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Splitting sets and weakly Matlis domains

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Abstract. An integral domain D is *weakly Matlis* if the intersection $D = \cap\{D_P \mid P \in t\text{-Max}(D)\}$ is independent of finite character. We investigate the question of when $D[X]$ or D_S is weakly Matlis.

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Call an integral domain D a *weakly Matlis domain* if D is of finite t-character and no two distinct maximal t-ideals of D contain a nonzero prime ideal. Recently Gabelli, Houston and Picozza [13] have studied polynomial rings over weakly Matlis domains and have shown that in some cases a polynomial ring over a weakly Matlis domain need not be weakly Matlis. The purpose of this paper is to indicate the use of splitting sets and t-splitting sets in the study of polynomial rings over weakly Matlis domains. We show for instance that if $K \subseteq L$ is an extension of fields and X an indeterminate over L, then the polynomial ring over $K + XL[X]$ is a weakly Matlis domain.

Let D be an integral domain with quotient field K and let $F(D)$ be the set of nonzero fractional ideals of D. A saturated multiplicative set S of D is said to be a *splitting set* if for all $d \in D\backslash\{0\}$ we can write $d = st$ where $s \in S$ and $tD \cap kD = tkD$ for all $k \in S$. (When $t, k \in D\backslash\{0\}$ are such that $tD \cap kD = tkD$ we say that t and k are *v-coprime*, because in this case $(t, k)_v = D$.) Splitting sets and their properties important for ideal theory were studied in [1]. Splitting sets have proved to be useful in many situations (see [20]). A saturated multiplicative set S is said to be a *t-splitting set* if for each $d \in D\backslash\{0\}$ we can write $(d) = (AB)_t$ where A and B are integral ideals such that $A_t \cap S \neq \phi$ and $(B, s)_t = D$ for all $s \in S$. Here the subscript v (resp., t) indicates the *v-operation* (resp., *t-operation*) defined on $F(D)$ by $A \mapsto A_v = (A^{-1})^{-1}$ (resp., $A_t = \cup\{F_v \mid F$ a finitely generated nonzero subideal of $A\}$). We shall freely use known facts about the v- and t-operations. A reader in need of a quick review on this topic may consult sections 32 and 34 of Gilmer's book [14]. Let us note for now that a proper integral ideal maximal with respect to being a t-ideal is a prime ideal called a *maximal t-ideal*. We note that if S is a splitting set or t-splitting set, then any prime t-ideal P intersecting S *intersects S in detail*, i.e., every nonzero prime ideal contained in P also intersects S (see [5, Proposition 2.8] and [2, Lemma 4.2]). Thus a splitting or t-splitting set induces a bifurcation of t-$Max(D)$, the set of maximal t-ideals, into those that intersect S (in detail) and those that are disjoint from S. The aim of this article is to show how the splitting sets and t-splitting sets can be used to prove useful results and provide interesting examples. Our focus will be on proving results about and providing examples of weakly Matlis domains which, as defined in [4], are domains D such that every nonzero nonunit is contained in at most a finite number of maximal t-ideals and no two distinct maximal t-ideals contain a common nonzero prime ideal. Indeed, as any nonzero prime ideal contains a minimal prime of a principal ideal which is necessarily a t-ideal, one can require that in a weakly Matlis domain no two maximal t-ideals contain a prime t-ideal.

A domain that satisfies ACC on integral divisorial ideals is called a *Mori domain*. In [13] Houston, Gabelli and Picozza give an example of a semiquasilocal one dimensional Mori domain (and hence a weakly Matlis domain) D such that the polynomial ring $D[X]$ is not a weakly Matlis domain. They also show that if D is a *t-local domain* (i.e., a quasilocal domain with maximal ideal a t-ideal) or a *UMT domain* (i.e., uppers to zero are maximal t-ideals, or, equivalently, t-invertible), then D is a weakly Matlis domain if and only if $D[X]$ is. One aim of this paper is to give a class of examples of one dimensional Mori domains D such that the polynomial ring $D[X]$ is a weakly Matlis domain. We do this by proving Theorem 1. We also provide a family of examples of non-UMT weakly Matlis domains such that polynomial rings over them are again weakly Matlis.

