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CONDENSED DOMAINS AND THE $D + XL[X]$ CONSTRUCTION

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Abstract. Let D be an integral domain with quotient field K and let $\mathcal{I}(D)$ be the set of nonzero ideals of D. Call, for $I, J \in \mathcal{I}(D)$, the product IJ of ideals *condensed* if $IJ = \{ij | i \in I, j \in J\}$. Call D a *condensed domain* if for each pair I, J the product IJ is condensed. We show that if a, b are elements of a condensed domain such that $aD \cap bD = abD$, then $(a, b) = D$. It was shown in [14] that a pre-Schreier domain is a $*$-domain, i.e., D satisfies $*$: For every pair $\{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n$ of sets of nonzero elements of D we have $(\cap(a_i))(\cap b_j) = \cap(a_ib_j)$. We show that a condensed domain D is pre-Schreier if and only if D is a $*$-domain. We also show that if $A \subseteq B$ is an extension of domains and $A + XB[X]$ is condensed, then B must be a field and A must be condensed and in this case $[B : K] < 4$. In particular we study the necessary and sufficient conditions for $D + XL[X]$ to be condensed, where D is a domain and L an extension field of K. It may be noted that if D is not a field $D[X]$ is never condensed. So for D condensed $D + XK[X]$ is a way of constructing new condensed domains from old.

Let D be an integral domain with quotient field K and let $\mathcal{I}(D)$ be the set of nonzero ideals of D, throughout. Call, for $I, J \in \mathcal{I}(D)$, the product IJ of ideals *condensed* if $IJ = \{ij | i \in I, j \in J\}$. We may call the ideals I, J a *condensed pair* if IJ is condensed. Call D a *condensed domain* if for each I, J the product IJ is condensed. While we are at it, let's call an element a *subtle* if $a \in IJ$ implies that $a = ij$ where $i \in I$ and $j \in J$. An element $a \in D$ is called irreducible or an atom if a is a nonzero non unit such that $a = xy$ implies x is a unit or y is, We show that if D is condensed a an atom and $b, c \in D$ with $(b, c) = D$, then $(a, b) = D$ or $(a, c) = D$. We also show that if a, b are elements of a condensed domain such that $aD \cap bD = abD$, then $(a, b) = D$. Call $x \in D\backslash\{0\}$ primal if for all $y, z \in D\backslash\{0\}$ $x|yz$ implies $x = rs$ where $r|y$ and $s|z$. A domain all of whose nonzero elements are primal was called a *pre-Schreier* domain in [14]. It was shown in [14] that a pre-Schreier domain D is a $*$-domain, i.e., D satisfies $*$: For every pair $\{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n$ of sets of nonzero elements of D we have $(\cap(a_i))(\cap b_j) = \cap(a_ib_j)$. We show that a condensed domain D is pre-Schreier if and only if D is a $*$-domain. We also show that if $A \subseteq B$ is an extension of domains and $A + XB[X]$ is condensed, then B must be a field and A must be condensed and in this case $[B : K] < 4$. In particular we study the necessary and sufficient conditions for $D + XL[X]$ to be condensed, where D is a domain and L an extension field of K. It may be noted that if D is not a field $D[X]$ is never condensed. So for D condensed $D + XK[X]$ is a way of constructing new condensed domains from old.

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This paper is my nth swan song and may well be my last.

