ASSOCIATED PRIMES OF PRINCIPAL IDEALS

J. W. BREWER AND W. J. HEINZER

If A is an ideal of the commutative ring R, then there are several ways of associating prime ideals of R to A. There are compelling reasons, however, why the "weakly" associated primes of Bourbaki are the most natural and in accord with Lazard [10; p. 92], "Cela (weakly) nous semble inutile, car, dans le cas noetherian, elles redonnent les notion classiques, et, dans le cas général, les notion classiques ont trés peu d'interêt." we shall call these primes the associated primes of A. Thus, a prime ideal P of R is said to be an *associated prime* of A if there exists $b \in R\backslash A$ with P a minimal prime of $A : bR$. This paper is concerned with the associated primes of principal ideals of R and of the polynomial ring $R[X]$. We shall prove that associated primes of regular elements, that is, nonzero-divisors, are well-behaved; for an integral domain D, using the representation $D = \cap\{D_{P_\alpha} \mid P_\alpha$ is an associated prime of a principal ideal of $D\}$, we prove a theorem on the finiteness of the ideal transform. It is also shown that the associated primes of regular elements of $R[X]$ are closely tied to associated primes of regular elements of R. This enables us to prove the stability under polynomial extension of a noetherian-like property of integral domains which is then used to obtain a result about "locally polynomial rings".

R will always denote a commutative unitary ring. A prime ideal P of R is called a *maximal prime* of the ideal A if P is maximal within the zero-divisors on R/A [9; p. 3]. A prime ideal Q of R is called a *prime divisor* of A if there exists a multiplicative system S in R such that QR_S is a maximal prime of AR_S. The maximal primes of A are precisely the maximal elements of the set of prime divisors of A [12; p. 19].

Our notation will be essentially as in [9].

We list here some facts we shall need and begin with perhaps the most useful fact about associated primes. They behave well under localization.

LEMMA 1 [11; p. 17, Proposition 5]. *Let A be an ideal of R with P a prime ideal of R. Assume that S is a multiplicative system in R and that $P \cap S = \varnothing$. If P is an associated prime of A, PR_S is an associated prime of AR_S. Conversely, if PR_P is an associated prime of AR_P, then P is an associated prime of A.*

It can happen that P is a maximal prime of an ideal A but PR_P does not consist of zero-divisors on AR_P [14], [3], and thus PR_P is not a maximal prime of AR_P. However, if we assume that an ideal has only finitely many associated primes, then the prime divisors and the associated primes coincide.

Received July 9, 1973. The authors were supported in part by grants from the National Science Foundation.

LEMMA 2 [7; p. 281, Proposition 3.5]. *Let A be an ideal of R and assume that A has only a finite number of associated primes. Then the set of prime divisors of A is precisely the set of associated primes of A.*

We come now to the first key result about associated primes of principal ideals.

THEOREM 3. *Let P be a prime ideal of R and assume that P is an associated prime of a regular element $x \in R$. Then P is an associated prime of each regular element contained in P.*

Proof. By localizing at P we may assume that R is quasi-local with maximal ideal P. There exists $z \in R$ such that P is a minimal prime of $xR : zR$. Since P is maximal, $P = \sqrt{(xR : zR)}$. Thus, if $y \in P$, there exists a positive integer n so that $y^n \in xR : zR$ and we may assume that n has been chosen minimally. There exists $d \in R$ such that $y^n z = xd$ and if y is a regular element, then $yR : dR \neq R$. By [12; p. 40, (12.2)], $P \supseteq [yR : dR] = [xR : y^{n-1}zR] \supseteq [xR : zR]$ and it follows that $P = \sqrt{(yR : dR)}$. Q.E.D.

We remark that "associated prime" cannot be replaced by "maximal prime" in the statement of Theorem 3 for Hochster [8] has given an example of a quasi-local domain D with maximal ideal M containing nonzero elements x and y such that M is a maximal prime of xD but M is not a maximal prime of yD.

