ON LOCALLY AGCD DOMAINS

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Abstract. Let D be an integral domain with quotient field K. Call D an *almost GCD domain* (AGCD domain) if for all $0 \neq a, b \in D$, there is an integer $n = n(a, b) \geq 1$ such that $a^n D \cap b^n D$ is principal. We say that D is a *locally AGCD domain* (resp., strongly locally AGCD domain) if D_M is an AGCD domain for all maximal ideals M of D (resp., for all $0 \neq a, b \in D$, there is an integer $n = n(a, b) \geq 1$ such that $a^n D \cap b^n D$ is locally principal). In this paper, we study some ring-theoretic properties of locally and strongly locally AGCD domains. Let X be an indeterminate over D. We use the ring $D + X^n K[X]$ for an integer $n \geq 2$ to give some examples of locally and strongly locally AGCD domains.

0. Introduction

0.1. **Motivation and Results.** Let D be an integral domain and $D^* = D \setminus \{0\}$. Then D is called a *GCD domain* if $aD \cap bD$ is principal for all $a, b \in D^*$. As in [33], we say that D is an *almost GCD domain* (AGCD domain) if for all $a, b \in D^*$, there is an integer $n = n(a, b) \geq 1$ such that $a^n D \cap b^n D$ is principal. Clearly, a GCD domain is an AGCD domain and a ring of fractions of an AGCD domain is an AGCD domain. Also, it is known that if D is an AGCD domain, then $Cl_t(D)$ is torsion [7, Theorem 3.4]. Moreover, if D is integrally closed, then D is an AGCD domain if and only if D is a PvMD with $Cl_t(D)$ torsion [33, Theorem 3.9]. (Necessary definitions will be provided later.) An *almost factorial domain* is a Krull domain with torsion divisor class group; hence, a Krull domain is an almost factorial domain if and only if it is an AGCD domain.

As in [3], we say that D is a *generalized GCD domain* (GGCD domain) if $aD \cap bD$ is invertible for all $a, b \in D^*$. A *locally GCD domain* D is an integral domain in which D_M is a GCD domain for all maximal ideals M of D. Hence, a locally GCD domain is integrally closed, and D is a locally GCD domain if and only if $aD \cap bD$ is locally principal for all $a, b \in D^*$ [13, Theorem 1.1]. Thus, a GGCD domain is a locally GCD domain. Following [30], we say that D is an *almost GGCD domain* (AGGCD domain) if for all $a, b \in D^*$, there exists an integer $n = n(a, b) \geq 1$ such that $a^n D \cap b^n D$ is invertible. Clearly, if D is an AGGCD domain, then D_M is an AGCD domain for all maximal ideals M of D. The purpose of this paper is to introduce and study two generalizations of AGCD domains. One of them is an integral domain D such that D_M is an AGCD domain for all maximal ideals M of D. Such an integral domain will be called a *locally AGCD domain*. The other is

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an integral domain D in which, for all $a, b \in D^*$, there is an integer $n = n(a, b) \geq 1$ such that $a^nD \cap b^nD$ is locally principal. We will call this type of integral domain a *strongly locally AGCD domain* (for short, *s-locally AGCD domain*).

Let M be a maximal ideal of D, and let $0 \neq \alpha, \beta \in D_M$. Then $\alpha D_M = aD_M$ and $\beta D_M = bD_M$ for some $a, b \in D^*$, and $\alpha^n D_M \cap \beta^n D_M = a^n D_M \cap b^n D_M = (a^nD \cap b^nD)D_M$ for all integers $n \geq 1$. Thus D is a locally AGCD domain if and only if for all $a, b \in D^*$ and all maximal ideals M of D, there is an integer $n = n(a, b, M) \geq 1$ such that $a^n D_M \cap b^n D_M$ is principal (cf. Theorem 2.2); and D is an s-locally AGCD domain if and only if for all $a, b \in D^*$, there is an integer $n = n(a, b) \geq 1$ such that $a^n D_M \cap b^n D_M$ is principal for all maximal ideals M of D. An invertible ideal is locally principal; hence we have the following implications.

$$\begin{tikzcd}
\text{GCD domain} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{GGCD domain} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{locally GCD domain} \arrow[d, Rightarrow] \\
\text{AGCD domain} \arrow[r, Rightarrow] & \text{AGGCD domain} \arrow[r, Rightarrow] & \text{s-locally AGCD domain} \arrow[d, Rightarrow] \\
 & & \text{locally AGCD domain.}
\end{tikzcd}$$

None of the implications is reversible except that we do not know an example of a locally AGCD domain that is not an s-locally AGCD domain.

In this paper, we study some ring-theoretic properties of s-locally AGCD domains and locally AGCD domains. Precisely, in Section 1, we first give examples of s-locally AGCD domains that are neither AGGCD domains nor locally GCD domains. We prove that the integral closure and flat overrings of an s-locally AGCD domain are both s-locally AGCD domains. We also prove that an s-locally AGCD domain D is a UMT domain if and only if D is an APvMD, if and only if D is an AGGCD domain. In Section 2, we show that if D is a locally AGCD domain, then the integral closure $\bar{D}$ of D is a locally AGCD domain; and D is of finite t-character if and only if $\bar{D}$ is a PvMD of finite t-character. We prove that if D is of finite character or if D is a PvMD, then D is a locally AGCD domain if and only if D is an s-locally AGCD domain, if and only if D is an AGGCD domain. We also prove that if D is integrally closed, then D is a locally AGCD domain if and only if the polynomial ring $D[X]$ over D is a locally AGCD domain. Finally, we prove that if K is the quotient field of D, then D is a locally AGCD domain if and only if $D + XK[X]$ is a locally AGCD domain.

