain.

Corollary 3.3. *Let D be a domain and $*$ a finite character star-operation on D. The following statements are equivalent for D.*

(1) *D is a $*$-almost super-SH domain of type 2.*
(2) *D is a $*$-Krull domain.*
(3) *If A is a finitely generated $*$-invertible ideal of D with $A_* \neq D$, then A_* is a $*$-product of $*$-almost super-homogeneous ideals of type 2.*
(4) *D is a $*$-SH domain of type 2.*

Proof. (1) ⇒ (4) Trivial.

(4) ⇔ (2) [10, Theorem 8].

(2) ⇒ (3) Suppose that A is a finitely generated $*$-invertible $*$-homogeneous ideal of D with $A_* \neq D$. Then $A_* = (A_1 \cdots A_k)_*$ by [10, Theorem 8], where each A_i is a $*$-invertible $*$-homogeneous ideal of type 2 . And since a $*$-Krull domain is a $*$-almost IRKT, each A_i is $*$-almost super-homogeneous by Theorem 3.2. Hence A_* is a $*$-product of $*$-almost super-homogeneous ideals of type 2.

(3) ⇒ (1) Clear. □

In [30, Theorem 3.6], it was shown that if a domain R is an APvMD, then $\overline{R}$ is a PvMD and $R \subseteq \overline{R}$ is a root extension. Thus it is natural to ask how to study the integral closure of an almost IRKT. Now we need to recall some concepts and terminology for w-modules. Let R be a commutative ring. As in [43], a nonzero ideal J of R is called a *Glaz-Vasconcelos ideal* (*GV-ideal*) if J is finitely generated and the natural homomorphism $\phi : J \to \mathrm{Hom}_R(J, R)$ is an isomorphism. Denote the set of GV-ideals of R by $\mathrm{GV}(R)$. Let M be an R-module. Then M is called GV-*torsion-free* if $Jx = 0$ with $J \in \mathrm{GV}(R)$ and $x \in M$ implies $x = 0$, and M is called GV-*torsion* if for any $x \in M$, there exists $J \in \mathrm{GV}(R)$ with $Jx = 0$. For a GV-torsion-free module M, set $M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\}$ which is called the *w-envelope* of M, where $E(M)$ is the injective hull of M. If $M = M_w$, then M is called a *w-module* (*over* R). Let $R \subseteq T$ be an extension of domains. Recall from [39] that T is called a *w-linked extension of* R if T as an R-module is a w-module. In case where $R \subseteq T \subseteq K := qf(R)$, T is said to be a *w-linked overring*. It is recorded in [38, Theorem 7.7.4] that $R \subseteq T$ is a w-linked extension of domains if and only if $J \in GV(R)$ implies $JT \in GV(T)$. Thus a w-linked extension $R \subseteq T$ of domains is precisely a t-linked extension as defined in [5]. In fact, the concept of a t-linked extension of domains defined in [5] is a generalization of a t-linked overring as defined in [19] and it has proved to be an important tool for studying extensions of domains. The concept of a w-linked extension can be viewed as a module-theoretic characterization of a t-linked extension. Next, we uniformize the terminology "t-linked extension".

Now let $R \subseteq T$ be a t-linked extension of domains. For any nonzero fractional ideal A of T, define

$$w_R : A \to A_w := \{x \in E(A) \mid xJ \subseteq A \text{ for some } J \in GV(R)\},$$

where A_w is the w-envelope of A as an R-module. Then w_R is a star-operation of finite character over T, which is called a *star-operation induced by the w-operation.* If $A = A_{w_R}$, then A is called a *w_R-ideal of* T. Let us denote the w-envelope of A as a T-module by A_W, which is different from the w-envelope $A_{w_R} = A_w$ of A as an R-module. Then $A_{w_R} = A_w \subseteq A_W$. Hence every w-ideal of T is always a w_R-ideal of T. However, a w_R-ideal of T is not necessarily a w-ideal of T (See [42, Example 2.7] or Example 3.7). Now let $R \subseteq T$ be a t-linked extension. Then T may be called *w_R-h-local* if $w_R\text{-}\max(T)$ is independent and of finite character, and further T is called a *w_R-weakly Krull domain* (*w_R-WKD*) if T is a w_R-h-local domain with $w_R\text{-}\max(T) = X^{(1)}(T)$. Let $*$ be a star-operation of finite character on a domain R. Define the *$*$-dimension* as follows:

$$*\text{-}\dim(R) = \sup\{\mathrm{ht}P \mid P \in *\text{-}\max(R)\}.$$

By [3, Corollary 2.17], we have $w\text{-}\max(R) = t\text{-}\max(R)$. So $t\text{-}\dim(R) = w\text{-}\dim(R)$ and a t-h-local domain is precisely w-h-local. We next uniformize the terminology "t-h-local". In particular, if $R \subseteq T$ is a t-linked extension of domains, then it is clear that $t\text{-}\dim(T) = w\text{-}\dim(T) \leq w_R\text{-}\dim(T)$.

Before we investigate the integral closure of an almost IRKT, that is before we study the integral closure of a t-h-local domain, we need to establish the following lemma.

Lemma 3.4. *Let $R \subseteq T$ be a t-linked extension of domains. Assume that $R \subseteq T$ is a root extension. Then we have the following facts.*

(1) $w\text{-}\max(R) = \{Q \bigcap R \mid Q \in w_R\text{-}\max(T)\}$.
(2) $w\text{-}\dim(R) = w_R\text{-}\dim(T)$.

Proof. (1) Let $Q' \in w_R\text{-}\max(T)$. Then $Q' \bigcap R$ is a w-ideal of R by [39, Theorem 3.8 (1)]. Noting that $Q' \bigcap R \neq R$, there exists $P \in w\text{-}\max(R) \subseteq \mathrm{Spec}(R)$ such that $Q' \bigcap R \subseteq P$. Since $R \subseteq T$ is a root extension, $\mathscr{Q} : \mathrm{Spec}(T) \to \mathrm{Spec}(R)$ given by $\mathscr{Q}(Q) = Q \bigcap R$ is an order isomorphism and a homeomorphism by [8, Theorem 2.1]. Hence there exists $Q = \sqrt{P} = \{s \in T \mid s^n \in P \text{ for some } n \in \mathbb{N}\} \in \mathrm{Spec}(T)$ and $Q' \subseteq Q$. We claim that $Q_{w_R} \neq T$. Otherwise, if $Q_{w_R} = T$, then there exists $J \in GV(R)$ such that $J \subseteq Q$. Hence $J \subseteq Q \bigcap R = P$. But then $P = P_w = R$, which is a contradiction. Now since $Q' \in w_R\text{-}\max(T)$ and $Q' \subseteq Q_{w_R} \neq T$, it follows that $Q' = Q_{w_R} \supseteq Q$. Hence $Q = Q'$ and $Q \bigcap R = Q' \bigcap R = P \in w\text{-}\max(R)$. So $w\text{-}\max(R) \supseteq \{Q \bigcap R \mid Q \in w_R\text{-}\max(T)\}$. Conversely, let $P \in w\text{-}\max(R)$. Then by [8, Theorem 2.1], there exists a unique $Q \in \mathrm{Spec}(T)$ lying over P. Since $P = Q \bigcap R$ is a proper w-ideal of R, Q is a proper w_R-ideal of T by [39, Theorem 3.8 (3)]. Thus there exists $Q' \in w_R\text{-}\max(T) \subseteq \mathrm{Spec}(T)$ such that $Q \subseteq Q'$. By [39, Theorem 3.8

(1)], we have that $Q'\bigcap R$ is a proper w-ideal of R and $P=Q\bigcap R\subseteq Q'\bigcap R$. Since $P\in w\text{-}\max(R)$, this forces $P=Q'\bigcap R$, where $Q'\in w_R\text{-}\max(R)$. So $\{Q\bigcap R\mid Q\in w_R\text{-}\max(T)\}\supseteq w\text{-}\max(R)$. Therefore $w\text{-}\max(R)=\{Q\bigcap R\mid Q\in w_R\text{-}\max(T)\}$.

(2) This follows from (1) and [8, Theorem 2.1]. □

Theorem 3.5. *Let $R\subseteq T$ be a t-linked extension of domains. Assume that $R\subseteq T$ is a root extension. Then*

(1) *R is a t-h-local domain (i.e., w-h-local) if and only if T is a w_R-h-local domain, and*

(2) *R is a WKD (i.e., w-WKD) if and only if $T$is a w_R-WKD.*

Proof. (1) $(\Rightarrow)$ Let $Q_1\neq Q_2$ with $Q_1,Q_2\in w_R\text{-}\max(T)$. Then $Q_1\bigcap R$ and $Q_2\bigcap R$ are both maximal w-ideals of R by Lemma 3.4 (1), and $Q_1\bigcap R\neq Q_2\bigcap R$ by [8, Theorem 2.1]. Suppose that $Q\in\operatorname{Spec}(R)$ such that $Q\subseteq Q_1\bigcap Q_2$. Then $Q\bigcap R\subseteq Q_1\bigcap Q_2\bigcap R=(Q_1\bigcap R)\bigcap(Q_2\bigcap R)$. Now since R is a t-h-local domain, we have $Q\bigcap R=0$. Hence $Q=\sqrt{0}=\{s\in T\mid s^n=0 \text{ for some } n\in\mathbb{N}\}=0$ by [8, Theorem 2.1]. So $w_R\text{-}\max(T)$ is independent. Next we prove that $w_R\text{-}\max(T)$ is of finite character. Let x be a nonzero nonunit of T. Since $R\subseteq T$ is a root extension, there exists some $n\in\mathbb{N}$ with $x^n\in R$. Now let $\{Q_i\mid i\in\Gamma\}$ be the set of maximal w_R-ideals of T containing x. Then x^n is a nonzero nonunit of R and $x^n\in Q_i\bigcap R\in w\text{-}\max(R)$ for each $i\in\Gamma$ by Lemma 3.4. Since R is a t-h-local domain, $w\text{-}\max(R)$ is of finite character. Hence $\{Q_i\bigcap R\mid i\in\Gamma\}$ is a finite set. By [8, Theorem 2.1], we have that $Q_i=Q_j$ if and only if $Q_i\bigcap R=Q_j\bigcap R$. So $\{Q_i\mid i\in\Gamma\}$ is a finite set. Thus $w_R\text{-}\max(T)$ is of finite character. Consequently, R is a w_R-h-local domain.