Both regularity assumptions of Theorem 3 are also essential. For this, let K be a field with X_1 and X_2 indeterminates. Set $R = K[[X_1, X_2]]/(X_1^2)$. Then $(X_1, X_2)/(X_1^2)$ is the maximal ideal of R and since (X_2, X_1^2) is primary for (X_1, X_2) in $K[[X_1, X_2]]$, M consists of zero-divisors on $X_2 + (X_1^2)$ in R. From Lemma 2 and [12; p. 20, (7.5)] it follows that in a noetherian ring the prime divisors and associated primes of an ideal coincide. Thus, M is an associated prime of $X_2 + (X_1^2)$, a regular element of R. However, M is not an associated prime of the zero-divisor $X_1 + (X_1^2)$ since $X_2 + (X_1^2)$ is not a zero-divisor on $X_1 + (X_1^2)$. Now let X_3 and X_4 be two additional indeterminates over K and consider $(K(X_1, X_2, X_3, X_4])_{(X_1, X_2, X_3, X_4)}$. Taking an appropriate homomorphic image and using the method of Gulliksen [5], one can show that an easy calculation yields an example of a local (noetherian) ring with maximal ideal M such that M is an associated prime of a zero-divisor but there exists a regular element $r \in M$ such that M is not an associated prime of r. (In fact, in conjunction with P. Montgomery, we have used Gulliksen's method to construct an example of the following kind. For each positive integer n there exists a local Macaulay ring R_n with maximal ideal M_n such that the dimension of R_n is n and M_n is an associated prime of a principal ideal.)

Theorem 3 has the following useful application.

PROPOSITION 4. *Let D be an integral domain with $\{P_\alpha\}$ the collection of associated primes of principal ideals of D. Then $D = \bigcap_\alpha D_{P_\alpha}$. Assume further that each principal ideal of D has only a finite number of associated primes. Then*

$D = \bigcap_\alpha D_{P_\alpha}$ *is a locally finite intersection. Moreover, if* S *is a multiplicative system of* D, *then* $D_S = \bigcap_\alpha \{D_{P_\alpha} \mid P_\alpha \cap S = \emptyset\}$.

Proof. Clearly, $D \subseteq \bigcap_\alpha D_{P_\alpha}$. Suppose that $a/b \in \bigcap_\alpha D_{P_\alpha}$. It suffices to show that $bD : aD = D$. Assuming that $bD : aD < D$, let P be a minimal prime of $bD : aD$. By definition, P is an associated prime of bD and so $a/b \in D_P$. Since b divides a in D_P, $bD_P : aD_P = D_P$ which is impossible since $PD_P \supseteq [bD : aD]D_P = bD_P : aD_P$.

Under the assumption that principal ideals of D have only a finite number of associated primes, let $x \in D$, $x \neq 0$. By Theorem 3, if $x \in P_\gamma$ for some γ, then P_γ is an associated prime of x. Since x has only finitely many associated primes, it follows that x is a unit in all but a finite number of the D_{P_α}.

Finally, by the first part of the proposition, $D_S = \cap \{(D_S)_{P_\gamma D_S} \mid P_\gamma D_S$ is an associated prime of a principal ideal of $D_S\} = \cap \{D_{P_\gamma} \mid P_\gamma \cap S = \emptyset$ and P_γ is an associated prime of a principal ideal of $D\}$ by Lemma 1. Q.E.D.

It is shown in [9; p. 34] that an integral domain D can also be represented as $D = \cap \{D_{P_\alpha} \mid P_\alpha$ is a maximal prime of a principal ideal of $D\}$. If principal ideals of D have only finitely many associated primes, then the maximal primes of principal ideals are associated primes of principal ideals and so in this case the first part of Proposition 4 follows from the above representation. However, if one uses only the maximal primes in the representation of D, then the good behavior under localization is lost. For example, if D is a two-dimensional non-Macaulay local domain, then the maximal ideal of D is an associated prime of a principal ideal, and therefore the representation in terms of maximal primes is just $D = D$. But as we shall demonstrate in the sequel, the good behavior under localization of the associated prime representation has some utility. On the other hand, the representation in terms of maximal primes has the following nice property.