Our basic tools come from the notion of star operations, as introduced in sections 32 and 34 of Gilmer's [12]. For our purposes we provide below a working introduction. Let D be an integral domain with quotient field K and let $F(D)$ denote the set of fractional ideals of D. Denote by A^{-1} the fractional ideal $D :_K A = \{x \in K | xA \subseteq D\}$. The function $A \mapsto A_v = (A^{-1})^{-1}$ on $F(D)$ is called the v-operation on D (or on $F(D)$). Associated to the v-operation is the t-operation on $F(D)$ defined by $A \mapsto A_t = \cup\{H_v |\ H$ ranges over nonzero finitely generated subideals of $A\}$. The v- and t-operations are examples of the so called star operations. Indeed $A \subseteq A_t \subseteq A_v$. A fractional ideal $A \in F(D)$ is called a v-ideal (resp., a t-ideal) if $A = A_v$ (resp., $A = A_t$) a v-ideal (resp., a t-ideal) of finite type if there is a finitely generated ideal B such $A = B_v$ (resp., $A = B_t$). An integral t-ideal maximal among integral t-ideals is a prime ideal called a *maximal* t-ideal. If A is a nonzero integral ideal with $A_t \neq D$ then A is contained in at least one maximal t-ideal. A prime ideal that is also a t-ideal is called a prime t-ideal. Every height one prime ideal is a t-ideal. Call $I \in F(D)$ v-invertible (resp., t-invertible) if $(II^{-1})_v = D$ (resp., $(II^{-1})_t = D$). A prime t-ideal that is also t-invertible was shown to be a maximal t-ideal in Proposition 1.3 of [13, Theorem 1.4]. Two elements $a, b \in D$ are said to be v-coprime if $(a, b)_v = D$. Indeed a, b are v-coprime if and only if a, b share no maximal t-ideals, if and only if $aD \cap bD = abD$.

Let X be an indeterminate over K. Given a polynomial $g \in K[X]$, let A_g denote the fractional ideal of D generated by the coefficients of g. A prime ideal P of $D[X]$ is called a prime upper to 0 if $P \cap D = (0)$. Thus a prime ideal P of $D[X]$ is a prime upper to 0 if and only if $P = h(X)K[X] \cap D[X]$, for a prime h in $K[X]$. It follows from [13, Theorem 1.4] that P a prime upper to zero of D is a maximal t-ideal if and only if P is t-invertible if and only if P contains a polynomial f such that $(A_f)_v = D$. A domain D all of whose prime uppers to zero are maximal t-ideals called a *UMT domain*, [13, Theorem 1.4]. Our terminology is standard as in [12] or is defined at the point of entry of the notion. We plan to split the paper into two sections. In Section 1, we collect basic properties of condensed domains, some of which are known, some known with simpler proofs and some new. Anderson and Dumitrescu in [2] studied condensedness for domains of the form $K + X^r L[[X]]$ where $K \subseteq L$ is an extension of fields. In Section 2, we study when a ring of the form $A + XB[X]$ is condensed and find the necessary and sufficient conditions for $D + XL[X]$ to be condensed, where L is an extension field of K. In particular we show that D is condensed if and only if $D + XK[X]$ is condensed.

1. Basics

An integrally closed pre-Schreier domain was originally called a Schreier domain in [6] where it was indicated that the group of divisibility of a Schreier domain is a Riesz group. Since the conclusion was based on the fact that the nonzero elements of a Schreier domain are primal, one can conclude that the group of divisibility of a pre-Schreier domain is a Riesz group too. In an earlier version of [14] this author indicated that one of the group theoretic characterizations of Riesz groups did not translate to domains as a characterization of pre-Schreier domains. The reason, in this author's opinion was the difference between the notions of products of ideals in semigroups and in rings, see Section 2 of [14]. This observation was related in an earlier version of [14]. Following the lead from that earlier version, D. F. Anderson and D. E. Dobbs [5] introduced the concept of a condensed integral domain, as defined in the introduction of this paper, see Corollary 2.6 of [5]. They showed that D is condensed if and only if every pair of two generated ideals of D are a condensed pair, if and only if every pair of finitely generated ideals is a condensed pair, and that every overring of a condensed domain is condensed. They also showed that a condensed domain D has $Pic(D) = 0$. Also, they showed that if a domain D is not a field then $D[X]$ is not condensed and that if F is a field $F[[X^2X^3]]$, is a condensed domain. Later, Anderson, J. T. Arnold and Dobbs [4] showed that an integrally condensed domain is Bezout. A number of other researchers have worked on concepts related to condensedness. An interested reader may find [2] a good source of information on this topic.

Lemma Let D be a condensed domain and let a be an atom in D.

(1) If b, c are co-maximal non-units of D, then a is co-maximal with b or with c.
(2) a belongs to a unique maximal ideal of D.