$(\Leftarrow)$ By Lemma 3.4 (1), we can assume that $Q_1\bigcap R$ and $Q_2\bigcap R$ are two different maximal w-ideals of R, where Q_1 and Q_2 are different maximal w_R-ideals of T. Let $P\in\operatorname{Spec}(R)$ with $P\subseteq(Q_1\bigcap R)\bigcap(Q_2\bigcap R)$. Then there exists $Q\in\operatorname{Spec}(T)$ such that $P=Q\bigcap R$ by [8, Theorem 2.1]. Thus $Q\bigcap R\subseteq Q_1\bigcap R$ and $Q\bigcap R\subseteq Q_2\bigcap R$. By [8, Theorem 2.1] again, we have $Q\subseteq Q_1\bigcap Q_2$. But since T is a t-h-local domain, $Q=0$ and so $P=0$. Thus $w\text{-}\max(R)$ is independent. Next we prove that $w\text{-}\max(R)$ is of finite character. Let x is a nonzero nonunit of R. Suppose that $\{P_i\mid i\in\Gamma\}$ is the set of all maximal w-ideal of R containing x. Set $P_i\in\{P_i\mid i\in\Gamma\}$. Then by Lemma 3.4 (1) and [8, Theorem 2.1], there exist unique $Q_i\in w_R\text{-}\max(R)$ such that $P_i=Q_i\bigcap R$. Hence $x\in P_i=Q_i\bigcap R\subseteq Q_i$. Also since T is a w_R-h-local domain, there is a finite number of maximal w_R-ideals of T containing x, say $\{Q_1,\dots,Q_n\}$. Then $\{Q_i\mid i\in\Gamma\}\subseteq\{Q_1,\dots,Q_n\}$, and hence Γ is a finite set. So R is of finite t-character. It follows that R is a t-h-local domain.

(2) $(\Rightarrow)$ Suppose that R is a WKD. Then R is a t-h-local domain with $w\text{-}\max(R)=X^{(1)}(R)$. Hence $w_R\text{-}\dim(T)=1$ by Lemma 3.4 (2). So $w_R\text{-}\max(T)\subseteq X^{(1)}(T)$. Now take $Q\in X^{(1)}(T)$. Then Q is a t-ideal and hence a w_R-ideal of T. So there exists $Q'\in w_R\text{-}\max(T)$ such that $Q\subseteq Q'$. But $Q'\in w_R\text{-}\max(T)\subseteq X^{(1)}(T)$, so $Q=Q'\in w_R\text{-}\max(T)$. Thus $w_R\text{-}\max(T)=X^{(1)}(T)$. Also since R is t-h-local, T is w_R-h-local by (1). Hence T is a w_R-WKD.

$(\Leftarrow)$ Suppose that T is a w_R-WKD. Then T is a w_R-h-local domain such that $w_R\text{-}\max(T)=X^{(1)}(T)$. Since $w\text{-}\dim(R)=w_R\text{-}\dim(T)$ by Lemma 3.4 (2), we have that $w\text{-}\dim(R)=1$. Hence $w\text{-}\max(R)\subseteq X^{(1)}(R)$. Let $P\in X^{(1)}(R)$. Then P is a

w-ideal of R. Hence there exists $P' \in w\text{-}\max(R)$ such that $P \subseteq P'$. Noting that $\mathrm{ht}P' = 1$, we have $P = P' \in w\text{-}\max(R)$. $\square$

We next give a characterization of the integral closure of an h-local domain (resp., a WKD).

Theorem 3.6. *Let $R \subseteq \overline{R}$ be a root extension. Then*

(1) *R is t-h-local (i.e., w-h-local) if and only if $\overline{R}$ is w_R-h-local, and*
(2) *R is a WKD (i.e., w-WKD) if and only if $\overline{R}$ is a w_R-WKD.*

Proof. Since $\overline{R}$ is integrally closed, $\overline{R}$ is root closed. Hence $\overline{R}$ is a t-linked overring of R by [5, Proposition 2.4]. Thus by Theorem 3.5, R is a t-h-local domain (resp., WKD) if and only if $\overline{R}$ is a w_R-h-local domain (resp., a w_R-WKD). $\square$

Let $R \subseteq T$ be a t-linked extension of domains. Because $w_R \leq w$, one may ask whether a w_R-h-local domain is a t-h-local domain (i.e., a w-h-local domain) . Next we point out that (1) a t-h-local domain is not necessarily a w_R-h-local domain, and (2) a w_R-h-local domain is not necessarily a t-h-local domain.

Example 3.7. Let $\mathbb{Z}$ be the ring of integers and let $\mathbb{Z}[X]$ be the polynomial ring over $\mathbb{Z}$. Then $\mathbb{Z} \subseteq \mathbb{Z}[X]$ is a t-linked extension of domains and $\mathbb{Z}[X]$ is a Krull domain. Hence $\mathbb{Z}[X]$ is a t-h-local domain by Corollary 3.3. Let p be a prime element of $\mathbb{Z}$. Then (p, X) is a maximal ideal of $\mathbb{Z}[X]$. Also since $(p, X) \bigcap \mathbb{Z} = p\mathbb{Z}$ is a w-ideal of $\mathbb{Z}$, (p, X) is a maximal w_R-ideal of $\mathbb{Z}[X]$ by [39, Theorem 3.8 (3)]. Thus

$$\{(p, X) \mid p \text{ is a prime element of } \mathbb{Z}\} \subseteq w_R\text{-}\max(\mathbb{Z}[X]),$$

and $(X) \subseteq (p, X)$ for each p. So $w_R\text{-}\max(\mathbb{Z}[X])$ is not of finite character and hence $\mathbb{Z}[X]$ is not a w_R-h-local domain.

From Example 3.7 it is also seen that $\mathbb{Z}[X]$ is a t-h-local domain but not an h-local domain. In [21, Corollary 3.2], it is proved that a t-local domain is precisely a quasilocal DW-domain. So every t-local domain is a quasilocal domain. However a quasilocal domain is not necessarily t-local. A counterexample is given by Fontana and the third author in [21, Example 2.9]. Now using this example, we can construct an example to show that a w_R-local domain is not necessarily a t-h-local domain. It also follows that an h-local domain is not necessarily t-h-local.

Example 3.8. Let $\mathbb{Q}$ be the field of rational numbers. As in [21, Example 2.9], set $T := \mathbb{Q}[[X, Y]] + Z\mathbb{Q}((X, Y))[[Z]]$, where $\mathbb{Q}((X, Y))$ is the quotient field of $\mathbb{Q}[[X, Y]]$. Then T is a 3-dimensional local PvMD with maximal ideal $M = (X, Y)\mathbb{Q}[[X, Y]] + Z\mathbb{Q}((X, Y))[[Z]]$ and T is not a t-local domain. So M is not a t-ideal of T. Also since $\mathbb{Q}[[X, Y]]$ is a local domain but not a t-local domain, $(X, Y)Q[[X, Y]]$ is not a maximal t-ideal of $\mathbb{Q}[[X, Y]]$. But $\mathbb{Q}[[X, Y]]$ is a 2-dimensional local Noetherian Krull domain, so $\mathrm{ht}(X\mathbb{Q}[[X, Y]]) = \mathrm{ht}(Y\mathbb{Q}[[X, Y]]) = 1$. Hence $X\mathbb{Q}[[X, Y]]$ and $Y\mathbb{Q}[[X, Y]]$ are maximal t-ideals of $\mathbb{Q}[[X, Y]]$. So $Q_1 := X\mathbb{Q}[[X, Y]] + Z\mathbb{Q}((X, Y))[[Z]]$ and $Q_2 := Y\mathbb{Q}[[X, Y]] + Z\mathbb{Q}((X, Y))[[Z]]$ are maximal t-ideals of T. Noting that $Z\mathbb{Q}((X, Y))[[Z]] \subseteq Q_1 \bigcap Q_2$, it follows that T is not a t-h-local domain. Now let $\mathbb{Z}$ be the ring of integers and let p be a prime element of $\mathbb{Z}$. Consider the following

$D+M$ construction

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\pi"'] & T \arrow[d, "\pi"] \\ \mathbb{Z}_{(p)} \arrow[r] & \mathbb{Q} \end{tikzcd} .$$

Then $R = \mathbb{Z}_{(p)} + (X,Y)\mathbb{Q}[[X,Y]] + Z\mathbb{Q}((X,Y))[[Z]]$ and $R \subseteq T$ is a t-linked extension of domains. Since M is a divisorial prime ideal of R, M a w-ideal of R. Hence M is a w_R-ideal of T by [39, Theorem 3.8 (3)]. So T is a w_R-h-local domain. Thus T is a w_R-h-local domain but not a t-h-local domain.

Based on Theorem 3.6, it is natural to ask when $\overline{R}$ is a t-h-local domain (resp., WKD) if R is a t-h-local domain (resp., WKD). In order to answer this question, we need the concept of a t-linked underring, which is the converse of the concept of a t-linked overring. Let T be an overring of a domain R. As in [8], R is called *t-linked under T* if for each nonzero finitely generated ideal of J of R, $(JT)_v = T$ implies $J_v = R$. We have the following lemma for a t-linked underring.

Lemma 3.9. *Let $R \subseteq T$ be a w-linked extension of domains. Then the following statements are equivalent.*

(1) *Each maximal w_R-ideal of T is a maximal w-ideal of T.*
(2) *If $J \in \mathrm{GV}(T)$, then there exists $J' \in \mathrm{GV}(R)$ such that $J' \subseteq J$.*
(3) *Each w_R-ideal of T is a w-ideal of T.*
(4) *For each prime ideal Q of T, $(Q \bigcap R)_t \subsetneq R$ implies $Q_t \subsetneq T$.*

Also, if $R \subseteq T$ is a root extension, then the conditions above are equivalent to the following condition.

(5) *R is t-linked under T.*

Proof. The equivalence of (1)-(4) is [41, Theorem 2.7].

(2) $\Rightarrow$ (5) Let J be a nonzero finitely generated ideal of R with $JT \in GV(T)$. Then there exists $J' \in GV(R)$ such that $J' \subseteq JT$. We claim that $J_w = R$. If $J_w \neq R$, there exists $P \in w\text{-}\max(R)$ such that $J \subseteq J_w \subseteq P$. Since $R \subseteq T$ is a root extension, there exists $Q \in \mathrm{Spec}(T)$ such that $Q \bigcap R = P$. Hence $J' \subseteq JT \subseteq PT \subseteq Q$. Also since P is a w-ideal, Q is a w_R-ideal by [39, Theorem 3.8 (3)]. But $J' \subseteq Q$, so we have $Q = Q_{w_R} = T$ and hence $P = R$, which is a contradiction. Hence $J_w = R$. So $J \in GV(R)$ and $J_v = R$.