Theorem 1. *Let $K \subseteq L$ be an extension of fields and let T be an indeterminate over L. The domain $D = K + TL[T]$ is a one dimensional Mori domain such that the polynomial ring $D[X]$ is a weakly Matlis domain.*

To facilitate the proof of this theorem we shall need a sequence of lemmas, which will find other uses as well.

Lemma 2. *Let S be a splitting set of D. If B is an integral t-ideal of D, then BD_S is an integral t-ideal. In fact, for a nonzero ideal A of D, $A_tD_S = (AD_S)_t$. If E is an integral t-ideal of D_S, then $E \cap D$ is a t-ideal of D. Consequently a maximal t-ideal of D that is disjoint from S extends to a maximal t-ideal of D_S, and every maximal t-ideal of D_S contracts to a maximal t-ideal of D. Hence t-$Max(D_S) = \{PD_S | P \in t$-$Max(D)$ and $P \cap S = \phi\}$.*

Proof. Only the "consequently" part is new. (See [1, section 3] for the other parts of the proof.) Thus suppose that P is a maximal t-ideal of D such that $P \cap S = \phi$, so PD_S is a proper t-ideal. Suppose that PD_S is not a maximal t-ideal. Let Q be a maximal t-ideal of D_S that properly contains PD_S. Thus $Q \cap D \supsetneq P$, but by the earlier part of the lemma, $Q \cap D$ is a t-ideal which contradicts the maximality of P. Further, if Q is a maximal t-ideal of D_S, then $Q \cap D = \mathcal{P}$ is a prime t-ideal of D. If $\mathcal{P}$ is not a maximal t-ideal, then $\mathcal{P}$ is properly contained in a maximal t-ideal M. There are two cases: $M \cap S = \phi$ and $M \cap S \neq \phi$. In the first case MD_S is a maximal t-ideal properly containing Q which contradicts the maximality of Q. In the second case, let $s \in M \cap S$ and let $p \in \mathcal{P} \backslash \{0\}$. Then since S is a splitting set, $p = s_1 t$ where $s_1 \in S$ and t is v-coprime to every element of S. Since $\mathcal{P} \cap S = \phi$, $t \in \mathcal{P}$. But then $t, s \in M$; so $M \supseteq (t, s)_v = D$, contradicting the assumption that M is a proper t-ideal. □

Note that the proof of Lemma 2 shows that the set t-$Max(D)$ is bifurcated by the splitting set S into two sets: those disjoint from S and those that intersect S in detail.

Lemma 2a. *Let S be a splitting set of a domain D with the following properties:*

(1) Every member of S belongs to only a finite number of maximal t-ideals of D,

(2) Every prime t-ideal intersecting S is contained in a unique maximal t-ideal of D.

Then D_S is a weakly Matlis domain if and only if D is.

Proof. Let D be a weakly Matlis domain and consider D_S. Take a nonzero nonunit $x \in D_S$. Then since S is a splitting set we conclude that $xD_S \cap D = dD$ a principal ideal [1]. Now $dD_S = xD_S$. Since every maximal t-ideal of D_S is extended from a maximal t-ideal of D disjoint from S, and because D, being weakly Matlis, is of finite t-character, we conclude that d and hence x belongs to only a finite number of maximal t-ideals of D_S. Since x is arbitrary, D_S is of finite t-character. Next suppose that there is a prime ideal P of D_S such that P is contained in two distinct maximal t-ideals M_1 and M_2 of D_S. Then $P \cap D \subseteq M_1 \cap D$, $M_2 \cap D$, two distinct maximal t-ideals of D, contradicting the fact that D is a weakly Matlis domain.

For the converse, suppose that D_S is a weakly Matlis domain and let d be a nonzero nonunit of D. Then $d = sr$ where $s \in S$ and r is v-coprime to every member of S. Now s and r being v-coprime do not share any maximal t-ideals. Next the number of maximal t-ideals containing s is finite because of (1) and the number of maximal t-ideals containing r is finite because D_S is weakly Matlis (and by Lemma 2). Next let P be a prime t-ideal of D such that P is contained in two distinct maximal t-ideals M and N. Then $P \cap S = \phi$ by (2). But then both M and N are disjoint from S (for if they intersect S, they intersect S in detail) and so the prime t-ideal PD_S is contained in the two maximal t-ideals MD_S and ND_S contradicting the assumption that D_S is a weakly Matlis domain. □