0.2. **Definitions related to the t-operation.** Let D be an integral domain with quotient field K, and let $F(D)$ be the set of all nonzero fractional ideals of D. For $A \in F(D)$, let $A^{-1} = \{x \in K \mid xA \subseteq D\}$; so $A^{-1} \in F(D)$. Let $A_v = (A^{-1})^{-1}$ and $A_t = \bigcup\{I_v \mid I \subseteq A, I \in F(D), \text{and } I \text{ is finitely generated}\}$ for all $A \in F(D)$. Hence if $* = v$ or t, then the map $A \mapsto A_*$ is a function from $F(D)$ into $F(D)$ such that for all $0 \neq x \in K$ and $I, J \in F(D)$, (i) $(xD)_* = xD$ and $(xI)_* = xI_*$, (ii) $I \subseteq I_*$ and $I \subseteq J$ implies $I_* \subseteq J_*$, and (iii) $(I_*)_* = I_*$ (i.e., $*$ is a star operation as defined in [25, p. 392]). Clearly, $I \subseteq I_t \subseteq I_v$, and if I is finitely generated, then $I_t = I_v$.

An $I \in F(D)$ is called a $*$-*ideal* if $I_* = I$, while a $*$-ideal is a *maximal* $*$-*ideal* if it is maximal among proper integral $*$-ideals of D. It is easy to see that if $I \in F(D)$ is invertible, then $I = I_t = I_v$; so $aD = (aD)_t = (aD)_v$ for all $a \in D^*$. Let $*$-Max(D) be the set of maximal $*$-ideals of D. It is well known that t-Max$(D) \neq \emptyset$ when D is not a field; each maximal t-ideal is a prime ideal; and $D = \bigcap_{P \in t\text{-Max}(D)} D_P$. However, this need not be true for v-Max(D) since a rank-one nondiscrete valuation domain D has v-Max$(D) = \emptyset$. An $I \in F(D)$ is said to be *t-invertible* if $(II^{-1})_t = D$. Let $T(D)$ be the group of t-invertible fractional t-ideals of D under $I * J = (IJ)_t$, and let $Prin(D)$ be its subgroup of principal fractional ideals. Then the *t-class group* $Cl_t(D)$ of D is the abelian group $Cl_t(D) = T(D)/Prin(D)$. Hence, if D is a Krull domain, then $Cl_t(D)$ is the usual divisor class group; and if D is a Dedekind domain, then $Cl_t(D) = Pic(D)$, the Picard group (or ideal class group) of D. For a reference on the t-class group, see [9].

A *Prüfer v-multiplication domain* (PvMD) is an integral domain in which all nonzero finitely generated ideals are t-invertible. We recall from [7] that D is an *almost valuation domain* if for all $a, b \in D^*$, there exists an integer $n = n(a, b) \geq 1$ such that either $a^n | b^n$ or $b^n | a^n$. Following [7], we say that D is an *almost Prüfer domain* (AP-domain) if for all $a, b \in D^*$, there exists an integer $n = n(a, b) \geq 1$ such that (a^n, b^n) is invertible. As the t-operation analogue, D is said to be an *almost Prüfer v-multiplication domain* (APvMD) if for all $a, b \in D^*$, there exists an integer $n = n(a, b) \geq 1$ such that (a^n, b^n) is t-invertible. Clearly, D is an AP-domain (resp., APvMD) if and only if D_M is an almost valuation domain for all maximal ideals (resp., maximal t-ideals) M of D [7, Theorem 5.8] (resp., [31, Theorem 2.3]). Also, almost valuation domain $\Rightarrow$ AP-domain $\Rightarrow$ AGGCD domain $\Rightarrow$ APvMD.

1. s-Locally AGCD domains

Let D be an integral domain with quotient field K, X be an indeterminate over D, and $D[X]$ be the polynomial ring over D.

We begin this section with an interesting proposition that motivates the study of the so-called s-locally AGCD domains. As we have already noted, AGGCD domains are s-locally AGCD domains, while the next proposition shows that an s-locally AGCD domain need not be an AGGCD domain.

Proposition 1.1. *Let D be a locally GCD domain with char$D = p \neq 0$, and let $R_n = D + X^n K[X]$ for an integer $n \geq 2$.*

1. *R_n is an AGCD domain if and only if D is an AGCD domain.*
2. *R_n is not a locally GCD domain.*
3. *For all $\alpha, \beta \in R_n$, $\alpha^{p^n} R_n \cap \beta^{p^n} R_n$ is locally principal. Thus R_n is an s-locally AGCD domain.*
4. *If D is not a GGCD domain (cf.* [13, Corollary 3.6] *for such an integral domain), then R_n is not an AGGCD domain.*

Proof. (1) This follows from [16, Corollary 2.12].

(2) This is an immediate consequence of the fact that a locally GCD domain is integrally closed, but R_n is not integrally closed.

(3) Let P be a prime ideal of D, and let $R = D_P + XK[X]$ (note that $D_P = K$ when $P = (0)$). Then D_P, and so $D_P + XK[X]$, is a GCD domain [19, Corollary

1.3]. Hence $\alpha R \cap \beta R = hR$ for some $h \in R$. Note that $\alpha^{p^n} R \cap \beta^{p^n} R = h^{p^n} R$ [33, Lemma 3.6]. Also note that since char$D = p$, then $f^{p^n} \in D_P + X^n K[X]$ for all $f \in R$. Thus, $\alpha^{p^n}(D_P + X^n K[X]) \cap \beta^{p^n}(D_P + X^n K[X]) = h^{p^n}(D_P + X^n K[X])$.

Let M be a maximal ideal of R_n, and let $P = M \cap D$. Then $(R_n)_M = (D_P + X^n K[X])_{M_{D \setminus P}}$, and thus $\alpha^{p^n}(R_n)_M \cap \beta^{p^n}(R_n)_M$ is principal. Hence, $\alpha^{p^n} R_n \cap \beta^{p^n} R_n$ is locally principal.

(4) Assume that D is not a GGCD domain, but R_n is an AGGCD domain. Then D is an AGGCD domain [15, Corollary 2.13], and hence D is a PvMD [31, Theorem 2.4] because an AGGCD domain is an APvMD and a locally GCD domain is integrally closed. Thus, D is a GGCD domain since a locally GCD domain is a PvMD if and only if it is a GGCD domain [13, Corollary 1.9], a contradiction. □

An *almost Dedekind domain* D is an integral domain such that D_M is a PID (DVR) for all maximal ideals M of D. Hence, an almost Dedekind domain is locally factorial, and thus is a locally GCD domain. Clearly, D is a Dedekind domain if and only if D is a Noetherian almost Dedekind domain.