(5) $\Rightarrow$ (2) Let $(a_1, \ldots, a_k) := J \in GV(T)$. Since $R \subseteq T$ is a root extension, there exists $n_j \in \mathbb{N}$ such that $a_j^{n_j} \in R$ for $j = 1, \ldots, k$. Set $n = \prod_{j=1}^{k} n_j$. Then $a_j^n \in R$ for $j = 1, \ldots, k$ and $J^{kn} \subseteq (a_1^n, \ldots, a_k^n)$ by Lemma 2.3. Since $J_V = T$ (V is the v-operation on T, which is different from the v-operation v on R), $(J^{kn})_V = T$. Hence $(a_1^n, \ldots, a_k^n)_V = T$. By (5), $(a_1^n, \ldots, a_k^n)_v = R$. So $(a_1^n, \ldots, a_k^n) \in GV(R)$. □

Theorem 3.10. *Let $R \subseteq \overline{R}$ be a root extension with R t-linked under $\overline{R}$. Then*

(1) *R is a t-h-local domain if and only if $\overline{R}$ is t-h-local domain, and*
(2) *R is a WKD if and only if $\overline{R}$ is a WKD.*

Proof. Since $R \subseteq \overline{R}$ is a root extension with R t-linked under $\overline{R}$, $w = w_R$ on $\overline{R}$ by Lemma 3.9. Hence by Theorem 3.6, R is a w-h-local domain (resp., WKD) if and only if $\overline{R}$ is a w-h-local domain (resp., WKD). □

In Theorem 3.10, it is natural to ask when $R \subseteq \overline{R}$ is a root extension with R t-linked under $\overline{R}$. In fact, if R is an AGCD-domain, then $R \subseteq \overline{R}$ is a root extension with R t-linked under $\overline{R}$. In [8, Theorem 5.9], the second author and the third author have proved that a domain R is an AGCD-domain if and only if $R \subseteq \overline{R}$ is a root extension with R t-linked under $\overline{R}$ and $\overline{R}$ is an AGCD-domain. And then, this result is extended to APvMDs. It is shown in [28, Proposition 2.5] that if a domain R is an APvMD, R is t-linked under $\overline{R}$. Thus by [30, Theorem 3.6 and 3.7], a domain R is an APvMD if and only if $R \subseteq \overline{R}$ is a root extension with R t-linked under $\overline{R}$ and $\overline{R}$ is a PvMD [28, Corollary 2.6]. Now from the point of view of the equivalence of operations, we give a direct proof for [28, Proposition 2.5]. It follows that $w_{\overline{R}} = w$ over the integral closure of an APvMD.

Proposition 3.11. *Let R be an APvMD. Then R is t-linked under $\overline{R}$.*

Proof. Since R is an APvMD, $R \subseteq \overline{R}$ is a root extension by [30, Theorem 3.8]. Hence $R \subseteq \overline{R}$ is a t-linked extension by [5, Proposition 2.4]. Now using Lemma 3.9, we claim that if $J' \in \mathrm{GV}(\overline{R})$, there exists $J \in \mathrm{GV}(R)$ such that $J \subseteq J'$. Let $J' = (x_1, \ldots, x_k) \in \mathrm{GV}(\overline{R})$ with $x_j \in \overline{R}$ ($j = 1, \ldots, k$). Since $R \subseteq \overline{R}$ is a root extension, there exists $n_j \in \mathbb{N}$ with $x_j^{n_j} \in R$ for each j. Set $n = \prod_{j=1}^{n} n_j$. Then $(x_1^n, \ldots, x_k^n) := (x_1^n, \ldots, x_k^n)R$ is a nonzero ideal of R. Since R is an APvMD, there exists some $m \in \mathbb{N}$ such that $(x_1^{nm}, \ldots, x_k^{nm})$ is t-invertible by [30, Theorem 2.3] and hence w-invertible. So $((x_1^{nm}, \ldots, x_k^{nm})(x_1^{nm}, \ldots, x_k^{nm})^{-1})_w = R$. Thus there exists $J \in GV(R)$ such that $J \subseteq (x_1^{nm}, \ldots, x_k^{nm})(x_1^{nm}, \ldots, x_k^{nm})^{-1}$. Also since $(J')^{nmk} = (x_1, \ldots, x_k)^{kmn} \subseteq (x_1^{mn}, \ldots, x_k^{nm})\overline{R}$ and $J' \in GV(\overline{R})$, we have $(J')^{nmk} \in \mathrm{GV}(\overline{R})$ and hence $(x_1^{mn}, \ldots, x_k^{nm})\overline{R} \in \mathrm{GV}(\overline{R})$. Thus

$$\begin{aligned}
J &\subseteq (x_1^{nm}, \ldots, x_k^{nm})(x_1^{nm}, \ldots, x_k^{nm})^{-1} \\
&\subseteq (x_1^{nm}, \ldots, x_k^{nm}) \cdot ((x_1^{nm}, \ldots, x_k^{nm})\overline{R})^{-1} \\
&= (x_1^{nm}, \ldots, x_k^{nm})\overline{R} \\
&\subseteq (x_1, \ldots, x_k) \\
&= J'.
\end{aligned}$$

□

Recall from [23] that a domain R is called a *ring of Krull type* (an *RKT*) if R is a PvMD of finite t-character. Accordingly, we define an *almost RKT* as an APvMD of finite t-character. Now we can characterize almost IRKTs and almost RKTs by their integral closures. (The concepts of almost GKD and t-afg-SH domains will be introduced in Section 4 and Section 5, respectively).

Theorem 3.12. *The following statements hold.*

(1) *A domain R is an almost IRKT if and only if $\overline{R}$ is an IRKT and $R \subseteq \overline{R}$ is a root extension with R t-linked under $\overline{R}$.*
(2) *A domain R is an almost RKT if and only if $\overline{R}$ is a RKT and $R \subseteq \overline{R}$ is a root extension with R t-linked under $\overline{R}$.*
(3) *A domain R is an almost RKT with torsion t-class group if and only if $\overline{R}$ is an RKT with torsion t-class group and $R \subseteq \overline{R}$ is a root extension with R t-linked under $\overline{R}$.*

(4) *A domain R is a t-almost super-SH domain if and only if $\overline{R}$ is a t-super-SH domain and $R \subseteq \overline{R}$ is a root extension with R t-linked under $\overline{R}$.*
(5) *A domain R is an almost GKD if and only if $\overline{R}$ is a GKD and $R \subseteq \overline{R}$ is a root extension with R t-linked under $\overline{R}$.*
(6) *A domain R is a type 1 t-almost super-SH domain if and only if $\overline{R}$ is a type 1 t-super-SH domain and $R \subseteq \overline{R}$ is a root extension with R t-linked under $\overline{R}$.*
(7) *A domain R is a t-afg-SH domain if and only if $\overline{R}$ is a t-af-SH domain and $R \subseteq \overline{R}$ is a root extension with R t-linked under $\overline{R}$.*
(8) *A domain R is a type 1 t-afg-SH domain if and only if $\overline{R}$ is a type 1 t-af-SH domain and $R \subseteq \overline{R}$ is a root extension with R t-linked under $\overline{R}$..*

Proof. (1) ($\Rightarrow$) Suppose that R is an almost IRKT. Then R is a t-h-local domain and an APvMD. Hence $R \subseteq \overline{R}$ is a root extension and $\overline{R}$ is a PvMD by [30, Theorem 3.6]. Hence $\overline{R}$ is a t-linked overring by [5, Proposition 2.4]. Also by Proposition 3.11, it follows that R is t-linked under $\overline{R}$. So $w = w_R$ over $\overline{R}$ by Lemma 3.9. Thus by Theorem 3.10, $\overline{R}$ is t-h-local. Thus $\overline{R}$ is an IRKT.

($\Leftarrow$) Suppose that $\overline{R}$ is an IRKT and $R \subseteq \overline{R}$ is a root extension with R t-linked under $\overline{R}$. Then $\overline{R}$ is a t-h-local PvMD. Hence R is an APvMD by [30, Theorem 3.7] and a t-h-local domain by Theorem 3.10.

(2) ($\Rightarrow$) Suppose that R is an almost RKT. Then R is an APvMD and $w\text{-}\max(R)$ is of finite character. Hence $\overline{R}$ is a PvMD and $R \subseteq \overline{R}$ is a root extension by [30, Theorem 3.6], and R is t-linked under $\overline{R}$ by Proposition 3.11. So $w = w_R$ over $\overline{R}$ by Lemma 3.9. We need to show that $w_R\text{-}\max(\overline{R})$ is of finite character. Let x be a nonzero nonunit of $\overline{R}$ and let $\{Q_i \mid i \in \Gamma\}$ be the set of maximal w_R-ideals of $\overline{R}$ containing x. Then there exists $n \in \mathbb{N}$ with $x^n \in R \bigcap Q_i \in w\text{-}\max(R)$ by Lemma 3.4 (1). But as $w\text{-}\max(R)$ is of finite character, $\{Q_i \bigcap R \mid i \in \Gamma\}$ is a finite set. By [8, Theorem 2.1], $\{Q_i \mid i \in \Gamma\}$ is also a finite set. So $w_R\text{-}\max(\overline{R})$ is of finite character.

($\Leftarrow$) Suppose that $\overline{R}$ is an RKT and $R \subseteq \overline{R}$ is a root extension with R t-linked under $\overline{R}$. Then R is an APvMD by [30, Theorem 3.6]. Let x be a nonzero nonunit of R and let $\{P_i \mid i \in \Gamma\}$ be the set of maximal w-ideals containing x. Then by [10, Theorem 2.1], there exists a unique $Q_i \in \operatorname{Spec}(\overline{R})$ such that $Q_i \bigcap R = P_i$ for each $i \in \Gamma$. By Lemma 3.4 (1), $Q_i \in w_R\text{-}\max(\overline{R})$. Also since R is t-linked under $\overline{R}$, $w = w_R$ over $\overline{R}$ by Lemma 3.9. Hence $Q_i \in w\text{-}\max(\overline{R})$. But as $w\text{-}\max(\overline{R})$ is of finite character, $\{Q_i \mid i \in \Gamma\}$ is a finite set. So by [10, Theorem 2.1], $\{P_i \mid i \in \Gamma\}$ is a finite set. Thus $w\text{-}\max(R)$ is of finite character and R is an almost RKT.

(3) ($\Rightarrow$) Suppose that R is an almost RKT with torsion t-class group. Then by (2), $R \subseteq \overline{R}$ is a root extension with R t-linked under $\overline{R}$ and $\overline{R}$ is an RKT. Also since R is an APvMD with torsion t-class group, R is an AGCD-domain by [30, Theorem 3.8 and 3.9]. Hence $\overline{R}$ is an AGCD-domain by [10, Theorem 5.9]. So the t-class group of $\overline{R}$ is torsion by [10, Theorem 3.4]. Thus $\overline{R}$ is an RKT with torsion t-class group.

($\Leftarrow$) By (2), it follows that R is an almost RKT. Also since $\overline{R}$ is a PvMD with torsion t-class group, $\overline{R}$ is an AGCD-domain. Hence by the hypothesis and [10, Theorem 5.9], R is an AGCD-domain. Thus by [10, Theorem 3.4], R is an almost RKT with torsion t-class group.