Proof. 1. (1) Let $I = (a, b)$ and $J = (a, c)$. Then $IJ = (a^2, ac, ab, bc) = (a, bc)$. Because D is condensed, $a = (ra + sb)(ua + vc)$. So, $ua + vc$ is a unit or $ra + sb$ is.

2. (2) If D is quasi local, then, clearly, a belongs to a unique maximal ideal. So let's assume that D is non-local and that a belongs to two maximal ideals M and N. Let $m \in M\backslash N$. So that $mD + N = D$. That is for some $n \in N$ we have m, n co-maximal. By (1), a is either co-maximal with m or with n. But that is impossible because a belongs to both M and N. Whence a belongs to a unique maximal ideal. □

As we shall see below v-coprime elements are co-maximal in a condensed domain. For this we begin by recalling from [8] some terminology. By an overring of D we mean a ring between D and its quotient field K. Let $D \subseteq R$ be an extension of domains. Then R is said to be t-linked over D if for each nonzero ideal I of D with $I^{-1} = D$ we have $(IR)^{-1} = R$ and D is t-linkative if every overring of D is t-linked over D.

Lemma 1.1. *Let D be condensed and let a, b be two nonzero non units of D. Then the following hold. (a) If $(a, b)_v = D$, then $(a, b) = D$. and (b) If I is a t-invertible ideal of D, then I is invertible and hence principal.*

Proof. (a) Every overring of a condensed domain is condensed by [5] and an integrally closed condensed domain is Bezout by [4], as already noted. So the integral closure of a condensed domain is Bezout, hence Prufer. Thus D is a t-linkative UMT domain, by Theorem 2.4 of [10] and every maximal ideal of D is a t-ideal by Lemma 2.1 of [10]. Now let $(x, y)_v = D$. Claim that $(x, y) = D$. For if not then (x, y) is contained in a maximal ideal M of D. But then $D = (x, y)_v \subseteq M$ a contradiction. For (b), let $II^{-1} \neq D$. Then, being a proper integral ideal, II^{-1} is contained in a maximal ideal M. Now because the integral closure of D is Bezout every maximal ideal of D is a maximal t-ideal as already noted. But then $II^{-1} \subseteq M$ gives a contradiction as above. Whence I is invertible. But an invertible ideal in a condensed domain is principal, by Proposition 2.5 of [5]. □

There is another interesting application of the above observations. But let us first record a simple fact which may be folklore, though I have not seen it.

Proposition 1. *An atom a in a domain D is a prime if and only if for all $b \in D$, $a \nmid b$ implies $(a,b)_v = D$. Consequently, an atom in a condensed domain is a prime if and only if for all $b \in D, a \nmid b$ implies $(a,b) = D$.*

Proof. Suppose for all $b \in D$ $a \nmid b$ implies $(a,b)_v = D$. Then for all $x, y \in D$ $a|xy$ implies a divides x or $a|y$. For if $a \nmid x$, then $(a,x)_v = D$ by the condition. Yet as $a|xy$ we have $(a) = (a, xy)$. This implies $(a) = (a,xy)_v = (a, ay, xy)_v = (a,(ay,xy))_v = (a,(a,x)y)_v = (a,(a,x)_v y)_v = (a,y)_v$. Now $(a) = (a,y)_v$ implies $y \in (a)$ which is equivalent to $a|y$. Conversely suppose that a is a prime and $a \nmid b$ for some, chosen, b. Then for each $h \in (a) \cap (b)$ we have $h = bt$ for $t \in D$. Since $a \nmid b$ we have $a|t$. But then $t = at'$ for some $t' \in D$ and so, for each $h \in (a) \cap (b)$ we have $h = abt'$. But this means $(a) \cap (b) = (ab)$ or $\frac{(a)\cap(b)}{ab} = D$, or $(a,b)^{-1} = D$ which is equivalent to saying that $(a,b)_v = D$. The "consequently" part follows from the fact that in a condensed domain $(a,b)_v = D$ is equivalent to $(a,b) = D$, by Lemma 1.1. □

The above Proposition can be put to use immediately as follows.

Corollary 1. *The following are equivalent for an atom a in an integral domain D.*

(1) a is a prime,
(2) a generates a maximal t-ideal,
(3) if a belongs to a prime ideal P, then a belongs to a maximal t-ideal contained in P
(4) if a belongs to a prime t-ideal P then P is a maximal t-ideal generated by a.