Proposition 2b. *Let X be an indeterminate over D and let D be of finite t-character. Then D and $D[X]$ are weakly Matlis if and only if for every pair of distinct maximal t-ideals P and Q of D, $P[X] \cap Q[X]$ does not contain a nonzero prime ideal.*

Proof. We know that D is of finite t-character if and only if $D[X]$ is [17, Proposition 4.2] (while this is stated for D being integrally closed, their proof does not use this hypothesis). Also, every maximal t-ideal M of $D[X]$ with $M \cap D \neq (0)$ is of the form $P[X]$ where P is a maximal t-ideal of D [16, Proposition 1.1]. Suppose that for every pair of distinct maximal t-ideals P and Q of D, $P[X] \cap Q[X]$ does not contain a nonzero prime ideal. Let us show that $D[X]$ is weakly Matlis. For this we must show that no two maximal t-ideals of $D[X]$ contain a nonzero prime ideal. Now the complement of the set of maximal t-ideals used in the condition is the set of maximal t-ideals that are uppers to zero and these are height-one prime ideals. But if at least one of a pair of maximal t-ideals is of height one, then obviously the condition is satisfied. So no pair of maximal t-ideals contains a nonzero prime ideal. The condition clearly indicates that D is also weakly Matlis. Conversely if both D and $D[X]$ are weakly Matlis, then the condition is obviously satisfied. □

There are examples of weakly Matlis domains D, such as weakly Krull domains, with the property that for every multiplicative subset S the ring of fractions D_S is weakly Matlis. Recall that an integral domain D is a *weakly Krull domain* if $D = \cap D_P$

is a locally finite intersection of localizations at height-one prime ideals of D. A *Krull domain* is a weakly Krull domain such that the localization at every height one prime ideal is a discrete rank one valuation domain. Now recall from [3] that D is a *weakly factorial domain* (i.e., every nonzero nonunit is a product of primary elements) if and only if every saturated multiplicative subset of D is a splitting set, if and only if D is a weakly Krull domain with zero t-class group. The t-class group is precisely the divisor class group for a Krull domain and a Krull domain with a zero divisor class group is a UFD, and there exist Krull domains that are not UFD's. So, in general, a saturated multiplicative set in a weakly Krull domain is not a splitting set. In other words there do exist weakly Matlis domains D such that D_S is a weakly Matlis domain while S is not a splitting set. There are other examples of weakly Matlis domains D that have nonsplitting saturated multiplicative sets S such that D_S is weakly Matlis. Now the question is: Is there a weakly Matlis domain D such that D_S is not weakly Matlis? The example below answers this question.

Example 2c. Let X and Y be indeterminates over the field of rational numbers $\mathbb{Q}$ and let $T = \mathbb{Q}[[X,Y]]$. The ring $R = \mathbb{Z}_{(p)} + (X,Y)\mathbb{Q}[[X,Y]]$, p a nonzero prime of $\mathbb{Z}$, is an integral domain of the general $D + M$ type [7] and obviously a quasilocal ring with the maximal ideal a principal ideal. Let K be the quotient field of R and let T be an indeterminate over K. Then the ring $S = R + TK[[T]] = \mathbb{Z}_{(p)} + (X,Y)\mathbb{Q}[[X,Y]] + TK[[T]]$ is a quasilocal ring with the maximal ideal a principal ideal. Since a principal ideal is a divisorial ideal and hence a t-ideal, we conclude that S is a t-local ring and hence is a weakly Matlis domain. But if $N = \{p^n\}_{n=0}^{\infty}$, then $S_N = \mathbb{Q}[[X,Y]] + TK[[T]]$ is a GCD domain such that every nonzero prime ideal of S_N contains $TK[T]$; so S_N cannot be a weakly Matlis domain.

A splitting set S of D is called *lcm splitting* if every element of S has an lcm with every member of $D\backslash\{0\}$. A splitting set generated by prime elements is obviously an lcm splitting set. This gives us a Nagata type theorem for weakly Matlis domains.

