Almost Bezout Domains, III

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Abstract. An integral domain R is an almost Bezout domain (respectively, almost valuation domain) if for each pair $a, b \in R\backslash\{0\}$, there is a positive integer $n = n(a, b)$ such that (a^n, b^n) is principal (respectively, $a^n \mid b^n$ or $b^n \mid a^n$). We show that a finite intersection of almost valuation domains with the same quotient field is an almost Bezout domain. This generalizes the result that a finite intersection of valuation domains with the same quotient field is a Bezout domain. We use our work to give a new characterization of Cohen-Kaplansky domains.

Let R be an integral domain with quotient field K. Call R an *almost Bezout (AB-) domain* if for each pair $a, b \in R\backslash\{0\}$ there is a positive integer $n = n(a, b)$ such that the ideal (a^n, b^n) is principal. Also, call R an *almost valuation (AV-) domain* if for each pair $a, b \in R\backslash\{0\}$ there is a positive integer $n = n(a, b)$ such that $a^n \mid b^n$ or $b^n \mid a^n$. It is easy to see that an AV-domain is a quasi-local AB-domain and it is known that the integral closure of an AV-domain (respectively, AB-domain) is a valuation domain (respectively, a Prüfer domain with torsion class group) [**5**]. Dedekind domains with torsion class groups are good examples of integrally closed AB-domains. As was shown in [**5**], the theory of almost Bezout domains runs along lines somewhat similar to that of Bezout domains (i.e., every two generated, or equivalently, every finitely generated, ideal is principal). To establish this similarity still further we show that if R is a domain with $R = R_1 \cap \cdots \cap R_n$ where $R_1, ..., R_n$ are AV-domains between R and K, then R is a semi-quasi-local AB-domain. This result is an analogue of the well known result that if R is a domain with $R = V_1 \cap \cdots \cap V_n$ where $V_1, ..., V_n$ are valuation domains between R and K, then R is a semi-quasi-local Bezout domain [**10**, Theorem 107]. Call R *atomic* if every nonzero nonunit x of R is expressible as a finite product of irreducible elements (atoms). In [**7**], I. Cohen and I. Kaplansky studied atomic integral domains with only a finite number of nonassociate atoms. These domains were later called CK-domains in [**3**] . As an application of the above result we show that R is a CK-domain if and only if R is a finite intersection of local CK-domains with the same quotient field.

Let us start with a brief introduction to AB-domains. The second author [**12**] called an integral domain R an *almost GCD-domain (AGCD-domain)* if for each pair $a, b \in R\backslash\{0\}$ there is a positive integer $n = n(a, b)$ such that $a^n R \cap b^n R$ (or equivalently, $(a^n, b^n)_v = R : (R : (a^n, b^n))$ is principal. The main purpose of [**12**]

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was to introduce a theory of almost factoriality which generalized the work of Störch [**11**] on almost factorial domains (fastfaktorielle ringe). It was shown in [**12**], among other things, that if R is an AGCD-domain with integral closure $\overline{R}$, then $\overline{R}$ is an AGCD-domain and for each $x \in \overline{R}$ there is a positive integer $n = n(x)$ such that $x^n \in R$. In the terminology of [**5**] if $R \subseteq S$ is an extension of domains such that for each $s \in S$ there is an $n = n(s) \geq 1$ with $s^n \in R$, then S is a *root extension of R*. In addition to the introduction of AB-domains and several other notions AGCD-domains were studied more thoroughly in [**5**]. One of the results that we shall need is a version of [**5**, Theorem 4.6] given below. Here by an *overring of R* we mean a ring S with $R \subseteq S \subseteq K$.

THEOREM 1. *Let R be an integral domain and S an overring of R with $R \subseteq S \subseteq \overline{R}$. Then R is an AB-domain if and only if S is an AB-domain and $\overline{R}$ is a root extension of R.*

Let us start putting to use the information that we have gathered.