PROPOSITION 5. *Assume that each principal ideal of the integral domain* D *has only a finite number of maximal primes and that each maximal prime of a principal ideal is an associated prime of a principal ideal. (This holds, for example, if principal ideals have only finitely many associated primes.) Then the representation* $D = \cap \{D_{P_\alpha} \mid P_\alpha$ *is a maximal prime of a nonzero principal ideal*$\}$ *is irredundant.*

Proof. Let P be a maximal prime of the principal ideal xD, $x \neq 0$. By hypothesis, xD has only finitely many maximal primes, so let $Q_1, \cdots, Q_n$ be its maximal primes distinct from P. Choose $d \in P \backslash (\bigcup_1^n Q_i)$ and let $y \in D \backslash xD$ be such that $dy \in xD$. Note that $xD : yD \not\subseteq Q_i$ for each i. But $xD : yD$ is contained in some maximal prime of xD and hence must be contained in P. By Theorem 3, $y/x \in \cap \{D_{P_\alpha} \mid P_\alpha$ is a maximal prime of a principal ideal, $P_\alpha \neq P\}$, for $y/x = d_1/d$ some $d_1 \in D$ and $y/x \in D_{P_\alpha}$ if P_α is not a maximal prime of xD and $d_1/d \in D_{P_\alpha}$ if $P_\alpha = Q_i$ for some i. Q.E.D.

The somewhat contrived hypothesis in Proposition 5 that each maximal prime of a principal ideal be an associated prime of a principal ideal appears to be necessary in view of the example of Hochster mentioned above.

We give now the promised application of the localization property of Proposition 4. Recall that if D is an integral domain with quotient field K and if A is an ideal of D, then $T = \bigcup_{n=1}^{\infty} A^{-n} = \{x \in K \mid xA^n \subseteq D \text{ for some } n\}$ is called the *A-transform* of D. The A-transform of D is said to be *finite* if T is a finitely generated ring extension of D. Finiteness of the A-transform of D was studied by Nagata in connection with his work on Hilbert's 14-th problem. Nagata showed in [13; p. 199] that for D a pseudo-geometric (noetherian) integrally closed domain and A a nonzero ideal of D, the A-transform of D is finite if and only if the AD_M-transform of D_M is finite for each maximal ideal M of D. The following theorem is an extension of this result.

THEOREM 6. *If D is a noetherian domain and A is a nonzero ideal of D, then the A-transform of D is finite if and only if the AD_M-transform of D_M is finite for each maximal ideal M of D.*

Proof. Let T denote the A-transform of D and let M be a maximal ideal of D. Since A is finitely generated, $T_{D\setminus M}$ is the AD_M-transform of D_M. From this it follows immediately that if the A-transform of D is finite, then the AD_M-transform is finite. To prove the converse it will suffice to show that for each maximal ideal M of D there exists $s \in D\setminus M$ such that $T[1/s]$ is a finitely generated ring extension of $D[1/s]$. Since $T_{D\setminus M}$ is a finitely generated ring extension of D_M, we can choose $t_1, \cdots, t_n \in T$ such that $T_{D\setminus M} = D_M[t_1, \cdots, t_n]$. If $D[t_1, \cdots, t_n] = R$, then R is noetherian and hence $R = \bigcap \{R_{P_\alpha} \mid P_\alpha \text{ is an associated prime of a principal ideal of } R\}$. Let a be a nonzero element of A; then $T \subseteq D[1/a] \subseteq R[1/a]$ and so $T \subseteq R_{P_\alpha}$ if P_α is not of the finite number of associated primes of aR. Let $P_1, \cdots, P_m$ be those associated primes of aR having the property that $T \nsubseteq R_{P_i}$. We observe that $P_i \cap D \nsubseteq M$, for $P_i \cap D \subseteq M$ implies that $D_M[t_1, \cdots, t_n] \subseteq R_{P_i}$ and $T_{D\setminus M} = D_M[t_1, \cdots, t_n]$, so $T \subseteq R_{P_i}$. Thus, $P_1, \cdots, P_m$ are precisely the associated primes of principal ideals of R such that $T \nsubseteq R_{P_i}$ and we can choose $s \in (P_1 \cap \cdots \cap P_m \cap D)\setminus M$. It follows that $T \subseteq R[1/s] = \bigcap_\beta \{R_{P_\beta} \mid P_\beta \text{ is an associated prime of a principal ideal of } R \text{ and } s \notin P_\beta\}$. Hence, $T[1/s] = R[1/s] = D[1/s, t_1, \cdots, t_n]$ is a finitely generated ring extension of $D[1/s]$, which completes the proof of the theorem. Q.E.D.