Lemma 3. *Let S be an lcm splitting set of D generated by primes. Then D_S is a weakly Matlis domain if and only if D is.*

Proof. Note that S satisfies (1) and (2) of Lemma 2a since a nonzero principal prime ideal is a maximal t-ideal. □

Proposition 4. *Let D be an integral domain that contains a splitting set S generated by primes such that $D_S[X]$ is a weakly Matlis domain. Then $D[X]$ is a weakly Matlis domain.*

Proof. The proof follows from the fact that if S is an lcm splitting set in D then S is an lcm splitting set in $D[X]$ [5, Theorem 2.2] and of course that $D_S[X] = D[X]_S$ which in turn makes the proof an application of Lemma 3. □

Proof of Theorem 1. The proof depends upon the fact that $TL[T]$ is a maximal ideal of $D = K + TL[T]$ of height one and every element of $K + TL[T]\backslash TL[T]$ is an associate of an element of the form $1 + Tf(T)$ which being common to both $L[T]$ and $K + TL[T]$

is a product of primes which are of height one and maximal [12]. From this it follows that $\mathcal{S}=K+TL[T]\backslash TL[T]$ is an lcm splitting set generated by primes. Also, because $D_{\mathcal{S}}$ is a one dimensional local domain, $D_{\mathcal{S}}[X]=D[X]_{\mathcal{S}}$ is a weakly Matlis domain, [6, page 389]. Now Proposition 4 applies. □

It is shown in [13] that if D is a UMT domain or a t-local domain, then D is a weakly Matlis domain if and only if the polynomial ring $D[X]$ is. Now a PVMD being an integrally closed UMT domain we conclude that for a PVMD and hence for a GCD domain D being weakly Matlis is equivalent to $D[X]$ being weakly Matlis. We shall use t-splitting sets to bring to light the behind the scenes goings on, in this matter, later. For now we shall show, that even a weakly Matlis domain that is neither t-local nor UMT can have a weakly Matlis polynomial ring. The example has already appeared in section 2 of [19]. So we shall briefly describe this example and let the reader check the details.

Example 5. Let V be a valuation domain of rank >1 and let Q be a nonzero non-maximal prime ideal of V. The domain $R=V+TV_Q[T]$ is a non-UMT weakly Krull domain such that $R[X]$ is a weakly Matlis domain.

Before we start to illustrate this example let us recall that an element x in $D\backslash\{0\}$ is called *primal* if for all $r,s\in D\backslash\{0\}$, $x|rs$ in D implies that $x=uv$ where $u|r$ and $v|s$. An integral domain D is a *Schreier domain* if D is integrally closed and every nonzero element of D is primal. Schreier domains were introduced by P.M. Cohn in [10] where it was shown that a GCD domain is Schreier and that every irreducible element in a Schreier domain is a prime. It was noted in [11, page 424] that if D is a GCD domain, S a multiplicative set in D and X is an indeterminate over D_S, then $D+XD_S[X]$ is a Schreier domain.

Illustration: That R is a Schreier domain that is not a GCD domain (and hence not a UMT domain) can be checked from [19, section 2]. Following [19] let us call $f\in R\backslash\{0\}$ *discrete* if $f(0)$ is a unit in V. Now according to Lemma 2.2 of [19] every nonzero nonunit f of R can be written uniquely up to associates as $f=gd$ where d is a discrete element and g is not divisible by a nonunit discrete element of R. Indeed it is also shown in [19] after Lemma 2.2 that every discrete element is a product of finitely many height-one principal primes. So the set $S=\{d\in R|d \text{ is discrete}\}$ is an lcm splitting set generated by primes. Next, as shown in Lemma 2.4 of [19] $M=R\backslash S$ is a prime t-ideal of R such that MR_M is a prime t-ideal. So, $R_M=R_S$ is t-local and according to [13] $R_S[X]=R[X]_S$ is a weakly Matlis domain. Now Lemma 3 facilitates the conclusion that $R[X]$ is a weakly Matlis domain.

Let us do some analysis here. Our main tool in Lemma 3 is the fact that we can split every nonzero nonunit x of D as a product $x=st$ where s is a finite product of height-one principal primes (coming from an lcm splitting set S) and hence is contained in a finite number of maximal t-ideals and t is not divisible by any such primes, i.e., t is coprime to every member of S. So, if we can show that each t for a general x belongs to at most a finite number of maximal t-ideals such that no two of those maximal t-ideals contain a nonzero prime ideal we have accomplished our task.