(4) This follows from (1), Theorem 3.2 and [10, Theorem 10].

(5) ($\Rightarrow$) Since R is an almost GKD, R is an almost IRKT and a WKD by Theorem 4.3. Hence R is an APvMD by Theorem 3.2. So R is t-linked under $\overline{R}$ by Proposition 3.11. Thus $w = w_R$ over $\overline{R}$ by Lemma 3.9, and so $\overline{R}$ is a WKD by Theorem 3.10. Also since $\overline{R}$ is an IRKT by (1), it follows from [10, Theorem 11] that $\overline{R}$ is a GKD.

($\Leftarrow$) Suppose that $\overline{R}$ is a GKD and $R \subseteq \overline{R}$ is a root extension with R t-linked under $\overline{R}$. Then R is an almost IRKT by (1) and a WKD by Theorem 3.10. So R is an almost GKD by Theorem 4.3.

(6) This follows from (5), Theorem 4.3 and [10, Theorem 11].

(7) This follows from (1), Theorem 5.7, [10, Theorem 13] and [8, Theroem 5.9].

(8) This follows from (6), (7) and Corollary 5.9. $\square$

Recall from [17] that a domain R is called a *w-LPI domain* (or *t-LPI domain*) if every nonzero t-locally principal ideal of R is t-invertible. It was shown in [17, Corollary 2.2] that if a domain R has finite t-character, then R is a w-LPI domain. Conversely, it is natural to ask whether R is of finite character if R is a w-LPI domain. In fact, Zafrullah in [46, Proposition 5] proved that a PvMD R has finite t-character if and only if every nonzero t-locally principal t-ideal is t-invertible. Thus a w-LPI PvMD has finite t-character. Now by [46, Proposition 5] and Theorem 3.12 (2), we have the following result.

Corollary 3.13. *Let R be an APvMD. Then the following statements are equivalent.*

(1) *R is of finite t-character.*
(2) *$\overline{R}$ is of finite t-character.*
(3) *$\overline{R}$ is a w-LPI domain.*
(4) *$\overline{R}$ is a w_R-LPI domain (i.e, each nonzero w_R-locally principal ideal of $\overline{R}$ is w_R-invertible).*

Proof. (1) $\Rightarrow$ (2) Suppose that R is of finite t-character. Then R is an almost RKT. Hence $\overline{R}$ is an RKT by Theorem 3.12 (2). So $\overline{R}$ is of finite t-character.

(2) $\Rightarrow$ (1) Since R is an APvMD by the hypothesis, $\overline{R}$ is a PvMD and $R \subseteq \overline{R}$ is a root extension with R t-linked under $\overline{R}$ by [28, Corollary 2.6]. Hence $\overline{R}$ is an RKT. So R is an almost IRKT by Theorem 3.12 (2). It follows that R is of finite t-character.

(2) $\Leftrightarrow$ (3) Noting that R is a PvMD by [28, Corollary 2.6], the result follows from [46, Proposition 5].

(3) $\Leftrightarrow$ (4) Since R is an APvMD, $w = w_R$ over $\overline{R}$ by Proposition 3.11 and Lemma 3.9. Hence $\overline{R}$ is a w_R-LPI domain if and only if $\overline{R}$ is a w-LPI domain. $\square$

Corollary 3.14. *Let R be an APvMD. If $\overline{R}$ is a w-LPI domain, then R is a w-LPI domain.*

Proof. This follows from Corollary 3.13 and [17, Corollary 2.2]. $\square$

For Corollary 3.14, one might ask whether $\overline{R}$ is a w-LPI domain if R is a w-LPI APvMD. By Corollary 3.13, this question is equivalent to asking whether R is of finite t-character if R is a w-LPI APvMD. Indeed, this question is closely related to [14, Problem 35]. It is asked whether R is of t-character if R is a w-LPI AGCD-domain. Recently, Chang and Hamdi considered [14, Problem 35] on APvMDs. They proved in [15, Theorem 2.4] that if R is an APvMD and each nonzero t-locally finitely generated w-ideal of R is of w-finite type, then R has finite t-character. By [17, Proposition 2.1], it follows that if each nonzero t-locally finitely generated w-ideal of a domain R is of w-finite type, then R is a w-LPI domain. So the condition in [15, Theorem 2.4] is stronger than the w-LPI property. Next we point out that a w-LPI APvMD has a kind of "weak finite t-character", i.e., every nonzero nonunit of a w-LPI APvMD is contained in only finitely many t-potent maximal t-ideals. Let $*$ be a finite character star-operation on a domain R. As in [25], a maximal $*$-ideal P of R is called *$*$-potent* if it contains a $*$-homogeneous ideal and R is called *$*$-potent* if each $P \in *\text{-}\max(R)$ is $*$-potent.

Theorem 3.15. *Let R be a w-LPI APvMD. Then every nonzero nonunit is contained in only finitely many t-potent maximal t-ideals.*

Proof. Let x be a nonzero nonunit of R and let $\{m_\alpha \mid \alpha \in \Gamma\}$ be the set of all t-potent maximal w-ideals containing x. Suppose that Γ is an infinite set. Then for given m_α, there exists a w-homogeneous ideal $A_\alpha := a_1R + \cdots + a_kR$ such that $A_\alpha \subseteq m_\alpha$. Since R_{m_α} is an AV-domain by [30, Theorem 2.3], R_{m_α} is an AB-domain by [8, Theorem 5.6]. Hence by [8, Lemma 4.3] there exists $n_\alpha \in \mathbb{N}$ with $(a_1^{n_\alpha}, \dots, a_k^{n_\alpha})R_{m_\alpha}$ principal. Set $B_\alpha = (a_1^{n_\alpha}, \dots, a_k^{n_\alpha})$ and $B_\alpha R_{m_\alpha} = q_\alpha R_{m_\alpha}$ with $q_\alpha \in B_\alpha R_{m_\alpha}$. Now if $m \in w\text{-}\max(R)$ and $m \neq m_\alpha$, then $A_\alpha \nsubseteq m$ since A_α is a w-homogeneous ideal of R. Take $a \in A_\alpha \setminus m$. Then $a^{kn_\alpha} \in A_\alpha^{kn_\alpha} \subseteq (a_1^{n_\alpha}, \dots, a_k^{n_\alpha}) = B_\alpha$ and $a^{kn_\alpha} \notin m$. Hence $1 = \frac{a^{kn_\alpha}}{a^{kn_\alpha}} \in B_\alpha R_m$. It follows that $B_\alpha R_m = R_m$. Now set $J = \sum_{\alpha\in\Gamma}(R : B_\alpha)$. If $m \in w\text{-}\max(R) \setminus \{m_\alpha \mid \alpha \in \Gamma\}$, then $J_m = \sum_{\alpha\in\Gamma}(R_m : B_\alpha R_m) = \sum_{\alpha\in\Gamma}(R_m : R_m) = R_m$ If $m = m_\alpha$ for some $\alpha \in \Gamma$, then $J_m = \sum_{\alpha\in\Gamma}(R_m : B_\alpha R_m) = (R_{m_\alpha} : B_\alpha R_{m_\alpha}) = (R_{m_\alpha} : q_\alpha R_{m_\alpha}) = \frac{1}{q_\alpha} R_{m_\alpha}$. Thus J is a w-locally principal fractional ideal of R. Noting that $(J_w)_w = J_w$, we have $JR_m = J_w R_m$ for each $m \in w\text{-}\max(R)$. Hence J_w is a w-locally principal w-ideal of R. Since R is a w-LPI domain, J_w is w-invertible. Hence J is w-invertible. So J is of w-finite type. Thus there exists a finitely generated subideal $C = (c_1, \dots, c_s)$ of J such that $J_w = C_w$. So there exists a finite subset Λ of Γ such that $C \subseteq \sum_{\alpha\in\Lambda}(R : B_\alpha) =: J' \subseteq J$. Since $J_w = C_w \subseteq J'_w \subseteq J_w$, $C_w = J'_w = J_w$. Hence $B_m = J'_m = J_m$ for each $m \in w\text{-}\max(R)$. Now take $\beta \in \Gamma \setminus \Lambda$. Then

$$\begin{aligned}(R_{m_\beta} : J_{m_\beta}) &= (R_{m_\beta} : J'_{m_\beta}) \\ &= (R_{m_\beta} : \sum_{\alpha\in\Lambda}(R : B_\alpha)_{m_\beta}) \\ &= (R_{m_\beta} : \sum_{\alpha\in\Lambda}(R_{m_\beta} : B_\alpha R_{m_\beta})) \\ &= (R_{m_\beta} : \sum_{\alpha\in\Lambda}(R_{m_\beta} : R_{m_\beta})) \\ &= R_{m_\beta}\end{aligned}$$

But this implies that $R_{m_\beta} = (R_{m_\beta} : J_{m_\beta}) = q_\beta R_{m_\beta} \subseteq m_\beta R_{m_\beta}$, a contradiction. $\square$

Corollary 3.16. *Let R be a t-potent APvMD. Then the following statements are equivalent.*

(1) *R is of finite t-character.*
(2) *R is a w-LPI domain.*
(3) *$\overline{R}$ is a w-LPI domain.*

Proof. (1) $\Leftrightarrow$ (3) Corollary 3.13.
(1) $\Leftrightarrow$ (2) This follows from Theorem 3.15 and Corollary 3.14. □

Next we give another sufficent condition to deicide when R is t-linked under $\overline{R}$.