Corollary 1.2. *Let D be an almost Dedekind domain with quotient field K and char$D \neq 0$, and let $R_n = D + X^n K[X]$ for an integer $n \geq 2$. Then R_n is an s-locally AGCD domain, but not a locally GCD domain. Moreover, if $Pic(D)$ is not torsion, then R_n is not an AGCD domain.*

Proof. The first result follows directly from Proposition 1.1. For the second result, note that $Pic(D)$ is a subgroup of $Cl_t(R_n)$. Hence, $Cl_t(R_n)$ is not torsion, and thus R_n is not an AGCD domain [7, Theorem 3.4]. □

By Proposition 1.1, if D is a locally GCD domain with char$D \neq 0$, then $D + X^n K[X]$ is an s-locally AGCD domain for $n \geq 2$, but not a locally GCD domain. Moreover, if D is not an AGCD domain, then $D + X^n K[X]$ is not an AGCD domain for $n \geq 2$. Hence, if D is a Dedekind domain such that $Pic(D)$ is not torsion and char$D \neq 0$ (see [17, Theorem 7] or [23, Theorem 14.10] for such a Dedekind domain), then $D + X^n K[X]$ for $n \geq 2$ is an s-locally AGCD domain that is neither a locally GCD domain nor an AGCD domain. Also, if D is not a GGCD domain (see [13, Corollary 3.6] for such an integral domain), then $D + X^n K[X]$ for $n \geq 2$ is an s-locally AGCD domain that is not an AGGCD domain.

Lemma 1.3. *Let $D \subseteq R$ be a root extension of integral domains such that R is root closed, and let $a, b, c \in D^*$. If $aD \cap bD = cD$, then $aR \cap bR = cR$.*

Proof. Note that $aD \cap bD = cD \Leftrightarrow \frac{c}{a}D \cap \frac{c}{b}D = (\frac{c}{a} \cdot \frac{c}{b})D$; $aR \cap bR = cR \Leftrightarrow \frac{c}{a}R \cap \frac{c}{b}R = (\frac{c}{a} \cdot \frac{c}{b})R$; and $\frac{c}{a}, \frac{c}{b} \in D$; so it suffices to show that $aR \cap bR = abR$ under the assumption that $aD \cap bD = abD$. Obviously, $abR \subseteq aR \cap bR$. For the reverse containment, let $x \in aR \cap bR$. Then $x = ar = bs$ for some $r, s \in R$, and since R is a root extension of D, there is an integer $n \geq 1$ such that $r^n, s^n \in D$. Hence, $x^n = a^n r^n = b^n s^n \in a^n D \cap b^n D = a^n b^n D$ because $(a, b)_v = D$ implies $(a^n, b^n)_v = D$, and so $x^n = a^n b^n \beta$ for some $\beta \in D$. Thus $(\frac{x}{ab})^n = \beta \in D \subseteq R$, and since R is root closed, $\frac{x}{ab} \in R$ or $x \in abR$. Therefore, $aR \cap bR \subseteq abR$. □

Let $\bar{D}$ be the integral closure of D in K. Although the authors of [7] didn't name an s-locally AGCD domain, the notion of s-locally AGCD domains was first studied in [7, Proposition 3.2]. There, it was shown that if D is an s-locally AGCD domain, then $\bar{D}$ is a root extension of D.

Proposition 1.4. *Let D be an s-locally AGCD domain with integral closure $\bar{D}$.*

1. *$\bar{D}$ is a root extension of D, and hence D is integrally closed if and only if D is root closed.*
2. *$\bar{D}$ is an s-locally AGCD domain.*
3. *A flat overring R of D is an s-locally AGCD domain.*

Proof. (1) [7, Proposition 3.2].

(2) Let $0 \neq x, y \in \bar{D}$. Then, by (1), there is an integer $k \geq 1$ such that $x^k, y^k \in D$, and since D is an s-locally AGCD domain, $x^{mk}D \cap y^{mk}D$ is locally principal for some integer $m = m(x^k, y^k) \geq 1$. Let M be a maximal ideal of $\bar{D}$, and let $M \cap D = P$. Then P is a maximal ideal of D such that M is the unique maximal ideal of $\bar{D}$ lying over P by (1); so $\bar{D}_M = \bar{D}_{D\setminus P}$ and $\bar{D}_M$ is a root extension of D_P. Note that $(x^{mk}D \cap y^{mk}D)D_P = x^{mk}D_P \cap y^{mk}D_P = cD_P$ for some $c \in D$, and hence, by Lemma 1.3, $x^{mk}\bar{D}_M \cap y^{mk}\bar{D}_M = (x^{mk}\bar{D} \cap y^{mk}\bar{D})\bar{D}_M = c\bar{D}_M$. Thus, $x^{mk}\bar{D} \cap y^{mk}\bar{D}$ is locally principal.

(3) Let $0 \neq a, b \in R$. Then there is a $0 \neq d \in D$ such that $da, db \in D$; so $I = d^n a^n D \cap d^n b^n D$ is locally principal for some integer $n = n(da, db) \geq 1$. Let M be a maximal ideal of R. Since R is flat over D, $R_M = D_{M\cap D}$ [32, Theorem 2], and so $IR_M = ID_{M\cap D}$ is principal. Thus, $(a^n R \cap b^n R)R_M = \frac{1}{d^n}IR_M$ is principal. $\square$

A nonzero prime ideal Q of $D[X]$ is an *upper to zero* in $D[X]$ if $Q \cap D = (0)$, and we say that D is a *UMT-domain* if every upper to zero in $D[X]$ is a maximal t-ideal. It is well known that D is an integrally closed UMT-domain if and only if D is a PvMD [27, Proposition 3.2]. As in [34], we say that D is *conditionally well behaved* if MD_M is a t-ideal of D_M for all maximal t-ideals M of D.