LEMMA 1. *Let R be a domain with $R = R_1 \cap \cdots \cap R_n$ where $R_1, ..., R_n$ are AGCD-domains between R and its quotient field K. Then $\overline{R} = \overline{R}_1 \cap \cdots \cap \overline{R}_n$ and $\overline{R}$ is a root extension of R.*

PROOF. Let $D = \overline{R}_1 \cap \cdots \cap \overline{R}_n$. We show that $D = \overline{R}$. For this first note that as D is integrally closed, being an intersection of integrally closed domains, and as $R \subseteq D$ we have $\overline{R} \subseteq D$. For the reverse containment, let $x \in D$. Then since $x \in \overline{R}_1 \cap \cdots \cap \overline{R}_n$ we have $x \in \overline{R}_i$ for $i = 1, ..., n$ and so there exist n_i such that $x^{n_i} \in R_i$ for each i. Now let $n = \prod n_i$. Then $x^n = (x^{n_i})^{n/n_i} \in R_i$ for each i, whence there is an $n = n(x) = \prod n_i$ such that $x^n \in R_1 \cap \cdots \cap R_n = R$. So x is integral over R and hence in $\overline{R}$. This gives $D \subseteq \overline{R}$ and consequently $D = \overline{R}$. The above proof also establishes that $\overline{R}$ is a root extension of R. □

More generally, if $R = \cap R_\alpha$ is a locally finite intersection of overrings with each $R_\alpha \subseteq \overline{R}_\alpha$ a root extension (e.g., each R_α is an AGCD-domain), then $\overline{R} = \cap \overline{R}_\alpha$.

Must the ring R in Lemma 1 be an AGCD-domain? The answer is "not in general". To see this let us call, as in [**1**], R *locally factorial* if R_x is a factorial domain for each nonunit $x \in R \backslash \{0\}$. Here R_x denotes the ring of fractions R_S where $S = \{x^i | i \text{ a nonnegative integer}\}$. Fossum [**8**, page 80] establishes the existence of locally factorial Dedekind domains which are not PID's. So let R be a locally factorial Dedekind domain, with $Cl(R) = \mathbb{Z}$, the group of integers, that is not a PID, see e.g. [**8**, Example 15.21]. Obviously R is not a DVR. So there exist nonunits $x, y \in R$ such that $(x, y) = R$. This implies that $R = R_x \cap R_y$ by [**1**, Corollary 2.2]. So R is a Dedekind domain with $Cl(R) = \mathbb{Z}$ not torsion and so R cannot be an AB-domain. But as both R_x and R_y are PID's we conclude that a finite intersection of AB-domains may not be an AB-domain. Since AB-domains are a special case of AGCD-domains we have the conclusion. Let us now prepare to prove another result that will be useful in the proof of our main theorem. For this we need to recall the following result from [**5**].

THEOREM 2. *([**5**, Theorem 2.1 and Corollary 2.2]) Suppose that $R \subseteq S$ is a root extension of commutative rings. The map $\theta : Spec(S) \to Spec(R)$ given by $\theta(Q) = Q \cap R$ is an order isomorphism and a homeomorphism. The inverse of θ is given by $\theta^{-1}(P) = \{s \in S | s^n \in P \text{ for some } n \geq 1\}$. Consequently $Spec(R)$ is treed if and only if $Spec(S)$ is treed.*

LEMMA 2. *Let D be a domain with quotient field K. If D has overrings $R_1, ..., R_n$ which are AV-domains with the $\overline{R}_i$ mutually comparable, then $R = R_1 \cap \cdots \cap R_n$ is an AV-domain.*

PROOF. By Lemma 1, $\overline{R} = \overline{R}_1 \cap \cdots \cap \overline{R}_n = \overline{R}_k$ for some k where $\overline{R}_k$ is a valuation domain and $\overline{R}_k$ is a root extension of R. So, by [**5**, Theorem 5.6], R is an AV-domain. □