Theorem 3.17. *Suppose that $R \subseteq \overline{R}$ is a root extension and $w_R\text{-}\dim \overline{R} = 1$. Then R is a t-h-local domain (resp., a WKD, an almost IRKT, an almost GKD, a t-afg-SH domain) if and only if $\overline{R}$ is a t-h-local domain (resp., a WKD, an IRKT, a GKD, a t-af-SH domain).*

Proof. Since $R \subseteq \overline{R}$ is a t-linked extension by [5, Proposition 2.4] and $w_R\text{-}\dim \overline{R} = 1$, $w\text{-}\dim(R) = w_R\text{-}\max(\overline{R}) = 1$ by Lemma 3.4 (2). By Theorem 3.10 and Theorem 3.12, we only need to prove that R is t-linked under $\overline{R}$. Let J be a nonzero finitely generated ideal of R with $J\overline{R} \in GV(\overline{R})$. If $J_w \neq R$, then there exists $P \in w\text{-}\max(R)$ such that $J_w \subseteq P$. Since $R \subseteq \overline{R}$ is a root extension, there exists $Q \in \operatorname{Spec}(\overline{R})$ such that $Q \bigcap R = P$ by [8, Theroem 2.1]. Hence Q is a w_R-ideal of $\overline{R}$ by [39, Theorem 3.8]. Since $w_R\text{-}\dim \overline{R} = 1$, $\mathrm{ht}Q = 1$. Hence Q is a t-ideal of $\overline{R}$. So Q is a w-ideal of $\overline{R}$. But since $J\overline{R} \subseteq J_w\overline{R} \subseteq P\overline{R} \subseteq Q$, we have $Q = Q_W = \overline{R}$ and hence $P = Q \bigcap R = R$, which is a contradiction. Thus $J_w = R$ and $J \in GV(R)$. So R is t-linked under $\overline{R}$ by Lemma 3.9. □

One might consider when an almost IRKT is integrally closed. For this question, we next give a sufficient condition. Let R be a commutative ring and let M and N be R-modules. Suppose that f is a homomorphism from M to N. As in [37], f is called a *w-monomorphism (resp., a w-epimorphism, a w-isomorphism)* if $f_P : M_P \to N_P$ is a monomorphism (resp., an epimorphism, an isomorphism) for each $P \in w\text{-}\max(R)$. A sequence $A \to B \to C$ is said to be *w-exact* if $A_P \to B_P \to C_P$ is exact for each $P \in w\text{-}\max(R)$. Recall from [26] that an R-module M is said to be *w-flat* if for any w-monomorphism $f : A \to B$, the induced sequence $1 \otimes f : M \otimes_R A \to M \otimes_R B$ is a w-monomorphism. Let R be a domain. As in [42], an overring T of R is called a *w-flat overring* of R if T as an R-module is w-flat. As usual, we denote $\{a \in R \mid ax \in Ry\}$ by $((y) : (x))$.

Theorem 3.18. *Let R be an APvMD. If $\overline{R}$ is a w-flat overring of R, then $R = \overline{R}$. Consequently R is a PvMD.*

Proof. Since R is an APvMD, $R \subseteq \overline{R}$ is a root extension and $\overline{R}$ is a PvMD by [30, Theorem 3.6]. Hence $\overline{R}$ is a t-linked overring by [8, Proposition 2.4]. Let $\frac{x}{y} \in \overline{R}$ with $x \in R$ and $0 \neq y \in R$. Since $\overline{R}$ is a w-flat overring, $(((y) : (x))\overline{R})_{w_R} = \overline{R}$ by [42, Theorem 2.2]. We claim that $((y) : (x))$ can not be contained in any proper prime w-ideal of R. If P is a proper prime w-ideal of R with $((y) : (x)) \subseteq P$, then there exists $Q \in w_R\text{-}\max(\overline{R})$ with $P = Q \bigcap R$ by Lemma 3.4(1). Since $P\overline{R} \subseteq Q\overline{R} = Q$, we have $(P\overline{R})_{w_R} \subseteq Q_{w_R} = Q \neq \overline{R}$. So $(P\overline{R})_{w_R} \neq \overline{R}$. But since $((y) : (x))\overline{R} \subseteq P\overline{R} \subseteq (P\overline{R})_{w_R}$, we have $\overline{R} = (((y) : (x))\overline{R})_{w_R} = (P\overline{R})_{w_R} \neq \overline{R}$.

Which is a contradiction. Thus $((y):(x))_w = R$. Hence there exists $J \in GV(R)$ such that $Jx \subseteq (y)$, and so $x \in (y)_w = (y)$. Hence $\frac{x}{y} \in R$. Consequently, $R = \overline{R}$ is a PvMD. □

Corollary 3.19. *Let R be an almost IRKT (resp., an almost GKD, an almost RKT and a t-afg-SH domain). If $\overline{R}$ as an R-module is w-flat, then R is an IRKT (resp., a GKD, an RKT and a t-af-SH domain).*

Proof. This follows from Theorem 3.18 and Theorem 3.12. □

Next we show that the condition of being a root extension in Theorem 3.5 is essential. Let D be a domain with quotient field K and $T = K + X^2K[X]$. Set $M = X^2K[X]$. Consider the following commutative diagram of rings and homomorphisms:

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\pi"'] & T \arrow[d, "\pi"] \\ D \arrow[r] & K \end{tikzcd} .$$

Then $R = D + X^2K[X]$.

Proposition 3.20. *For the above $D+M$ construction, the following statements hold.*

(1) *T is an h-local DW-domain and hence a t-h-local domain.*
(2) *R is a t-h-local domain if and only if D is a t-local domain.*
(3) *R is an almost IRKT if and only if D is an AV-domain with char$D \neq 0$.*
(4) *$R := D + X^2K[X]$ is a t-almost super-SH domain if and only if R is a t-afg-SH domain (i.e., an almost IRKT and an AGCD-domain (Theorem 5.7)).*
(5) *If D is not an AV-domain, then $D + X^2K[X] \subseteq K + X^2K[X]$ is not a root extension.*

Proof. (1) Since $T = K + X^2K[X]$ is a Gorenstein Dedekind domain [40, Example 2.2], we have $\dim T = 1$. Hence T is an h-local domain. By [40, Proposition 2.10], T is a DW-domain. So T is a t-h-local domain.

(2) ($\Rightarrow$) Let $P_1, P_2 \in t\text{-}\max(D)$. Set $Q_1 := P_1 + M$ and $Q_2 = P_2 + M$. If $P_1 \neq P_2$, then $Q_1, Q_2 \in t\text{-}\max(R)$ and $M \subseteq Q_1 \bigcap Q_2$. Since R is a t-h-local domain, this forces $Q_1 = Q_2$. Hence $P_1 = \pi(Q_1) = \pi(Q_2) = P_2$, which is a contradiction. Thus D has a unique maximal t-ideal.

($\Leftarrow$) By [16, Corollary 11], $R = D + X^2K[X]$ is always of finite t-character. Set

$$A = \{P + M \mid P \text{ is a maximal ideal of } D\} \text{ and}$$

$$B = \{Q \bigcap R \mid Q \in \max(T), Q \text{ is not comparable to } M\}$$

Hence $\mathrm{Max}(R) = A \bigcup B$ by [18, Theorem 1.3]. We first claim that $\mathrm{Max}(R)$ is independent. Since D is a t-local domain, D is a quasilocal domain by [21, Corollary 2.3]. Hence A only contain an element $P+M$, where P is the unique maximal ideal of D. So we only need to consider two cases as follows.

case 1. Let $Q_1 \bigcap R, Q_2 \bigcap R \in B$ and let $P \subseteq (Q_1 \bigcap R) \bigcap (Q_2 \bigcap R)$. Then $P \bigcap D = 0$. Otherwise, if $P \bigcap D \neq 0$, then $T = PT$ and hence $T = PT \subseteq Q_1$ by [18, Lemma 1], which is a contradiction. Since $P \bigcap D = 0$, there exists $Q \in \mathrm{Spec}(T)$

such that $Q \bigcap R = P \subseteq (Q_1 \bigcap R) \bigcap (Q_2 \bigcap R)$ by [18, Theorem 1.3]. By [18, Theorem 1.3] again, we have that $Q \subseteq Q_1 \bigcap Q_2$. Also since T is an h-local DW-domain, this forces $Q = 0$. Hence $P = 0$.

case 2. Let $Q \bigcap R \in B$ and let $P' \subseteq (P + M) \bigcap (Q \bigcap R)$ with $P' \in \mathrm{Spec}(R)$. Then $P' = Q' \bigcap R$ with $Q' \in \mathrm{Spec}(T)$. Hence $Q' \subseteq Q$ by [18, Theorem 1.3]. But $T = K + X^2K[X]$ is a Gorenstein Dedekind domain, and hence $\dim T = 1$. So $Q' = 0$ and $P' = 0$.

Thus $\mathrm{Max}(R)$ is independent. Also since D and T are DW-domains, R is a DW-domain by [32, Theorem 3.1 (3)]. Hence $w\text{-}\max(R) = \max(R)$. So $\mathrm{Max}(R)$ is also of finite character. It follows that R is a h-local domain. Since R is a DW-domain, R is a t-h-local domain.

(3) ($\Rightarrow$) Let R be an almost IRKT. Then R is an APvMD and a t-h-local domain by Theorem 3.2. Hence D is an APvMD with $\mathrm{char} D \neq 0$ by [16, Theorem 2.6], and D is a t-local domain by (2). Since D has a unique maximal t-ideal P, $D = D_P$ is an AV-domain by [30, Theorem 2.3].

($\Leftarrow$) Suppose that D is an AV-domain with $\mathrm{char} D \neq 0$. Then R is an APvMD by [16, Theorem 2.6]. Also since D is t-local, R is t-h-local by (2). Hence R is an almost IRKT by Theorem 3.2.

(4) By (3), D is an AV-domain with $\mathrm{char} D \neq 0$. Hence D is an AGCD-domain with $\mathrm{char} D \neq 0$. So $R = D + X^2K[X]$ is an AGCD-domain by [16, Corollary 2.10]. Thus R is an almost IRKT by Theorem 3.2. The converse is clear by Theorem 5.7.

(5) Since T is an h-local DW-domain, $w = w_R$ on T. Hence T is a w_R-h-local domain. If $D + X^2K[X] \subseteq K + X^2K[X]$ is a root extension, R is a t-h-local domain by Theorem 3.5. Hence D is an AV-domain, which is a contradiction. $\square$

Now we show that Theorem 3.5 does not hold if the condition of being a root extension is deleted.

Example 3.21. As in [16, Example 2.15 (4)], let D_1 be a PvMD with $\mathrm{char} D_1 \neq 0$. Then $D_1[Y]$ is a PvMD that is not an AGCD-domain. Set $D = D_1[Y]$ with quotient field K. Then $R := D + X^2K[X]$ is an APvMD that is not an AGCD-domain. Hence R is not a t-h-local domain. Otherwise, if R is a t-h-local domain, then R is an almost IRKT by Theorem 3.2. Hence D is an AV-domian by Proposition 3.20 (3) and hence an AGCD-domain, which is a contradiction. Also since $T = K + X^2K[X]$ is a t-h-local DW-domain, we have $d = w_R = w$ over T. Hence T is a w_R-h-local domain. Now by [38, Theorem 8.2.19 (1)], it is clear that $R \subseteq T$ is a t-linked extension of domains. If $R \subseteq T$ is a root extension, R is a t-h-local domain by Theorem 3.5, which is a contradiction.

4. $*$-ALMOST GENERALIZED KRULL DOMAINS

In this section we introduce $*$-almost GKDs and we prove that a domain D is a $*$-almost GKD if and only if D is a type 1 $*$-almost super-SH domain.

Definition 4.1. Let $*$ be a finite character star-operation on the domain D. Then D is called a *$*$-almost generalized Krull domain* (*$*$-almost GKD*) if D satisfies the following three conditions:

(1) $D = \bigcap \{D_P \mid P \in X^{(1)}(D)\}$ is locally finite,

(2) $*\text{-}\max(D) = X^{(1)}(D)$, and
(3) D_P is an AV-domain for each $P \in *\text{-}\max(D)$.