We know that an APvMD is a UMT-domain [31, Theorem 3.8], and since an AGGCD domain is an APvMD, an AGGCD domain is also a UMT-domain. Thus, an AGCD domain is a UMT-domain (cf. [10, Lemma 3.1]), while an s-locally AGCD domain need not be a UMT-domain. For example, let D be a locally GCD domain that is not a PvMD (see [13, Example 3.11] for such an integral domain). Then D is not a UMT domain, and thus a locally GCD domain (so also an s-locally AGCD domain) need not be a UMT domain. The next result is the s-locally AGCD domain analogue of [13, Corollary 1.9] that a locally GCD domain D is a PvMD if and only if D is a GGCD domain.

Proposition 1.5. *The following statements are equivalent for an s-locally AGCD domain D.*

1. *D is a UMT domain.*
2. *D is conditionally well behaved.*
3. *D is an APvMD.*
4. *D is an AGGCD domain.*

Proof. (1) $\Rightarrow$ (2) This follows from [22, Theorem 1.5].

(2) $\Rightarrow$ (3) Let $a, b \in D^*$. Then $a^n D \cap b^n D$ is locally principal for some integer $n = n(a, b) \geq 1$. Note that $a^n D \cap b^n D = a^n b^n (a^n, b^n)^{-1}$; so for every maximal t-ideal M of D, $((a^n, b^n)D_M)^{-1} = (a^n, b^n)^{-1}D_M$ is principal. Thus, $((a^n, b^n)D_M)_v$

is principal, and since MD_M is a t-ideal of D_M by assumption, $(a^n, b^n)D_M$ is principal. This means that D_M is an almost valuation domain, and hence D is an APvMD.

(3) $\Rightarrow$ (4) Let $a, b \in D^*$. Since D is an APvMD, there is an integer $n = n(a, b) \geq 1$ such that (a^n, b^n) is t-invertible. Also, since D is an s-locally AGCD domain, $a^{nk}D \cap b^{nk}D$ is locally principal for some integer $k = k(a^n, b^n) \geq 1$. Note that $((a^n, b^n)^k)_t = (a^{nk}, b^{nk})_t$ [7, Lemma 3.3]; so (a^{nk}, b^{nk}), and thus $a^{nk}D \cap b^{nk}D$, is t-invertible. Hence, $a^{nk}D \cap b^{nk}D$ is invertible [2, Theorem 2.1].

(4) $\Rightarrow$ (1) This is clear by the comments before the proposition. $\square$

Recall that D is said to be of *finite t-character* if every nonzero nonunit of D is contained in only a finite number of maximal t-ideals of D. It is known that integral domains for which $t = v$ (e.g., Noetherian domains) are of finite t-character [26, Theorem 1.3] and the polynomial ring over an integral domain of finite t-character is of finite t-character ([28, Proposition 4.2] or [5, Corollary 3.4]).

Corollary 1.6. *Let D be an integral domain of finite t-character. Then D is an s-locally AGCD domain if and only if D is an AGGCD domain.*

Proof. An AGGCD domain is always an s-locally AGCD domain. Conversely, if D is of finite t-character, then D is conditionally well behaved [5, Theorem 1.1]. Thus, the result follows directly from Proposition 1.5. $\square$

2. Locally AGCD domains

We say that an integral domain D is a *locally AGCD domain* if D_M is an AGCD domain for all maximal ideals M of D. Hence, AGCD domains, locally GCD domains, and s-locally AGCD domains are locally AGCD domains.

Let D be a Dedekind domain with $Pic(D)$ not torsion (see, for example, [17, Theorem 7] or [23, Theorem 14.10] for such a Dedekind domain). Then D is a locally factorial domain (hence locally GCD domain), but D is not an AGCD domain because the t-class group (Picard group) of an AGCD domain is torsion. Next, note that a locally GCD domain is integrally closed while an AGCD domain does not have to be integrally closed. We begin this section with an example of integrally closed AGCD domains that are not locally GCD domains.

Example 2.1. Let D be a PID, $p \in D$ be a prime element, $m \geq 2$ be an integer, X be an indeterminate over D, and $R = D[X; \frac{p^m}{X}]$. Then R is an integrally closed AGCD domain that is not a locally GCD domain.

Proof. Since D is a PID (and hence a Krull domain), R is a Krull domain [4, Theorem 8], but R is not a locally GCD domain [4, Theorem 9] because $p^m D_{pD}$ is not a prime ideal. Also, note that $Cl_t(R) = \mathbb{Z}/m\mathbb{Z}$ [4, Theorem 18], and so $Cl_t(R)$ is torsion. Hence, as a Krull domain is a PvMD, R is an integrally closed AGCD domain. $\square$

We next give a nice characterization of locally AGCD domains.

Theorem 2.2. *The following statements are equivalent for an integral domain D.*

(1) *D is a locally AGCD domain.*
(2) *D_P is an AGCD domain for all prime ideals P of D.*

(3) *For all* $a, b \in D^*$ *and for all maximal ideals* M *of* D*, there exists an integer* $n = n(a, b, M) \geq 1$ *such that* $a^n D_M \cap b^n D_M$ *is principal.*

Proof. (1) $\Rightarrow$ (2) This follows from the fact that a ring of fractions of an AGCD domain is an AGCD domain.

(2) $\Rightarrow$ (3) Let $a, b \in D^*$ and M be a maximal ideal of D. Then D_M is an AGCD domain by (2), and hence there is an integer $n = n(a, b) \geq 1$ such that $(a^n D \cap b^n D)D_M = a^n D_M \cap b^n D_M$ is principal. Clearly, n also depends on M.