THEOREM 3. *Let D be an integral domain with quotient field K and let $R_1, ..., R_n$ be a finite set of AV-overrings of R. Then $R = R_1 \cap \cdots \cap R_n$ is a semi-quasi-local almost Bezout domain with at most n maximal ideals. Moreover, if the $\overline{R}_i$ are mutually incomparable and if Q_i is the maximal ideal of $\overline{R}_i$, then $R = \bigcap R_{P_i}$ where $P_i = Q_i \cap R$.*

PROOF. By Lemma 1, $\overline{R} = \overline{R}_1 \cap \cdots \cap \overline{R}_n$ where each of $\overline{R}_i$ is a valuation domain, because each R_i is an almost valuation domain and of course $\overline{R}$ is a root extension of R. So, by Theorem 1, R is an AB-domain. Since $\overline{R}$ is a semi-quasi-local Bezout domain [**10**, Theorem 107], R is a semi-quasi-local AB-domain. Next, by using Lemma 2, we can reduce $R = R_1 \cap \cdots \cap R_n$ to $R = S_1 \cap \cdots \cap S_m$ where the S_i are AB-domains with the $\overline{S}_i$ mutually incomparable valuation domains. So $\overline{R} = \overline{S}_1 \cap \cdots \cap \overline{S}_m$ is an intersection of mutually incomparable valuation domains and according to [**10**, Theorem 107] has precisely m maximal ideals where obviously $m \leq n$. Now if the $\overline{R}_i$ are mutually incomparable, then by [**10**, Theorem 107] $\overline{R}_i = \overline{R}_{Q'_i}$, after re-ordering, where $Q'_1, ..., Q'_n$ are the maximal ideals of $\overline{R}$ (and hence $Q'_i\overline{R}_{Q'_i} = Q_i$, the maximal ideal of $\overline{R}_i$) and so the number of maximal ideals of $\overline{R}$ is n. By Theorem 2, the number of distinct maximal ideals of R is n as well and each of them is given by $P_i = Q'_i \cap R = Q_i \cap R$. Hence $R = R_{P_1} \cap \cdots \cap R_{P_n}$. □

REMARK 1. *We were was unable to prove in the above theorem that $R_i = R_{P_i}$. Later in Theorem 5, it is shown however, that if $htP_i = 1$, $R_i = R_{P_i}$.*

An integral domain R is called *atomic* if every nonzero nonunit x of R is expressible as a finite product of irreducible elements (atoms). In [**7**] I. Cohen and I. Kaplansky studied atomic integral domains with only a finite number of nonassociate atoms. (Some care must be taken since they called irreducible elements "primes".) Their work was revived in [**3**] by the first author and Mott where they called the rings studied in [**7**] *Cohen-Kaplansky (CK-) domains*. They included a full treatment of the rings studied in [**7**] and added a lot more. For instance they showed in [**3**, Theorem 4.3] that R is a CK-domain if and only if R is a Noetherian AB-domain with $G(R)$, the group of divisibility of R, finitely generated. To facilitate the reading of the following material we recall some terminology and basic information.

(i) Call an element x of R a *primary element* if xR is a primary ideal. Call R a *weakly factorial domain (WFD)* if each nonzero nonunit of R is expressible as a finite product of primary elements. WFD's were discussed in [**2**].