Let D be a domain. If $D = \bigcap\{D_P \mid P \in X^{(1)}(D)\}$ is locally finite and for each $P \in X^{(1)}(D)$, D_P is an AV-domain, then D is said to be an *almost generalized Krull domain* (almost GKD). Hence a domain D is a $*$-almost GKD if and only if D is an almost GKD and $*\text{-}\max(D) = X^{(1)}(D)$. Thus a t-almost GKD is the same thing as an almost GKD (see Proposition 4.2), while a d-almost GKD is just a one-dimensional finite character AP-domain.

Let $\mathrm{Ass}(K/D)$ be the set of associated primes of principal ideals of a domain D, $\mathrm{Ass}(K/D) = \{P \in \mathrm{Spec}(D) \mid P$ is minimal over $((a) : (b))$ for some $a, b \in D\}$. Then $\mathrm{Ass}(K/D)$ is a defining family for D by [35, Theorem E(i)]. The function g defined for all $A \in F(D)$ by $A \to \bigcap\{A_P \mid P \in \mathrm{Ass}(K/D)\}$ is a star-operation, which is the so-called *g-operation.* This star-operation is also called the ρ-operation in [45]. In particular, g_s is a star-operation called the *f-operation* in [40]. Recall from [33], a domain D is called a *GW-domain* if $A_g = A_w$ for any $A \in F(D)$. It is shown in [33, Theorem 1.5] that D is a GW-domain if and only if ($t\text{-}\max(D) =$) $w\text{-}\max(D) \subseteq \mathrm{Ass}(K/D)$, if and only if g is of finite character, i.e., $g = f$. Next, we point out that a $*$-almost GKD is a GW-domain.

Proposition 4.2. *Let D be a domain and $*$ a finite character star-operation on D.*

(1) *If $X^{(1)}(D) = *\text{-}\max(D)$, then $X^{(1)}(D) = \mathrm{Ass}(K/D) = t\text{-}\max(D) = w\text{-}\max(D)$.*
(2) *If D is a $*$-almost GKD, then $g = *_w = f = w$.*

Proof. (1) We first prove that $\mathrm{Ass}(K/D) = X^{(1)}(D)$. It is clear that $X^{(1)}(D) \subseteq \mathrm{Ass}(K/D)$. For the reverse inclusion, let $P \in \mathrm{Ass}(K/D)$. Then P is a minimal prime ideal over $\mathrm{ann}_D(\frac{b}{a} + D)$ for some $\frac{b}{a} \in K$, where $K = qf(D)$. Since $\mathrm{ann}_D(\frac{b}{a} + D) = ((a) : (b))$ is a v-ideal of D, P is a t-ideal of D by [24, Proposition 1.1 (5)]. Hence P is a $*_w$-ideal of D. Also, since $X^{(1)}(D) = *\text{-}\max(D)$, it follows from [3, Corollary 2.10] that $*_w$ is induced by $\{D_Q \mid Q \in X^{(1)}(D)\}$. Hence by [1, Theorem 1 (5)], there exists $Q \in X^{(1)}(D)$ such that $P \subseteq Q$. But as $\mathrm{ht}Q = 1$, this forces $P = Q \in X^{(1)}(D)$. Hence $\mathrm{Ass}(K/D) \subseteq X^{(1)}(D)$, and so $X^{(1)}(D) = *\text{-}\max(D) = \mathrm{Ass}(K/D)$. Now by [35, Theorem E(i)], we have $D = \bigcap\{D_P \mid P \in \mathrm{Ass}(K/D) = *\text{-}\max(D)\}$. Thus $g = *_w$ has finite character. On the other hand, since $X^{(1)}(D) = \mathrm{Ass}(K/D)$, we have $\mathrm{Ass}(K/D) = t\text{-}\max D = w\text{-}\max D$ by [25, Lemma 5.1 (5)(d)]. Consequently, $X^{(1)}(D) = \mathrm{Ass}(K/D) = t\text{-}\max(D) = w\text{-}\max(D)$.

(2) By definition we have $X^{(1)}(D) = *\text{-}\max(D)$. Hence $X^{(1)}(D) = \mathrm{Ass}(K/D) = t\text{-}\max(D) = w\text{-}\max(D)$ by (1). So $g = *_w = w = f$ by [33, Theorem 5]. □

Theorem 4.3. *Let D be a domain and $*$ a finite character star-operation on D. Then the following statements are equivalent for D.*

(1) *D is a $*$-almost GKD.*
(2) *D is an AP$*$MD and a $*$-WKD.*
(3) *D is a $*$-potent AP$*$MD with $X^{(1)}(D) = *\text{-}\max(D)$.*
(4) *D is an $*$-almost IRKT and a $*$-WKD.*
(5) *D is an $*$-almost IRKT and every $*$-invertible $*$-homogeneous ideal has type 1.*

(6) *D is a $*$-h-local domain and every $*$-invertible $*$-homogeneous ideal is $*$-almost super-homogeneous and has type 1.*
(7) *D is a type 1 $*$-almost super-SH domain.*
(8) *If A is a finitely generated $*$-invertible ideal of D with $A_* \neq D$, then A_* is a $*$-product of type 1 $*$-almost super-homogeneous ideals.*

Proof. (1) $\Rightarrow$ (2) This follows from the definitions and [27, Theorem 2.4].

(2) $\Rightarrow$ (3) Since D is a $*$-WKD, D is a $*$-h-local domain and $X^{(1)}(D) = *\text{-}\max(D)$. Hence D is a $*$-SH domain by [10, Theorem 4]. Let $P \in *\text{-}\max(D)$. Take $0 \neq x \in P$. Then $(A_1 \cdots A_k)_* = xD$ where each A_i is a $*$-homogeneous ideal of D. Hence $A_1 \cdots A_k \subseteq P$. It follows that there exists some A_j such that $A_j \subseteq P$. Hence P is $*$-potent. So D is a $*$-potent domain.

(3) $\Rightarrow$ (4) Since $X^{(1)}(D) = *\text{-}\max(D)$ by (3), we have $\text{Ass}(K/D) = X^{(1)}(D) = *\text{-}\max(D)$ by Proposition 4.2. Hence $D = \bigcap\{D_P \mid P \in X^{(1)}(D)\}$ by [35, Theorem E(ii)]. Also, since D is a $*$-potent domain with $X^{(1)}(D) = *\text{-}\max(D)$, D has finite $*$-character by [25, Theorem 5.3]. Hence $D = \bigcap\{D_P \mid P \in X^{(1)}(D)\}$ is locally finite. It follows that D is a WKD. So D is a $*$-h-local domain by [10, Theorem 7]. On other hand, since D is an AP$*$MD, D_P is an AV-domain for each $P \in *\text{-}\max(D)$ by [27, Theorem 2.4]. So D is a $*$-almost IRKT.

(4) $\Rightarrow$ (1) Obvious.

(4) $\Rightarrow$ (5) Since D is a $*$-WKD, every $*$-homogeneous ideal of D has type 1 by [10, Theorem 7]. Hence every $*$-invertible $*$-homogeneous ideal of D has type 1.

(5) $\Rightarrow$ (6) This follows from Theorem 3.2.

(7) $\Rightarrow$ (4) This follows from Theorem 3.2 and [10, Theorem 7].

(6) $\Rightarrow$ (8) Since D is $*$-h-local, D is a $*$-SH domain. Hence by [10, Theorem 6] $A_* = (A_1 \cdots A_k)_*$ where each A_i is $*$-homogeneous. Since A is $*$-invertible, each A_i is also $*$-invertible. Hence A_i is $*$-almost super-homogeneous and has type 1 by (6). It follows that A is a $*$-product of type 1 $*$-almost super-homogeneous ideals.

(8) $\Rightarrow$ (7) This is clear. □

Now we give an example to show that (1) an almost GKD is not necessarily a GKD, (2) an almost IRKT is not necessarily an IRKT (i.e., a t-almost super-SH domain is not necessarily a t-super-SH domain), and (3) a t-almost super-homogeneous ideal is not necessarily a t-super-homogeneous ideal.

Example 4.4. Let $\mathbb{F}$ be a field of characteristic 2. Set $D = \mathbb{F}[X^2, X^3] = \mathbb{F} + X^2\mathbb{F}[X]$. Then D is an AGCD-domain by [12, Corollary 3.2], and D is a WKD by [13, Corollary 4.6]. Hence D is a WKD and an APvMD by [30, Theorem 3.1]. So D is an almost GKD (i.e., t-almost GKD) by Theorem 4.3. On the other hand, since D is never integrally closed, D is not a GKD by [25, Theorem 5.9]. So D is an almost GKD, but not a GKD. By Theorem 4.3 and [10, Theorem 11] it follows that D is an almost IRKT (i.e., t-almost IRKT), but not an IRKT. Also, since D is an almost GKD but not a GKD, D is a t-almost super-SH domain but not a t-super-SH domain. Hence there exists a t-almost super-homogeneous ideal of D that is not a t-super-homogeneous ideal.

Further, there exists a non-integrally closed almost IRKT that is neither an almost GKD nor an IRKT. From the following example, it can be seen that (1) a

t-h-local domain is not necessarily a WKD, and (2) a t-almost super-SH domain is necessarily a type 1 t-almost super-SH domain.

Example 4.5. As in [6, Example 3.6], let D be a non-integrally closed AV-domain with quotient field K. Then D is a quasilocal AB-domain by [8, Theorem 5.6] and a DW-domain by [15, Corollary 1.3]. Hence D is a quasilocal APvMD. Set $R = D + XK[[X]]$. Then R is a quasilocal APvMD by [29, Corollary 2.6]. Also since D and $K[[X]]$ are DW-domains, $R = D + XK[[X]]$ is a DW-domain by [32, Theorem 3.1]. Hence R is a t-local APvMD and so R is a t-h-local APvMD. Thus R is an almost IRKT by Theorem 3.2. Since D is not a field, R is not a WKD by [13, Theorem 2.3]. Hence R is not an almost GKD. Also since R is not integrally closed, R is not an IRKT. So R is a non-integrally closed almost IRKT that is neither an almost GKD nor an IRKT.

It is well-known that if D is a GKD, then the polynomial ring $D[X]$ of D is also a GKD. Next, we point out that if D is an almost GKD, then $D[X]$ the polynomial ring $D[X]$ over D is not necessarily an almost GKD.