(3) $\Rightarrow$ (1) Let M be a maximal ideal of D, and let $0 \neq \alpha, \beta \in D_M$. Then as $\alpha = \frac{c}{s}$ and $\beta = \frac{d}{t}$ for some $c, d \in D$ and $s, t \in D \backslash M$, we have that $\alpha D_M = cD_M$ and $\beta D_M = dD_M$. Note that by (3), there is an integer $n = n(c, d, M) \geq 1$ such that $c^n D_M \cap d^n D_M = (c^n D \cap d^n D)D_M$ is principal. Thus, $\alpha^n D_M \cap \beta^n D_M$ is principal. $\square$

Corollary 2.3. *Let* D *be an integral domain with integral closure* $\bar{D}$ *such that* $D[X] \subseteq \bar{D}[X]$ *is a root extension (e.g.,* D *is integrally closed). If* D *is a locally AGCD domain, then* $D[X]$ *is a locally AGCD domain.*

Proof. Let M be a maximal ideal of $D[X]$, and let $P = M \cap D$. Then D_P is an AGCD domain by Theorem 2.2, $\bar{D}_{D \backslash P}$ is the integral closure of D_P, and $D_P[X] \subseteq \bar{D}_{D \backslash P}[X]$ is a root extension. Hence, $D_P[X]$ is an AGCD domain [6, Theorem 3.4], and thus $D[X]_M = D_P[X]_{M_{D \backslash P}}$ is an AGCD domain. $\square$

Corollary 2.4. *Let* D *be a locally AGCD domain. If* R *is a flat overring of* D *(e.g., a fraction ring of* D*), then* R *is a locally AGCD domain.*

Proof. Let Q be a maximal ideal of R, and let $P = Q \cap D$. Then R_Q is an AGCD domain because $R_Q = D_P$ [32, Theorem 2] and D_P is an AGCD domain by Theorem 2.2. $\square$

We next give the locally AGCD domain analogue of [33, Theorem 3.4] that the integral closure of an AGCD domain is an AGCD domain.

Corollary 2.5. *Let* D *be a locally AGCD domain with integral closure* $\bar{D}$.

(1) $\bar{D}$ *is a locally AGCD domain.*
(2) *If* D *is of finite* t*-character, then* D *is an APvMD.*
(3) D *is of finite* t*-character if and only if* $\bar{D}$ *is a PvMD of finite* t*-character.*
(4) D *is integrally closed if and only if* D_M *is a PvMD with* $Cl_t(D_M)$ *torsion for all maximal ideals* M *of* D.

Proof. (1) Let M be a maximal ideal of $\bar{D}$, and let $P = M \cap D$. Then D_P is an AGCD domain by Theorem 2.2, $\bar{D}_{D \backslash P}$ is the integral closure of D_P, and $\bar{D}_M = (\bar{D}_{D \backslash P})_{M_{D \backslash P}}$. Hence, $\bar{D}_{D \backslash P}$ is an AGCD domain [33, Theorem 3.4], and thus $\bar{D}_M$ is an AGCD domain.

(2) Let P be a maximal t-ideal of D. Then D_P is an AGCD domain by Theorem 2.2 and PD_P is a t-ideal of D_P [5, Theorem 1.1]. Hence, D_P is an almost valuation domain [7, Theorem 5.6]. Thus, D is an APvMD [31, Theorem 2.3].

(3) By (2) and [31, Theorem 3.6], we may assume that $\bar{D}$ is a PvMD. Let M be a nonzero prime ideal of $\bar{D}$, and let $P = M \cap D$. Then D_P is an AGCD domain and $\bar{D}_{D \backslash P}$ is the integral closure of D_P. Hence, $\bar{D}_{D \backslash P}$ is a root extension of D_P, and so $M\bar{D}_{D \backslash P}$ is the unique prime ideal of $\bar{D}_{D \backslash P}$ lying over PD_P. Thus, M is the

unique prime ideal of $\bar{D}$ that contracts to P; whence it suffices to show that M is a t-ideal of $\bar{D}$ if and only if P is a t-ideal of D. (This implies that M is a maximal t-ideal if and only if P is a maximal t-ideal.)

Assume that M is a t-ideal of $\bar{D}$, and note that $\bar{D}$ is a PvMD. Hence $M\bar{D}_{D\setminus P}$ is a t-ideal of $\bar{D}_{D\setminus P}$, and since D_P is an AGCD domain and $M\bar{D}_{D\setminus P} \cap D_P = PD_P$, we have that PD_P is a t-ideal of D_P (see the proof of [20, Proposition 3.1(b)]). Thus, $P = PD_P \cap D$ is a t-ideal of D. Conversely, assume that P is a t-ideal of D. Then $\bar{D}_{D\setminus P}$ is a Prüfer domain by [22, Theorem 1.5] and [31, Theorem 3.8], and so $M\bar{D}_{D\setminus P}$ is a t-ideal of $\bar{D}_{D\setminus P}$. Hence, $M = M\bar{D}_{D\setminus P} \cap \bar{D}$ is a t-ideal of $\bar{D}$.

(4) Clearly, D is integrally closed if and only if D_M is integrally closed for all maximal ideals M of D. Thus, the result follows since D is an integrally closed AGCD domain if and only if D is a PvMD with $\mathrm{Cl}_t(D)$ torsion [33, Theorem 3.9]. □

An integral domain D is said to be of *finite character* if every nonzero nonunit of D is contained in only a finite number of maximal ideals of D. It is clear that AGGCD domains are locally AGCD domains. The next result shows that if D is of finite character, then D is a locally AGCD domain if and only if D is an AGGCD domain. This result is the locally AGCD domain analogue of [5, Corollary 1.3] that if D is of finite t-character, then D is a locally GCD domain if and only if D is a GGCD domain. (Also, see Corollary 1.6.)

Corollary 2.6. *The following statements are equivalent for an integral domain D of finite character.*

(1) *D is a locally AGCD domain.*
(2) *D is an s-locally AGCD domain.*
(3) *D is an AGGCD domain.*

Proof. (1) $\Rightarrow$ (2) Assume that D is a locally AGCD domain. Let $a, b \in D^*$, and let $I = aD \cap bD$. Then there are only finitely many maximal ideals of D containing I, say, $M_1, \ldots, M_k$, because D is of finite character. By assumption and Theorem 2.2, there is an integer $n_i = n_i(a, b, M_i) \geq 1$ for $i = 1, ..., k$ such that $a^{n_i}D_{M_i} \cap b^{n_i}D_{M_i}$ is principal. Hence, if we let $n = n_1 \cdots n_k$, then $a^nD_{M_i} \cap b^nD_{M_i} = (a^{n_i}D_{M_i} \cap b^{n_i}D_{M_i})^{\frac{n}{n_i}}$ is principal, where the equality holds because $xD \cap yD = dD \Leftrightarrow (\frac{d}{x}, \frac{d}{y})_v = D \Leftrightarrow (\frac{d^m}{x^m}, \frac{d^m}{y^m})_v = D \Leftrightarrow x^mD \cap y^mD = d^mD$ for all integers $m \geq 1$ and $x, y, d \in D^*$. Also, it is clear that if M is a maximal ideal of D with $M \neq M_i$ for $i = 1, \ldots, k$, then $a^nD_M \cap b^nD_M = D_M$. Hence, $a^nD \cap b^nD$ is locally principal.