Following Theorem 4.9 of [2] we can prove the following lemma similar to Lemma 2.

Lemma 6. *Let D be an integral domain and S a t-splitting set of D. If B is an (integral) t-ideal of D, then BD_S is an (integral) t-ideal of D_S. In fact, for a nonzero ideal A of D, $A_tD_S = (AD_S)_t$. If E is a t-ideal of D_S, then $E \cap D$ is a t-ideal of D. Consequently, if P is a maximal t-ideal of D with $P \cap S = \phi$, then PD_S is a maximal t-ideal of D_S. Hence t-*$\text{Max}(D_S) = \{PD_S | P \in t\text{-Max}(D)$ *and* $P \cap S = \phi\}$.

Indeed the “consequently" part of Lemma 6 can be handled in precisely the same manner as we did in the proof of Lemma 2. For the other parts of the proof the reader may consult [2, Theorem 4.9].

Let S be a t-splitting set of D and let $\tau = \{A_1A_2 \cdots A_n |\ A_i = d_iD_S \cap D\}$ be the multiplicative set generated by ideals that are contractions of dD_S to D for each nonzero $d \in D$. Call τ a *t-complement* of S. Also, let D_τ be the *τ-transform*, i.e., $D_\tau = \{x \in K | xA \subseteq D$ for some $A \in \tau\}$. It is easy to show that $D = D_S \cap D_\tau$ and as shown in Theorem 4.3 of [2], $D_S = \cap D_P$ where P ranges over the maximal t-ideals P of D with $P \cap S = \phi$ and $D_\tau = \cap D_Q$ where Q ranges over the maximal t-ideals Q of D with Q intersecting S in detail.

This discussion leads to the following result.

Lemma 7. *Let S be a t-splitting set of D, $F = \{P \in t\text{-}Max(D) | P \cap S = \phi\}$ and $G = \{Q \in t\text{-}Max(D) | Q \cap S \neq \phi\}$. Suppose that D_S is a ring of finite t-character and every nonzero nonunit of D belongs to at most a finite number of members of G. Then D is a ring of finite t-character. If in addition D_S is a weakly Matlis domain and no two members of G contain a nonzero prime ideal, then D is a weakly Matlis domain. Moreover, if S is a t-splitting set and D is a ring of finite t-character (resp., weakly Matlis domain), then D_S is a ring of finite t-character (resp., weakly Matlis).*

Let D be an integral domain, X an indeterminate over D, and $S = \{f \in D[X] | (A_f)_v$ = $D\}$. It is easy to check that the set S is multiplicative and saturated. Our next result gives an alternate proof to parts of Lemma 2.1 and Proposition 2.2 of [13]; also see Corollary 3.5 of that paper.

Proposition 8. *Let D be an integral domain, X be an indeterminate over D, and $S = \{f \in D[X] | (A_f)_v = D\}$. If $D[X]_S$ is a ring of finite t-character (resp., weakly Matlis domain), then $D[X]$ is a ring of finite t-character (resp., weakly Matlis domain) and so is D. Moreover, if $D[X]$ is a ring of finite t-character (resp., weakly Matlis domain), then so is $D[X]_S$.*

Proof. It has been shown in [8, Proposition 3.7] that S is a t-splitting set. So all we have to do for the first part is to check that the requirements of Lemma 7 are met. For this set $F = \{P \in t\text{-}Max(D[X]) | P \cap S = \phi\}$ and $G = \{Q \in t\text{-}Max(D[X]) | Q \cap S \neq \phi\}$. Now every nonzero nonunit of $D[X]$ belongs to at most a finite number of members of G, because every nonzero nonunit of $D[X]$ belongs to at most a finite number of uppers to zero. The other requirement is met automatically because the members of G are all height-one primes. □

The bifurcation induced by the t-splitting set $S = \{f \in D[X] | (A_f)_v = D\}$ does indeed shed useful light on the construction $D[X]_{N_v}$ where $N_v = S = \{f \in D[X] | (A_f)_v = D\}$ by B.G. Kang [18]. For details the reader may consult [8] and [9].

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