Proof. (1) $\Rightarrow$ (2). Let $\wp = (a) = \{ar | r \in D\}$. Obviously, being a principal ideal (a) is a t-ideal. Let M be a maximal t-ideal containing $\wp$ and let $x \in M \backslash \wp$ and so $a \nmid x$, by construction. But by Proposition 1 $a \nmid x$ implies that $(a,x)_v = D$ and this contradicts the assumption that M is a t-ideal. Whence there is no $x \in M\backslash\wp$ and $\wp = M$ a maximal t-ideal.

(2) $\Rightarrow$ (3). Because $a \in P$ implies that $(a) \subseteq P$ and by (2) (a) is a maximal t-ideal.

(3) $\Rightarrow$ (4). Obvious because $a \in P$ implies that $(a) \subseteq P$ and (a) is a maximal t-ideal. Whence $(a) = P$.

(4) $\Rightarrow$ (1). Obvious because a generates a prime. □

Note here that for an atom a, $a \nmid b$ does not necessarily mean that $(a,b)_v = D$. For example, let D be a one-dimensional (Noetherian) local domain and let a, b be two non-associate atoms. Then $a \nmid b$ yet $(a,b)_v \neq D$ for the following two reasons. First: $a|b^n$ for some positive integer n, because D is quasi-local and one dimensional and $(a,b)_v = D$ if and only if $(a,b^n)_v$ for every positive integer n (cf. [15]). And second: D is a one dimensional quasi-local domain and so its maximal ideal is a t-ideal. For a concrete example note that if F is a field and X an indeterminate over F, then $D = F[[X^2, X^3]]$ is a one dimensional (Noetherian) local domain and of course X^2 and X^3 are two non associate atoms. (This domain is condensed, as already noted.)

Corollary 2. *In each of the following situations every prime element generates a maximal ideal. (a) When every maximal ideal of D is a t-ideal, i.e. when D is t-linkative* [8]. *(b) When D has a Prufer integral closure* [10].

Proof. Observe that (a) is obvious by Theorem 2.6 of [8] and for (b) one can recall from Theorem 2.4 of [10], that D has Prufer integral closure if and only if D is a t-linkative UMT domain. □

Now as we know that the integral closure of a condensed domain is Bezout we have for the record the following corollary.

Corollary 3. *In a condensed domain, every prime element generates a maximal ideal and consequently $D[[X]]$ is a condensed domain if and only if D is a field.*

As already mentioned, Cohn [6] called an integrally closed integral domain D Schreier if each nonzero element of D is primal. A domain whose nonzero elements are primal was called pre-Schreier in [14]. Note that in a pre-Schreier domain every irreducible element (atom) is a prime. (In fact a primal atom in any domain, is prime. For let p be an irreducible element that is also primal and let $p|ab$. So $p = rs$ where $r|a$ and $s|b$, because p is primal. But as p is also an atom, r is a unit or s is a unit. Whence $p|a$ or $p|b$. In studying pre-Schreier domains, I came across a property that I called the property $*$. It was defined in the introduction.

It was shown in Theorem 1.6 of [14] that D is a pre-Schreier domain if and only if for each pair $\{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n$ of sets of nonzero elements of D and for all $x \in D\backslash\{0\}$ $a_ib_j|x$ implies $x = rs$ where $a_i|r$ and $b_j|s$, $i = 1...m$ and $j = 1...n$. This result can be used to prove the following proposition.