(2) $\Rightarrow$ (3) Let $a, b \in D^*$. Then there is an integer $n = n(a, b) \geq 1$ such that $a^nD \cap b^nD$ is locally principal. Also, since D is of finite character, $a^nD \cap b^nD$ is invertible [8, Theorem 4].

(3) $\Rightarrow$ (1) Clear. □

An integral domain D is called a *v-finite conductor domain* if $(a) \cap (b)$ is a v-ideal of finite type for all $a, b \in D^*$. Clearly, PvMDs, Mori domains, Noetherian domains, and GCD domains are v-finite conductor domains. (A *Mori domain* is an integral domain that satisfies the ascending chain condition on integral v-ideals.) We next prove that a v-finite conductor locally AGCD is an AGGCD domain. For this, we first need a lemma whose proof is similar to that of [7, Lemma 5.7].

Lemma 2.7. *Let $x, y \in D$, and assume that for every maximal ideal M, there is an integer $n_M \geq 1$ such that $((x^{n_M}) \cap (y^{n_M}))D_M = a_M D_M$ for some $a_M \in (x^{n_M}) \cap (y^{n_M})$ and $f_M((x^{n_M}) \cap (y^{n_M})) \subseteq a_M D$ for some $f_M \in D \backslash M$. Then there is an integer $N \geq 1$ such that $(x^N) \cap (y^N)$ is locally principal.*

Proof. Let M be a maximal ideal of D, and let $D_{f_M} = D[\frac{1}{f_M}]$. Then $((x^{n_M}) \cap (y^{n_M}))D_{f_M} = a_M D_{f_M}$ since $f_M \in D \backslash M$, and thus $((x^{n_M k}) \cap (y^{n_M k}))D_{f_M} = a_M^k D_{f_M}$ for every integer $k \geq 1$ [33, Lemma 3.6]. Clearly, $D = (\{f_{M'} \mid M'$ is a maximal ideal of $D\})$, and hence $D = (f_1, ..., f_m)$ for some maximal ideals $M_1, ..., M_m$ with $f_i = f_{M_i} \in D \backslash M_i$. So if we let $N = n_{M_1} \cdots n_{M_m}$, then $((x^N) \cap (y^N))D_{f_i} = a_{M_i}^{\frac{N}{n_{M_i}}} D_{f_i}$. Since $(f_1, ..., f_m) = D \nsubseteq M$, we have $f_i \notin M$ for some i. Thus, $((x^N) \cap (y^N))D_M$ is principal for every maximal ideal M of D. □

Proposition 2.8. *The following statements are equivalent for a v-finite conductor domain D (e.g., D is a PvMD).*

1. *D is a locally AGCD domain.*
2. *D is an s-locally AGCD domain.*
3. *D is an AGGCD domain.*

Proof. (1) ⇒ (3) Let $x, y \in D^*$. It suffices to show that there is an integer $N \geq 1$ such that $(x^N) \cap (y^N)$ is locally principal because D a v-finite conductor domain implies that $(x^N) \cap (y^N)$ is invertible [2, Theorem 2.1]. Let M be a maximal ideal of D. Then there is an integer $n_M \geq 1$ such that $((x^{n_M}) \cap (y^{n_M}))D_M = a_M D_M$ for some $a_M \in (x^{n_M}) \cap (y^{n_M})$. Since $(x^{n_M}) \cap (y^{n_M})$ is of finite type by assumption, there is a finitely generated ideal I of D such that $(x^{n_M}) \cap (y^{n_M}) = I_v$. Hence, $D_M = (a_M D_M : ((x^{n_M}) \cap (y^{n_M}))D_M) = (a_M D_M : (I_v D_M)_v) = (a_M D_M : (ID_M)_v) = (a_M D_M : ID_M) = (a_M D : I)D_M = (a_M D : I_v)D_M = (a_M D : ((x^{n_M}) \cap (y^{n_M})))D_M$, which shows that there is an $f_M \in D \backslash M$ such that $f_M((x^{n_M}) \cap (y^{n_M})) \subseteq a_M D$. Thus, by Lemma 2.7, there is an integer $N \geq 1$ such that $(x^N) \cap (y^N)$ is locally principal.

(3) ⇒ (2) ⇒ (1) Clear. □

We next show that if D is integrally closed, then D is a locally AGCD domain if and only if $D[X]$ is a locally AGCD domain. To do this, we first need the notion of Nagata rings. Let $S = \{f \in D[X] \mid c(f) = D\}$, where $c(f)$ is the ideal of D generated by the coefficients of f. Then S is a saturated multiplicative subset of $D[X]$, and hence $D[X]_S$ is an overring of $D[X]$. The ring $D[X]_S$, called the *Nagata ring* of D, is denoted by $D(X)$. It is known that D is a Prüfer domain if and only if $D(X)$ is a Prüfer domain [11, Theorem 4], if and only if $D(X)$ is a Bezout domain [12, Theorem 2.2]. Since a Krull domain is a PvMD, the next lemma also generalizes [23, Proposition 8.9] that if D is a quasi-local Krull domain, then $Cl_t(D) = Cl_t(D(X))$.

Lemma 2.9. *Let D be an integral domain.*

1. *D is a PvMD if and only if $D(X)$ is a PvMD.*
2. *If D is a quasi-local PvMD, then $Cl_t(D) = Cl_t(D(X))$.*

Proof. (1) (⇒) Let Q be a maximal t-ideal of $D(X)$. Then $Q \cap D[X]$ is a prime t-ideal of $D[X]$; so if $Q \cap D \neq (0)$, then $Q = (Q \cap D)(X)$ and $Q \cap D$ is a maximal

t-ideal of D (cf. [27, Proposition 1.1]). Hence, $D_{Q\cap D}$ is a valuation domain, and thus $D(X)_Q = D_{Q\cap D}(X)$ is a valuation domain. Next, assume that $Q \cap D = (0)$. Then $D(X)_Q = K[X]_{Q\cap K[X]}$, and thus $D(X)_Q$ is a rank-one DVR. Hence, $D(X)$ is a PvMD.