(ii) Denote by $X^{(1)}(R)$ the set of height-one prime ideals of R, call an ideal generated by an atom an *atomic ideal*, and call R a *weakly Krull domain (WKD)* if $R = \bigcap_{P \in X^{(1)}(R)} R_P$ where the intersection is locally finite. It is well known that if $|X^{(1)}(R)| < \infty$, then each member of $X^{(1)}(R)$ is a maximal ideal [**10**, Theorem 105]. It was shown in [**2**] that a WFD is a WKD. A WKD is a WFD if and only if $xR_P \cap R$ is principal for every $P \in X^{(1)}(R)$ and for every $x \in P$ [**4**, (6) of Theorem]. So in a weakly factorial domain for every nonzero nonunit x we have $xR = (xR_{P_1} \cap R) \cap \cdots \cap (xR_{P_n} \cap R)$ where $\{P_1, ..., P_n\}$ is the set of all the height-one primes containing x. Of course $x_iR = xR_{P_i} \cap R$ is a primary ideal, and as shown in the proof of [**4**, Theorem] $x = ux_1 \cdots x_n$ where u is a unit and $x_iR \cap x_jR = x_ix_jR$ for $i \neq j$. Let us call elements x_i *mutually v-coprime* if $x_iR \cap x_jR = x_ix_jR$ for $i \neq j$. We see that if R is a WFD, then every nonzero nonunit x of R can be expressed as a product of mutually v-coprime primary elements. This discussion facilitates the following theorem.

THEOREM 4. *Let R be a weakly factorial domain and let $P \in X^{(1)}(R)$. Let $A(Y)$ denote the set of atomic ideals in Y. Then $|A(P)| = |A(PR_P)|$ and for any prime ideal $Q \in X^{(1)}(R)$ with $Q \neq P$, we have $A(P) \cap A(Q) = \phi$.*

PROOF. Let x be an atom of R. Since x is a product of primary elements, x is P-primary. This ensures that x and hence xR cannot be in any $Q \in X^{(1)}(R) \backslash \{P\}$. Next let $x \in P$, x is an atom in R, and suppose that $xR_P = yzR_P$ where y, z are both nonunits in R_P. Now y and z can both be assumed to be in P. Next let $yR_P \cap R = y_1R$ and let $zR_P \cap R = z_1R$ where both y_1R, z_1R are P-primary and so y_1z_1R is P-primary. This forces $y_1z_1R_P \cap R = y_1z_1R$. But as $y_1z_1R_P = yzR_P$ we have $xR = xR_P \cap R = yzR_P \cap R = y_1z_1R_P \cap R = y_1z_1R$ where both y_1, z_1 are nonunits, a contradiction. Thus if xR is an atomic ideal in P, then xR_P is an atomic ideal in R_P. Next let α be an atom in R_P. Then there is $b \in P$ such that $\alpha R_P = bR_P$. Now as $bR_P \cap R$ is principal, we have $\alpha R_P \cap R = cR$. If $c = rs$ where both r, s are nonunits, we must have both $r, s \in P$ since cR is P-primary. But then $\alpha R_P = (\alpha R_P \cap R)R_P = cR_P = rsR_P$, a contradiction. Thus each atomic ideal of R_P contracts to an atomic ideal in P. □

COROLLARY 1. *Let $A(Y)$ denote the set of all atomic ideals in Y. If R is a weakly factorial domain, then $A(R) = \bigcup_{P \in X^{(1)}(R)} A(P)$.*

(iii) A CK-domain R is a one dimensional semi-local domain such that R_P is a CK-domain for each nonzero prime ideal P [**3**, Theorem 2.1]. Thus a CK-domain is a finite intersection of local CK-domains and so is weakly Krull. If R is CK, then $Pic(R) = 0$, and hence by [**2**, Theorem 12] a CK-domain is weakly factorial.

THEOREM 5. *An integral domain R is a CK-domain if and only if R is an intersection of a finite number of local CK-overrings.*