Example 4.6. Following [6, Example 3.8], let $K \subset L$ be a pair of finite fields. Then $R = K + YL[[Y]]$ is a local API-domain (i.e., for any $\{a_\alpha\}_{\alpha\in\Lambda}$, there exists an $n \in \mathbb{N}$ with $(\{a_\alpha^n\}_{\alpha\in\Lambda})$ principal). Hence R is an APvMD. Also, since $L[[Y]]$ is a DVR and K is a subfield of L, R is a WKD by [13, Theorem 2.3]. So R is an almost GKD by Theorem 4.3. Also, since $R[X] \subseteq \overline{R}[X]$ is not a root extension by [6, Example 3.8], $R[X]$ is not an APvMD by [30, Theorem 3.13]. So $R[X]$ is not a almost GKD by Theorem 4.3.

5. $*$-almost factorial general-SH domains

In this section we introduce $*$-almost factorial general-SH domains. We prove that a domain D is a $*$-almost factorial general-SH domain if and only if D is a $*$-IRKT and an AGCD-domain.

Definition 5.1. Let $*$ be a finite character star-operation on a domain D.

1. (1) An ideal A of D is called *$*$-almost factorial general-homogeneous* (*$*$-afg-homogeneous*) if there exists a maximal $*$-ideal P of D such that A is a $*$-invertible $*$-homogeneous ideal and given $b_1, \dots, b_s \in P$ with $A^r \subseteq (b_1, \dots, b_s)_*$ for some $r \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ with $(b_1^n, \dots, b_s^n)$ principal.
2. (2) A domain D is called a *$*$-almost factorial general-SH domain* (*$*$-afg-SH domain*) if every nonzero proper principal ideal of D is a $*$-product of $*$-afg-homogeneous ideals.

Remark 5.2. In Definition 5.1 (1), the n depends on $b_1, \dots, b_s$.

Proposition 5.3. *Let $*$ be a finite character star-operation on a domain D and A a P-$*$-afg-homogeneous ideal of D. Then the following statements hold.*

1. (1) *A is $*$-almost super-homogeneous.*
2. (2) *$(A^n)_*$ is a principal ideal of D for some $n \in \mathbb{N}$.*
3. (3) *If $(b_1, \dots, b_s)$ is a P-$*$-homogeneous ideal of D, then $A^n \subseteq (b_1^n, \dots, b_s^n)_*$ or $(b_1^n, \dots, b_s^n) \subseteq (A^n)_*$ for some $n \in \mathbb{N}$.*

(4) *If B is a P-$*$-afg-homogeneous ideal of D, then $A^n \subseteq (B^n)_*$ or $B^n \subseteq (A^n)_*$ for some $n \in \mathbb{N}$.*

(5) *If B is a P-$*$-afg-homogeneous ideal of D, then so is AB.*

(6) *A^n is P-$*$-afg-homogeneous for each $n \in \mathbb{N}$.*

Proof. (1) This follows from the definitions.

(2) Let $A = (a_1, \ldots, a_k)$. Then $(a_1^n, \ldots, a_k^n)_*$ is a principal ideal of D for some $n \in \mathbb{N}$. Since A is $*$-invertible, we have $(A^n)_* = (a_1^n, \ldots, a_k^n)_*$ by [27, Lemma 2.2]. Hence $(A^n)_*$ is a principal ideal of D.

(3) Since A is $*$-afg-homogeneous, A is $*$-almost super-homogeneous by (1). Hence $A^n \subseteq (b_1^n, \ldots, b_s^n)_*$ or $(b_1^n, \ldots, b_s^n) \subseteq (A^n)_*$ by Proposition 2.5(1).

(4) Suppose that $B = (b_1, \ldots, b_s)$. Then by (3) $A^n \subseteq (b_1^n, \ldots, b_s^n)_*$ or $(b_1^n, \ldots, b_s^n) \subseteq (A^n)_*$ for some $n \in \mathbb{N}$. Since B is $*$-afg-homogeneous, B is $*$-invertible. Hence, $(B^n)_* = (b_1^n, \ldots, b_s^n)_*$ by [27, Lemma 2.2]. So $A^n \subseteq (B^n)_*$ or $B^n \subseteq (A^n)_*$.

(5) By (4) we have $A^n \subseteq (B^n)_*$ or $B^n \subseteq (A^n)_*$ for some $n \in \mathbb{N}$. Let $C = (c_1, \ldots, c_l)$ be P-$*$-homogeneous with $(AB)^r \subseteq (c_1, \ldots, c_l)_*$ for some $r \in \mathbb{N}$. Then by Lemma 2.3 $C^{nl} \subseteq (c_1^n, \ldots, c_l^n)$. Hence $A^{nrl}B^{nrl} \subseteq (A^{nrl}B^{nrl})_* \subseteq (C^{nl})_* \subseteq (c_1^n, \ldots, c_l^n)_*$. If $A^n \subseteq (B^n)_*$, then $A^{2nrl} \subseteq (c_1^n, \ldots, c_l^n)_*$. Since A is $*$-afg-homogeneous, $(c_1^{mn}, \ldots, c_l^{mn})_*$ is principal for some $m \in \mathbb{N}$. Similarly if $B^n \subseteq (A^n)_*$, then $(c_1^{sn}, \ldots, c_l^{sn})_*$ is principal for some $s \in \mathbb{N}$. Also, it is clear that AB is $*$-invertible and similar to both A and B by [10, Proposition 2]. Hence AB is P-$*$-afg-homogeneous.

(6) This follows from (5). $\square$

Corollary 5.4. *If D is a $*$-afg-SH domain, then D is a $*$-almost super-SH domain.*

Proof. This follows from Proposition 5.3. $\square$

Now by Proposition 5.3 (5) a product of similar $*$-afg-homogeneous ideals is again $*$-afg-homogeneous. Thus the proof of Theorem 2.9 gives the corresponding uniqueness result for $*$-products of $*$-afg-homogeneous ideals.

Theorem 5.5. *Let $*$ be a finite character star-operation on a domain D and let $A_1, \ldots, A_n$ be $*$-afg-homogeneous ideals of D. Then the $*$-product $(A_1 \cdots A_n)_*$ can be expressed uniquely, up to order, as a product of pairwise $*$-comaximal $*$-afg-homogeneous ideals.*

Let $*$ be a finite character star-operation on a domain D. The set $*$-Inv(D) of $*$-invertible fractional $*$-ideals forms a group under the $*$-product $I * J := (IJ)_*$ with subgroup Prin(D), the set of nonzero principal fractional ideals of D. The quotient group $Cl_*(D) := *\text{-Inv}(D)/\text{Prin}(D)$ is called the $*$-*class group* of D in [11]. If $*_1 \leq *_2$ are finite character star-operations on D, then $Cl_{*_1}(D) \subseteq Cl_{*_2}(D)$. Let $*$ be a finite character star-operation on a domain D. Then D is called a $*$-*almost Bézout domain* in [10] if for $0 \neq a, b \in D$, there exists an $n = n(a,b) \in \mathbb{N}$ with $(a^n, b^n)_*$ principal. It follows from [27, Theorem 3.4] that a domain D is a $*$-almost Bézout domain if and only if D is an AP$*$MD with $Cl_*(D)$ torsion. If $*_1 \leq *_2$ are finite character star-operations on D, then D $*_1$-almost Bézout implies D is $*_2$-almost Bézout. Next we characterize $*$-afg-SH domains. We need the following lemma.

Lemma 5.6. *Let $*$ be a finite character star-operation on a domain D. If D is a $*$-almost IRKT, then D is an AGCD-domain if and only if D is a $*$-almost Bézout domain.*

Proof. ($\Rightarrow$) Let D be an AGCD-domain. Then $Cl_t(D)$ is torsion by [30, Theorem 3.1]. Since $Cl_*(D) \subseteq Cl_t(D)$, it follows that $Cl_*(D)$ is torsion. Also since D is a $*$-almost IRKT, D is a AP$*$MD. Hence D is a $*$-almost Bézout domain by [27, Theorem 3.4].

($\Leftarrow$) Let D be a $*$-almost Bézout domain. Then D is a t-almost Bézout domain and hence D is an AGCD-domain. □

Theorem 5.7. *Let D be a domain and $*$ a finite character star-operation on D. The following statements are equivalent.*

1. *D is a $*$-afg-SH domain.*
2. *D is a $*$-almost IRKT and an AGCD-domain.*
3. *D is a $*$-almost IRKT with $Cl_*(D)$ torsion.*
4. *D is a $*$-h-local domain and each $*$-invertible $*$-homogeneous ideal is $*$-afg-homogeneous.*

Proof. (1) $\Rightarrow$ (2) Since D is a $*$-afg-SH domain, D is a $*$-almost super-SH domain by Corollary 5.4. Hence D is a $*$-almost IRKT by Theorem 3.2. We only need to show that D is an AGCD-domain. Let c be nonzero nonunit of D. Then $cD = (C_1 \cdots C_l)_*$, where the C_i are mutually $*$-comaximal and $*$-afg-homogeneous. Hence by Proposition 5.3 (2), $(C_i^{n_i})_* = c_i'D$ for some $n_i \in \mathbb{N}$ and $c_i' \in D$ ($i = 1, \dots, l$). Set $n = \prod_{i=1}^{l} n_i$ and $c_i = (c_i')^{n/n_i}$. Then $(C_i^n)_* = c_iD$ is $M(C_i)$-$*$-homogeneous. Hence $c^nD = (c_1D \cdots c_lD)_* = c_1D \cdots c_lD$, and the c_iD are mutually $*$-comaximal. So $c^nD = c_1D \cdots c_lD = c_1D \bigcap \cdots \bigcap c_lD$. Let a and b be nonzero nonunits of D. Let $P_1, \dots, P_s$ be the maximal $*$-ideal containing a or b. Then for suitable m $a^mD = a_1D \bigcap \cdots \bigcap a_sD = a_1D \cdots a_sD$ and $b^mD = b_1D \bigcap \cdots \bigcap b_sD = b_1D \cdots b_sD$ where either a_iD (resp., b_iD) is P_i-$*$-homogeneous or $a_i = 1$ (resp., $b_i = 1$). Thus $a_i^{m_i}D \subseteq b_i^{m_i}D$ or $a_i^{m_i}D \supseteq b_i^{m_i}D$ for some $m_i \in \mathbb{N}$ ($i = 1, \dots, s$) by Proposition 5.3(4). It follows that $a_i^{m_i}D \bigcap b_i^{m_i}D$ is principal and P_i-homogeneous ($i = 1, \dots, s$). Set $n = \prod_i m_i$ and $n_i = \prod_{j \neq i} m_j$. Then $a^{mn}D = (a_1^{m_1}D)^{n_1} \bigcap \cdots \bigcap (a_s^{m_s}D)^{n_s}$ and $b^{mn}D = (b_1^{m_1}D)^{n_1} \bigcap \cdots \bigcap (b_s^{m_s}D)^{n_s}$. Hence $a^{mn}D \bigcap b^{mn}D = (a_1^{m_1n_1}D \bigcap b_1^{m_1n_1}D) \bigcap \cdots \bigcap (a_s^{m_sn_s}D) \bigcap (b_s^{m_sn_s}D)$ and $(a_i^{m_i}D)^{n_i} \bigcap (b_i^{m_i}D)^{n_i}$ is principal and P_i-$*$-homogeneous ($i = 1, \dots, r$). So $a^{mn}D \bigcap b^{mn}D$ is a product of principal ideals and hence is a principal ideal. Consequently, D is an AGCD-domain.