($\Leftarrow$) If P is a maximal t-ideal of D, then $P(X)$ is a prime t-ideal of $D(X)$ [29, Proposition 2.2]. Hence, $D(X)_{P(X)} = D[X]_{P[X]}$ is a valuation domain, and since $D[X]_{P[X]} \cap K = D_P$, D_P is a valuation domain.

(2) For a t-invertible t-ideal J of an integral domain R, let $[J] \in Cl_t(R)$ be the equivalence class of $T(R)$ containing J. We claim that $\varphi : Cl_t(D) \to Cl_t(D(X))$ defined by $\varphi([I]) = [ID(X)]$ is a group isomorphism. It is routine to check that φ is a group homomorphism. Let M be the maximal ideal of D, and let I be a nonzero ideal of D such that $ID(X)$ is principal. Then $D(X) = (ID(X))(ID(X))^{-1} = (ID(X))(I^{-1}D(X)) = (II^{-1})D(X)$, and hence $II^{-1} \nsubseteq M$. Thus, I is invertible, and since D is quasi-local, I is principal. Hence, φ is injective. Finally, let A be an ideal of $D[X]$ and $S = \{f \in D[X] \mid c(f) = D\}$ such that A_S is t-invertible. Then there is a nonzero finitely generated ideal B of $D[X]$ such that $(A_S)_t = (B_S)_t$. Note that $D[X]$ is a PvMD; so B is t-invertible, and since $Cl_t(D) = Cl_t(D[X])$ [24, Theorem 3.6], there is a t-invertible t-ideal I of D and $0 \neq u \in K(X)$ such that $B_t = uID[X]$. Thus, $\varphi([I]) = [(B_S)_t]$, and so φ is surjective. $\square$

Corollary 2.10. *Let D be an integrally closed domain. Then D is a locally AGCD domain if and only if $D[X]$ is a locally AGCD domain.*

Proof. ($\Rightarrow$) Corollary 2.3.

($\Leftarrow$) Let M be a maximal ideal of D. Then $M[X]$ is a prime ideal of $D[X]$, and hence $D[X]_{M[X]} = D_M(X)$ is an AGCD domain by Theorem 2.2. Note that $D_M(X)$ is an integrally closed quasi-local domain; so $D_M(X)$ is a PvMD with $Cl_t(D_M(X))$ torsion by Corollary 2.5. Thus, by Lemma 2.9, D_M is a PvMD with $Cl_t(D_M)$ torsion, and so D_M is an AGCD domain [33, Theorem 3.9]. $\square$

We next give the locally AGCD domain analogue of [1, Proposition 1] that if D is a locally GCD domain, then D is a Prüfer domain if and only if $M((a) \cap (b)) = Ma \cap Mb$ for all maximal ideals M of D and all $a, b \in D^*$.

Proposition 2.11. *The following statements are equivalent for a locally AGCD domain D.*

1. *D is an almost Prüfer domain.*
2. *For all maximal ideals M of D and all $a, b \in D^*$, there exists an integer $n = n(a, b, M) \geq 1$ such that $M(a^nD \cap b^nD) = Ma^n \cap Mb^n$.*
3. *For all $a, b \in D^*$, there exists an integer $n = n(a, b) \geq 1$ such that $M(a^nD \cap b^nD) = Ma^n \cap Mb^n$ for all maximal ideals M of D.*
4. *D is an AGGCD domain and for all $a, b \in D^*$, there exists an integer $n = n(a, b) \geq 1$ such that $(a^n, b^n)_v = (a^n, b^n)$.*

Proof. (1) $\Rightarrow$ (3) Suppose that D is an almost Prüfer domain and M is a maximal ideal of D. Then (a^n, b^n) is invertible for some integer $n = n(a, b) \geq 1$, and so $(a^n, b^n)D_M$ is a^nD_M or b^nD_M [25, Proposition 7.4]. Hence, if M' is a maximal ideal of D, then

$$\begin{aligned}(M(a^nD\cap b^nD))D_{M'} &= MD_{M'}\cdot(a^nD_{M'}\cap b^nD_{M'})\\ &= (MD_{M'}\cdot a^nD_{M'})\cap(MD_{M'}\cdot b^nD_{M'})\\ &= (Ma^n)D_{M'}\cap(Mb^n)D_{M'}\\ &= (Ma^n\cap Mb^n)D_{M'},\end{aligned}$$

where the second equality follows because $MD_{M'} = D_{M'}$ when $M \neq M'$. Thus, $M(a^nD_M \cap b^nD_M) = Ma^n \cap Mb^n$.

(3) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (1) Let $a, b \in D^*$. We show that (a^j, b^j) is invertible for some integer $j \geq 1$. By [7, Lemma 5.7], we may assume that D is quasilocal with maximal ideal M; so D is an AGCD domain. Let $c \in D^*$ and $n \geq 1$ be an integer such that $(a^n, b^n)_v = cD$. Replacing (a, b) by $(a^n/c, b^n/c)$, we may assume that $(a, b)_v = D$, that is, $aD \cap bD = abD$. By hypothesis, there exists an integer $m \geq 1$ such that $M(a^mD \cap b^mD) = Ma^m \cap Mb^m$. Again, replacing (a, b) by (a^m, b^m), we may assume that $m = 1$. Suppose that (a, b) is not invertible. By [1, Lemma 2], $aD\cap bD = aM\cap bM = M(aD\cap bD)$, which gives the contradiction $M = D$ because $aD \cap bD = abD$.

(1) $\Leftrightarrow$ (4) Note that D is an AGGCD domain if and only if for all $a, b \in D^*$, there is an integer $n = n(a, b) \geq 1$ such that $(a^n, b^n)_v$ is invertible [30, Theorem 3.2]. Thus, the result follows because an invertible ideal is a t-ideal. □

Let S be a multiplicative subset of D^* and n be a positive integer. We next study the prime ideals of the ring $D + X^nD_S[X]$ that intersect S. This result is already known for the ring $D + XD_S[X]$ ([21, Lemma 3.7], [19, Theorem 2.1], and [5, Lemma 2.1]).