Proposition 2. *A domain D is a pre-Schreier domain if and only if D is a $*$-domain such that for every pair $\{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n$ of sets of nonzero elements of D $(\cap(a_i)), (\cap(b_j))$ is a condensed pair.*

Proof. Let D be a pre-Schreier domain. That D is a $*$-domain follows from (1) of [14, Corollary 1.7]. Now let $\{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n$ be a pair of sets of nonzero elements of D such that $a_ib_j|x$. Then $x = rs$ where $a_i|r$ and $b_j|s$. Now as $a_ib_j|x$ if and only if $x \in \cap(a_ib_j)$ and $a_i|r$ and $b_j|s$ if and only if $r \in \cap(a_i)$ and $s \in \cap(b_j)$. Thus by the pre-Schreier property $x \in \cap(a_ib_j$ implies that $x = rs$ where $r \in \cap(a_i)$ and $s \in \cap(b_j)$. But as we already have established that D has the $*$-property, $\cap(a_ib_j) = (\cap(a_i))(\cap(b_j))$. Thus $x \in (\cap(a_i))(\cap(b_j))$ implies that $x = rs$ where $r \in \cap(a_i)$ and $s \in \cap(b_j)$ and $(\cap(a_i)), (\cap(b_j))$ is a condensed pair. For the converse suppose that D is a $*$-domain and for $\{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n \subseteq D$, $(\cap(a_i)), (\cap(b_j))$ is a condensed pair. Because $(\cap(a_i)), (\cap(b_j))$ is a condensed pair, $x \in (\cap(a_i))(\cap(b_j))$ implies that $x = rs$ where $r \in (\cap(a_i))$ and $s \in (\cap(b_j))$. But since D has the $*$-property, $(\cap(a_i))(\cap(b_j)) = \cap(a_ib_j)$. Thus $x \in \cap(a_ib_j)$ implies that $x = rs$ where $r \in (\cap(a_i))$ and $s \in (\cap(b_j))$. which translates to $a_ib_j|x$ implies $x = rs$ where $a_i|r$ and $b_j|s$ and according to Theorem 1.6 of [14] this is the characterizing property of pre-Schreier domains. □

The above Proposition can be used to prove the following result.

Proposition 3. *If D is condensed and a $*$-domain, then D is a pre-Schreier domain.*

Proof. If D is condensed, then every pair of nonzero ideals of D is condensed and so is $(\cap(a_i)), (\cap(b_j))$, for any pair $\{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n$ of sets of nonzero elements of D. But then, being a $*$-domain makes D a pre- Schreier domain. □

Now these simple observations have the following somewhat interesting implications.

Corollary 4. *An atomic condensed domain D is a PID if and only if D has the $*$-property. Consequently a non-integrally closed atomic condensed domain D does not have the $*$-property.*

Proof. Let D be atomic and condensed and suppose that D has the $*$-property. Then, by Proposition 3, D is pre-Schreier. But every atom is a prime in a pre-Schreier domain. So, being an atomic domain, D is a UFD. But then D is integrally closed and an integrally closed condensed domain is Bezout, [4]. Whence D is a PID. Of course a PID has the $*$-property and is condensed. □

Example 1.2. Let K be a field, let X be an indeterminate over K and let $D = K[[X^2, X^3]]$. Then D does not satisfy $*$. The reasons are (a) D is Noetherian, (b) according to [5] D is condensed and (c) D is not integrally closed.

Now recall the "number crunching" I had to do in Example 2.8 of [14] to establish that $K[[X^2, X^3]]$ was not a $*$-domain. (Of course the above approach offers a simpler and direct route compared to the alternate suggested in [14].) There may arise a question here: Is a pre-Schreier domain condensed? The answer is: generally, it is not the case. For example if D is a Schreier domain then it is well known that $D[X]$ is Schreier (cf. [6]) and Schreier is integrally closed pre-Schreier. Now if D is not a field then, as we have noted above (see Proposition 4 below as well), $D[X]$ can never be a condensed domain.

Usually, D having the $*$-property does not mean that D is integrally closed and this is established by the existence of a pre-Schreier domain that is not Schreier [14], yet there are situations where the presence of the $*$-property in D ensures that D is ("more than" integrally closed. Call an integral domain D v-coherent if for each nonzero finitely generated ideal I of D we have I^{-1} a v-ideal of finite type. Also call D a generalized GCD (GGCD) domain if for each pair of nonzero elements of D we have $aD \cap bD$ invertible. It is well known that a GGCD domain is a locally GCD domain, i.e. D_M is a GCD domain for each maximal ideal M, and hence is integrally closed, [1]. So a condensed GGCD domain being Bezout is as given as a a Prufer domain becoming Bezout being condensed. However the following may well be an improvement on Corollary 2.6 of [5]. For this recall that D is a v-finite conductor domain if for every pair of nonzero elements a, b of D the ideal $aD \cap bD$ is a v-ideal of finite type.