(2) $\Rightarrow$ (3) Since D is an AGCD-domain and a $*$-almost IRKT by (2), D is a $*$-almost Bézout domain by Lemma 5.6. Hence $Cl_*(D)$ is torsion by [30, Theorem 3.1].

(3) $\Rightarrow$ (4) Since D is a $*$-almost IRKT, it is clear that D is a $*$-h-local domain. Now suppose that A is a $*$-invertible and P-$*$-homogeneous ideal of D. Then A is $*$-almost super-homogeneous by Theorem 3.2. Let $(b_1, \dots, b_s)$ be P-$*$-homogeneous with $A^r \subseteq (b_1, \dots, b_s)_*$ for some $r \in \mathbb{N}$. Then $(b_1^n, \dots, b_s^n)$ is $*$-invertible for some $n \in \mathbb{N}$. Hence $((b_1^n, \cdots, b_s^n)^m)_*$ is principal for some $m \in \mathbb{N}$ because $Cl_*(D)$ is torsion. Since $(b_1^n, \dots, b_s^n)$ is $*$-invertible, $((b_1^n, \dots, b_s^n)^m)_* = (b_1^{mn}, \dots, b_s^{mn})_*$ by Lemma 2.4. It follows that $(b_1^{mn}, \dots, b_s^{mn})_*$ is principal. So A is $*$-afg-homogeneous.

(4) $\Rightarrow$ (1) Clear. □

Recall from [10] that a $*$-homogeneous ideal A of a domain D is called *$*$-almost factorial-homogeneous* (*$*$-af-homogeneous*) if for each $*$-homogeneous ideal $B \supseteq A$, there exists some $n \in \mathbb{N}$ with $(B^n)_*$ principal. The domain D is a *$*$-af-SH domain* in [10] if for each nonzero nonunit $x \in D$, xD is expressible as a $*$-product of finitely many $*$-af-homogeneous ideals. Next we point out that every $*$-af-SH domain is a $*$-afg-SH domain.

Corollary 5.8. *Every $*$-af-SH domain is a $*$-afg-SH domain.*

Proof. Let D be a $*$-af-SH domain. Then D is a $*$-IRKT and an AGCD-domain by [10, Theorem 13]. Hence D is a $*$-almost IRKT and an AGCD-domain. It follows from Theorem 5.7 that D is a $*$-afg-SH domain. □

Corollary 5.9. *Let D be a domain and $*$ a finite character star-operation on D. The following statements are equivalent.*

(1) *D is a $*$-afg-SH domain of type 1.*
(2) *D is a $*$-almost GKD and an AGCD-domain.*
(3) *D is a $*$-h-local domain and each $*$-invertible $*$-homogeneous ideal is a $*$-afg-homogeneous ideal of type 1.*
(4) *If A is a finitely generated $*$-invertible ideal of D with $A_* \neq D$, then A_* is a $*$-product of $*$-afg-homogeneous ideals of type 1.*
(5) *D is a $*$-almost GKD with $Cl_*(D)$ torsion.*

Proof. (1) ⇒ (2) Since D is a $*$-afg-SH domain of type 1, D is a $*$-almost super-SH domain of type 1 by Corollary 5.4. Hence D is a $*$-almost GKD by Theorem 4.3. Also, since D is a $*$-afg-SH domain, D is an AGCD-domain by Theorem 5.7. So D is a $*$-almost GKD and an AGCD-domain.

(2) ⇒ (3) Since D is a $*$-almost GKD and an AGCD-domain, D is a $*$-almost IRKT and an AGCD-domain. Hence by Theorem 5.7 D is a $*$-h-local domain and each $*$-invertible $*$-homogeneous ideal is $*$-afg-homogeneous. By Theorem 4.3 it follows that each $*$-invertible $*$-homogeneous ideal has type 1. So D is a $*$-SH domain and each $*$-invertible $*$-homogeneous ideal is a $*$-afg-homogeneous ideal of type 1.

(3) ⇒ (4) Suppose that A is a finitely generated $*$-invertible ideal of D with $A_* \neq D$. Then by [10, Theorem 6] $A = (A_1 \cdots A_k)_*$ where each A_i is $*$-homogeneous. Since A is $*$-invertible, each A_i is $*$-invertible. Hence each A_i is $*$-afg-homogeneous ideals of type 1.

(4) ⇒ (1) Clear.

(2) ⇔ (5) This follows from Theorem 5.7. □

Corollary 5.10. *Let D be a domain and $*$ a finite character star-operation on D. The following statements are equivalent.*

(1) *D is a $*$-afg-SH domain of type 2.*
(2) *D is a $*$-Krull domain and an AGCD-domain.*
(3) *If A is a finitely generated $*$-invertible ideal of D with $A_* \neq D$, then A_* is a $*$-product of $*$-afg-homogeneous ideals of type 2.*
(4) *D is a $*$-Krull domain with $Cl_*(D)$ torsion.*
(5) *D is a $*$-af-SH domain.*

Proof. (1) $\Rightarrow$ (5) Trivial.

(5) $\Leftrightarrow$ (4) $\Leftrightarrow$ (2) [10, Theorem 15].

(2) $\Rightarrow$ (3) Suppose that A is a finitely generated $*$-invertible ideal of D with $A_* \neq D$. Then $A_* = (A_1 \cdots A_k)_*$ by [10, Theorem 8], where each A_i is a $*$-invertible $*$-homogeneous ideal of type 2 . Also since D is a $*$-almost IRKT and an AGCD-domain, each A_i is $*$-afg-homogeneous. Hence A_* is a $*$-product of $*$-afg-homogeneous ideals of type 2.

(3) $\Rightarrow$ (1) Clear. $\square$

Next, we give an example to show that (1) a t-afg-SH domain is not necessarily a t-af-SH domain, and (2) a t-afg-homogeneous ideal is not necessarily a t-af-homogeneous ideal.

Example 5.11. Let K be a field of characteristic $p > 0$ and let $K \subset L$ be a purely inseparable field extension. Set $D = K + XL[X]$. Then D is a WKD by [13, Corollary 3.11]. Hence D is a t-h-local domain. Since D is an AB-domain by [8, Example 4.14], D_P is an AV-domain for each $P \in t\text{-}\max(D)$ by [8, Theorem 5.8]. It follows that D is an almost IRKT. Thus D is a t-afg-SH domain by Theorem 5.7. Also since the integral closure of D is precisely $L[X]$, it follows that D is not integrally closed. Hence D is not an IRKT. So D is not a t-af-SH domain by [10, Theorem 13]. Thus, there exists a t-afg-homogeneous ideal in D that is not a t-af-homogeneous ideal.

Further, there exists a non-integrally closed t-afg-SH domain that is not an almost GKD. By the following example, it is seen that a t-afg-SH domain is necessarily a t-afg-SH domain of type 1.

Example 5.12. Let D be a non-integrally closed AV-domain with quotient field K. Then D is a t-local AGCD-domain by [8, Theorem 5.6]. Let $R = D + XK[X]$. Then R is an AGCD-domain by [4, Corollary 3.13] and $\overline{R} = \overline{D} + XK[X]$. Hence R is a non-integrally closed APvMD. Also since D is t-local, R is of finite t-character by [2, Corollary 17]. Set $A := \{Q \bigcap R \mid 0 \neq Q \in \operatorname{Spec}(K[X]) \text{ and } Q \neq XK[X]\}$ and $B := \{P + XK[X] \mid P \in t\text{-}\max(D)\}$. Then $t\text{-}\max(R) = A \bigcup B$ by [25, Lemma 4.1 (3)]. Since D is an AV-domain, $B = \{m + XK[X]\}$ where m is the unique maximal t-ideal of D. Now let $P = Q \bigcap R \in A \subseteq t\text{-}\max(R)$. Then P is not comparable to $XK[X]$. Hence by [18, Lemma 1.2 and 1.5], P is of the form $(1 + Xg(X))R$ and $\operatorname{ht}(1 + Xg(X))R = 1$. Hence no two maximal t-ideals of R contain a nonzero prime ideal. This makes R t-h-local. So R is a non-integrally closed t-afg-SH domain by Theorem 5.7. On other hand, since D is not a field, D is not a WKD. Hence D is not an almost GKD. So D is not a t-afg-SH domain of type 1 by Theorem 5.9.

Finally, we use the following diagram to summarize the relationships between the domains introduced in this paper.

$$\begin{tikzcd}
AGCD\text{-}domain & \text{type 1 t-SH domain} \arrow[r] \arrow[d, equal] & \text{t-h-local domain} \arrow[d, equal] \\
\text{t-afg-SH domain} \arrow[u] \arrow[drr] & WKD \arrow[r] & \text{t-SH domain} \\
\text{type 1 t-afg-SH domain} \arrow[u] \arrow[r] \arrow[ur] \arrow[dr] & \text{type 1 t-almost super-SH domain} \arrow[u] \arrow[r] \arrow[d, equal] & \text{t-almost super-SH domain} \arrow[u] \arrow[d, equal] \\
\text{type 2 t-afg-SH domain} \arrow[u] \arrow[r] \arrow[d, equal] & \textit{almost } GKD \arrow[r] & \text{almost } IRKT \\
\text{type 2 t-af-SH domain} \arrow[r] \arrow[d] & GKD \arrow[u] \arrow[r] \arrow[d, equal] & IRKT \arrow[u] \arrow[d, equal] \\
\text{type 2 t-super-SH domain} \arrow[r] \arrow[d, equal] & \text{type 1 t-super-SH domain} \arrow[r] & \text{t-super-SH domain} \\
\text{Krull} \arrow[r, equal] & \text{type 2 t-SH domain} \arrow[u] \arrow[r, equal] & \text{type 2 t-almost super-SH domain} \arrow[u]
\end{tikzcd}$$

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(Xing) College of Applied Mathematics, Chengdu University of Information Technology, Chengdu, Sichuan 610225, China

E-mail address: sqxing@yeah.net

(Anderson) Department of Mathematics, The University of Iowa, Iowa City, IA 52242-1419, USA

E-mail address: dan-anderson@uiowa.edu

(Zafrullah) Department of Mathematics, Idaho State University, Pocatello, ID 83209-8085, USA

E-mail address: mzafrullah@usa.net