Lemma 2.12. *Let S be a multiplicative subset of D^*, $\mathfrak{P}$ be the set of prime ideals of D intersecting S, n be a positive integer, and $R_n = D + X^nD_S[X]$.*

1. *If A is an ideal of R_n such that $A \cap S \neq \emptyset$, then $A = (A \cap D)R_n = (A \cap D) + X^nD_S[X]$. Moreover, $A_t = (A \cap D)_t + X^nD_S[X]$.*
2. *$\{P + X^nD_S[X] \mid P \in \mathfrak{P}\}$ is the set of prime ideals of R_n intersecting S.*
3. *Let $P \in \mathfrak{P}$. Then $P + X^nD_S[X]$ is a prime ideal (resp., maximal ideal, prime t-ideal, maximal t-ideal) of R_n if and only if P is a prime ideal (resp., maximal ideal, prime t-ideal, maximal t-ideal) of D.*

Proof. (1) Let $s \in A \cap S$. Then, for every $f \in D_S[X]$, we have $\frac{f}{s} \in D_S[X]$, and so $X^nf = s \cdot X^n\frac{f}{s} \in A$. Thus, $(A \cap D) + X^nD_S[X] = A$. Also, if $g = a + X^nh \in A$, where $a \in A\cap D$ and $h \in D_S[X]$, then $g \in (a, s)R_n$. Hence, $A \subseteq (A\cap D)R_n$. Thus, if $A \cap S \neq \emptyset$, then $A = (A \cap D)R_n = (A \cap D) + X^nD_S[X]$.

Next, to show that $A_t = (A \cap D)_t + X^nD_S[X]$, we recall from [14, Lemma 2.1] that if I is a nonzero finitely generated ideal of D, then $(IR_n)^{-1} = I^{-1}R_n$. Thus, we have the result by an argument similar to the proof of [21, Lemma 3.7].

(2) and (3) Clearly, P is a prime ideal of D if and only if $P + X^nD_S[X]$ is a prime ideal of R_n. Hence, this follows directly from (1) above. □

Recall from [16, Corollary 2.12] (resp., [15, Corollary 2.14]) that if $n \geq 2$, then $R_n = D + X^nK[X]$ is an AGCD domain (resp., AGGCD domain) if and only if D

is an AGCD domain (resp., AGGCD domain) and char$D \neq 0$. We next give the locally AGCD domain analogue.

Proposition 2.13. *Let D be an integral domain with quotient field K, and let $R_n = D + X^nK[X]$ for an integer $n \geq 1$.*

(1) *R_1 is a locally AGCD domain if and only if D is a locally AGCD domain.*
(2) *If char$D \neq 0$, then the following statements are equivalent.*
 (a) *D is a locally AGCD domain.*
 (b) *R_n is a locally AGCD domain for some $n \geq 1$.*
 (c) *R_n is a locally AGCD domain for all $n \geq 1$.*

Proof. (1) ($\Rightarrow$) See the proof of ((b) $\Rightarrow$ (a)) of (2) below.

($\Leftarrow$) Let M be a maximal ideal of R_1, and let $P = M \cap D$. If $P = (0)$, then $(R_1)_M = K[X]_{M_{D^*}}$, and since $K[X]$ is a PID (thus an AGCD domain), $(R_1)_M$ is an AGCD domain. Next, if $P \neq (0)$, then $(R_1)_{D\setminus P} = D_P + XK[X]$. Hence, $(R_1)_{D\setminus P}$ is an AGCD domain [6, Corollary 3.11] because D_P is an AGCD domain by Theorem 2.2. Thus, $(R_1)_M = ((R_1)_{D\setminus P})_{M_{D\setminus P}}$ is an AGCD domain.

(2) $(a) \Rightarrow (c)$ An argument similar to the proof of Proposition 1.1(3) also shows that if D is a locally AGCD domain, then R_n is a locally AGCD domain for all $n \geq 1$.

$(c) \Rightarrow (b)$ Clear.

$(b) \Rightarrow (a)$ Let P be a maximal ideal of D. Then $M = P + XD_S[X]$ is a maximal ideal of R_n by Lemma 2.12 such that $(R_n)_{D\setminus P} = D_P + X^nK[X]$ and $(R_n)_M = ((R_n)_{D\setminus P})_{M_{D\setminus P}}$. So, by replacing $D + X^nK[X]$ with $D_P + X^nK[X]$, we may assume that D is quasi-local with maximal ideal P. Let $T = R_n \setminus (P + X^nK[X])$. Then $T = \{a + X^nf \mid a$ is a unit in D and $f \in K[X]\}$, and by (b), $(R_n)_T$ is an AGCD domain. Let $a, b \in D^*$. Then there is an integer $m \geq 1$ such that $a^m(R_n)_T \cap b^m(R_n)_T = g(R_n)_T$ for some $g = c + X^nh \in R_n$ with $c \in D$, and it is routine to check that $a^mD \cap b^mD = cD$. Thus, D is an AGCD domain. ☐

We do not know an example of a locally AGCD domain that is not an s-locally AGCD domain. In fact, in the proof of Proposition 2.8, as $((x^{n_M}) \cap (y^{n_M}))D_M = a_MD_M$, where $a_M \in (x^{n_M}) \cap (y^{n_M})$, we can take $a_M = \frac{s_Mx^{n_M}y^{n_M}}{h}$, where $s_M \in D \setminus M$ and $h \in D$ is such that $hD_M = ((x^{n_M}, y^{n_M})D_M)_v$. But how we can manage an $f_M \in D\setminus M$ such that $f_M((x^{n_M}) \cap (y^{n_M})) \subseteq a_MD$, when $(x^{n_M}) \cap (y^{n_M})$ is not of finite type, is elusive at best. Of course, there are examples of locally GCD domains that are not PvMDs (cf. [13, Example 3.11]) in which $(x) \cap (y)$ is locally principal, despite the fact that $(x) \cap (y)$ is not of finite type. So we end this paper with the following question.

Question 2.14. *Is there a locally AGCD domain that is not an s-locally AGCD domain?*

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