PROOF. If R is a CK-domain with maximal ideals $P_1, \ldots, P_n$, then $R = \bigcap_{i=1}^{n} R_{P_i}$ where each R_{P_i} is a local CK-domain. This follows from (iii) above. Conversely, suppose that R is an intersection of local CK-domains $R_1, ..., R_n$, that is, $R = R_1 \cap \cdots \cap R_n$. Let M_i be the maximal ideal of R_i. Because the intersection of two CK-domains with the same integral closure is a CK-domain [**6**, Proposition 2.3], we can assume that the $\overline{R}_i$ are mutually incomparable discrete rank-one valuation domains with maximal ideals Q_i. So, as in the proof of Theorem 3, R has distinct maximal ideals $P_1, ..., P_n$ each of height-one where $P_i = Q_i \cap R$. This gives $R = \bigcap_{i=1}^{n} R_{P_i}$. To be able to ascertain that R is indeed a CK-domain we need to show that for each i, $R_i = R_{P_i}$. We do it for $i = 1$ and leave the rest to the reader. Let $S = R \backslash P_1$. Since the P_i are mutually incomparable, $P_i \cap S \neq \phi$ and hence $M_i \cap S \neq \phi$ for each $i > 1$. This makes $(R_i)_S = K$ for each $i > 1$. Now according to [**9**, Proposition 43.5] and the above observations, $R_{P_1} = (R_1)_S \cap \cdots \cap (R_n)_S = (R_1)_S$. And as S is contained in the set of units of R_1; we conclude that $(R_1)_S = R_1$. Thus $R_1 = R_{P_1}$is a CK-domain. So each R_{P_i} is a CK-domain. This makes R a Noetherian weakly factorial domain. Next as each of R_{P_i} has finitely many atomic ideals, each of P_i has finitely many atomic ideals by Theorem 4 and so, by Corollary 1, $|A(R)| = |\cup A(P_i)|$ is finite. Thus R is a CK-domain. □

Example 1. *Let K be a finite field with char$K = p$, $f_1, \cdots, f_n \in K[X]$ nonassociate irreducible polynomials, and $m_1, \cdots, m_n$ positive integers. For each i, let $T_i = K[X]/(f_i^{m_i})$; so T_i is a finite local ring, and let S_i be a subring of T_i. Define $R_i = \pi_i^{-1}(S_i)$ where $\pi_i : K[X]_{(f_i)} \longrightarrow K[X]_{(f_i)}/(f_i^{m_i})_{(f_i)} \approx T_i$ is the natural map. By* [**3**, Theorem 4.4]*, R_i is a local CK-domain with $\overline{R}_i = K[X]_{(f_i)}$. Let $R = R_1 \cap \cdots \cap R_n$. Note that $\mathbb{Z}_p + f_i^{m_i} K[X] \subseteq R_i$ and hence $\mathbb{Z}_p + f_1^{m_1} \cdots f_n^{m_n} K[X] \subseteq R_1 \cap \cdots \cap R_n = R$. Thus R has quotient field $K(X)$. By Theorem 5, R is a CK-domain with $\bar{R} = \bar{R}_1 \cap \cdots \cap \bar{R}_n = K[X]_{(f_1)} \cap \cdots \cap K[X]_{(f_n)}$. Note that in this case $R = \pi^{-1}(S_1 \times \cdots \times S_n)$ where $\pi : K[X]_{(f_1)} \cap \cdots \cap K[X]_{(f_n)} \to K[X]_{(f_1)} \cap \cdots \cap K[X]_{(f_n)}/f_1^{m_1} \cdots f_n^{m_n} \left(K[X]_{(f_1)} \cap \cdots \cap K[X]_{(f_n)}\right) \approx (K[X]/f_1^{m_1}K[X]) \times \cdots \times (K[X]/f_n^{m_n}K[X]) = T_1 \times \cdots \times T_n$ is the natural map.*

Example 2. *Let $F \subsetneq K$ be finite fields and let $R_1 = F + X^2K[[X]]$ and $R_2 = F + FX + FX^2 + X^3K[[X]]$. Then R_1 and R_2 are local CK-domains with common integral closure $\bar{R}_1 = \bar{R}_2 = K[[X]]$. Then $R = R_1 \cap R_2 = F + FX^2 + X^3K[[X]]$ is a local CK-domain with $\bar{R} = \bar{R}_1 \cap \bar{R}_2 = \bar{R}_1 = \bar{R}_2$.*

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