We now turn our attention to polynomial rings. Thus, let X denote an arbitrary collection of indeterminates.

THEOREM 7. *If P is an associated prime of (0) in $R[X]$, then $P = Q[X]$ for some associated prime Q of (0) in R.*

Proof. Let $Q = P \cap R$. By localizing at $R\setminus Q$ we may assume that R is quasi-local with maximal ideal Q. But then P cannot properly contain $Q[X]$, for no element in $R[X]\setminus Q[X]$ can be a zero-divisor. As for the second assertion, first observe that $P = \sqrt{((0) : hR[X])}$ for some nonzero polynomial $h \in R[X]$. To see this, there does exist $h \in R[X] \setminus (0)$ such that P is a minimal prime of $(0) : hR[X]$. Moreover, if P_1 is a minimal prime of $(0) : hR[X]$, then P_1 is an associated prime of (0) by definition. By the first part of the theorem, $P_1 = Q_1[X]$ for some prime ideal Q_1 of R. But $Q_1 \subseteq Q$ and so $Q_1[X] \subseteq Q[X] = P$. From this it follows that $Q = \sqrt{([(0) : hR[X]] \cap R)}$. Let a be a nonzero coefficient of h. Then $Q \supseteq (0) : aR \supseteq [(0) : hR[X]] \cap R$ and so $Q = \sqrt{((0) : aR)}$. Thus, Q is an associated prime of (0). Q.E.D.

Note that the corresponding statement is not valid for maximal primes. To prove this it suffices to find a quasi-local ring R having a maximal ideal M such that M consists entirely of zero-divisors but some finitely generated ideal of R has zero annihilator. For let $a_0, \cdots, a_n \in M$ be such that $\text{Ann}(a_0, \cdots, a_n) = (0)$. Then $a_0 + a_1X + \cdots + a_nX^n \in M[X]$ is a regular element of $R[X]$ and so no maximal prime of (0) in $R[X]$ can lie over M. But M is the unique maximal prime of (0) in R. To exhibit the R, consider $(K[X, Y])_{(X,Y)} = D$, $I = (X, Y)$ and proceed as in [9; pp. 62–63, *Exercises* 6–7], viz., let E be the D-module $\bigoplus \{D/P \mid P$ is a rank one prime of $D\}$. Then I is the set of zero-divisors on E and if one takes R to be the idealization of E, R provides the desired example.

COROLLARY 8. *Suppose that P is an associated prime of a regular element of $R[X]$ and let $Q = P \cap R$. If Q contains a regular element, then $P = Q[X]$ and Q is an associated prime of a regular element. Thus, if $R = D$ is an integral domain and P is an associated prime of a nonzero polynomial in $D[X]$, then $P \cap D = (0)$ or $P = (P \cap D)[X]$ and $P \cap D$ is an associated prime of a principal ideal.*

Proof. Let $a \in Q$, a regular. Then $a \in P$ and by Theorem 3, P is an associated prime of a. Thus, there exists $h \in R[X]$ such that P is a minimal prime of $aR[X] : hR[X]$. So, in $R[X]/aR[X] \simeq (R/aR)[X]$ the image of P, say $\bar{P}$, is an associated prime of (0). By Theorem 7, $\bar{P}$ is the extension to $(R/aR)[X]$ of an associated prime of (0) in R/aR, say $\bar{P}$ is the extension of Q/aR, Q prime in R. From this we have that Q is an associated prime of aR and that $P = Q[X]$. Q.E.D.

COROLLARY 9. *Let D be an integral domain and let X be a collection of indeterminates over D. If principal ideals of D have only finitely many associated primes, the same is true of $D[X]$.*

Proof. Let $f \in D[X] \setminus (0)$ and let P be an associated prime of f. If $Q = P \cap D = (0)$, then P is the contraction of an associated prime of f in $K[X]$, where K denotes the quotient field of D. Since $K[X]$ is a UFD, f has only finitely many associated primes in $K[X]$. If $Q \neq (0)$, then $P = Q[X]$ where Q is an associated prime of one, and hence all, of its regular elements. If a is a nonzero coefficient of f, then $a \in Q \setminus (0)$ and so Q is an associated prime of a. But by hypothesis, there are only finitely many such Q's. Q.E.D.

If $D \subseteq R$ are integral domains, then following [2] we shall say that R is *locally a polynomial ring* over D if for each maximal ideal M of D, $R_{D \setminus M}$ is a polynomial ring over D_M. Even if the transcendence degree of R over D is one, then R locally a polynomial ring over D does not imply that R is a finitely generated ring extension of D; for example, as in [2; p. 425], let D be the ring of integers and let $R = D[\{X/p \mid p \text{ is a prime integer}\}]$, with X an indeterminate. In this example R is not even contained in a finitely generated ring extension of D. However, if D is a noetherian domain and R is locally a polynomial ring over D which is contained in a finitely generated ring extension of D, then R itself is a finitely generated ring extension of D [6; p. 377]; moreover, as remarked in [6; p. 379], the noetherian assumption on D in this result can be relaxed to the condition that polynomial rings over D have the properties of Proposition 4. Therefore, Corollary 9 yields the following.

PROPOSITION 10. *Let D be an integral domain having the property that principal ideals in D have only finitely many associated primes. If R is locally a polynomial ring over D and is contained in a finitely generated ring extension of D, then R is a finitely generated ring extension of D.*

It would be nice if Proposition 10 were true under the assumption that for each maximal ideal M of D, R_{D-M} is merely a finitely generated ring extension of D_M rather than a polynomial ring over D_M. If the noetherian-like condition on principal ideals assumed in Corollary 9 were preserved under finitely generated ring extensions, then this could be shown. However, the condition need not be preserved as the following example shows.

Example. We present an example of a unique factorization domain D (and thus a domain in which principal ideals have only a finite number of associated primes) and a finite integral extension R of D such that some principal ideal of R has an infinite number of associated primes. Let k be a field of characteristic 0 and let X and Y be indeterminates. There exists an algebraic field extension L of $k(X, Y)$ such that the integral closure, D, of $k[X, Y, X^{-1}, Y^{-1}]$ in L has the following properties. D is a non-noetherian, two-dimensional unique factorization domain such that for each maximal ideal M of D, D_M is a two-dimensional regular local ring and $MD_M = (M \cap k[X, Y])D_M$. One way to realize this is to take D to be the group ring over k of a nonfinitely generated torsion-free abelian group of type $(0, 0, \cdots)$ and rank 2 [1], [4]. Since D is not noetherian, some maximal ideal M of D is not finitely generated. But MD_M is finitely generated, and so M cannot be the radical of a finitely generated ideal. Hence, there must exist infinitely many maximal ideals of D lying over $M \cap k[X, Y] = (\alpha, \beta)k[X, Y]$. Therefore, D is a two-dimensional UFD having elements α, β such that the ideal $(\alpha, \beta)D$ is contained in an infinite number of maximal ideals of D and for each maximal ideal P of D which contains (α, β) we have that $(\alpha, \beta)D_P = PD_P$. Let $R = D[\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}, \beta^{\frac{1}{2}}, \alpha^{3/2}]$. Then the principal ideal $\beta^{\frac{1}{2}}R$ has an infinite number of associated primes. For $\beta^{\frac{1}{2}}R : \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}R$ contains (α, β), is contained in each maximal ideal of R lying over $(\alpha, \beta)k[X, Y]$, and therefore has an infinite number of minimal primes.

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Brewer: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KANSAS 66045

Heinzer: DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907