Corollary 5. *Let D be a condensed domain that is also a v-finite conductor domain. Then D is a Bezout domain if and only if D is a $*$-domain.*

That a Bezout domain is a $*$-domain follows from the fact that every GCD domain is Schreier [6] and hence a $*$-domain [14, Theorem 3.6]. For the converse note that by Proposition 3 a condensed $*$-domain is pre-Schreier and a pre-Schreier v-finite conductor domain is a GCD domain [14, Theorem 3.6] and a GCD domain is integrally closed.

It may be noted, however, that a condensed v-finite conductor domain, even a condensed Noetherian domain may not be Bezout, as the example of $K[[X^2, X^3]]$ indicates. If you go chasing the facts they will take you further a field, with negative results as it were. Here's a slightly advanced form of Noetherian domains, recently introduced by this author in [17]. Call D a *dually compact domain* (*DCD*) if for each set $\{a_\alpha\}_{\alpha \in I} \subseteq K\backslash\{0\}$ with $\cap a_\alpha D \neq (0)$ there is a finite set of elements $\{x_1, ..., x_r\} \subseteq K\backslash\{0\}$ such that $\cap a_\alpha D = \cap_{i=1}^r x_i D$, or equivalently for each $I \in F(D)$, the ideal

$I_v = (I^{-1})^{-1}$ is a finite intersection of principal fractional ideals of D. Indeed a DCD can be condensed without being Bezout. The reason is that a DC domain will become v-G-Dedekind, only if it is a $*$-domain, as shown in Theorem 3.3 of [17]. Here a domain D is a v-G-Dedekind domain if I_v is invertible for each $I \in F(D)$. But as soon as you add the condensed property, you get a Bezout domain, because a v-G-Dedekind domain is integrally closed. On the other hand make a DC domain as condensed as you want, it won't become Bezout unless it is a $*$-domain.

2. New condensed domains from old

The following is a known result (see e.g. [5]), but our proof may be very simple.

Proposition 4. *Let D be an integral domain and X an indeterminate over D. Then $D[X]$ is condensed if and only if D is a field. Consequently if D is a domain such that D is not a field, then $D[X]$ is never condensed.*

Proof. Certainly X is irreducible and hence, by (2) of Lemma A, must belong to a unique maximal ideal of $D[X]$. But that is possible only if D is a field. (Alternatively note that X is a prime in $D[X]$ and if $D[X]$ is condensed, then X must generate a maximal ideal which is possible only if D is a field.) Conversely if D is a field, then $D[X]$ is PID and hence, obviously, a condensed domain. The consequently part is obvious. □

Proposition 5. *Let $A \subseteq B$ be an extension of domains such that $(A : B) \neq (0)$. If A is condensed, then so is B.*

Proof. Let $I, J \in \mathcal{I}(B)$. Then for some $\alpha, \beta \in [A : B]$ we have $\alpha\beta IJ = (\alpha I)(\beta J)$, where $(\alpha I), (\beta J) \in \mathcal{I}(A)$. So for $x \in IJ$, we have $\alpha\beta x \in \alpha\beta IJ = (\alpha I)(\beta J)$, forcing $\alpha\beta x = rs$ where $r \in \alpha I$ and $s \in \beta J$, because A is condensed. This gives $r/\alpha \in I$ and $s/\beta \in J$. But as $\alpha\beta x = rs$, we have $x = (r/\alpha)(s/\beta)$. □

Proposition 6. *Let $A \subseteq B$ be an extension of domains. If $A + XB[X]$ is a condensed domain, then B is a field and A is a condensed domain.*

Proof. Since $(A + XB[X] : B[X]) \supseteq XB[X]$, we conclude from Proposition 5 that $B[X]$ is condensed. But by Proposition 4, B must be a field. Next, let $I, J \in \mathcal{I}(A)$ and let $a \in IJ$. Since B is a field $I(A + XB[X]) = I + XB[X]$, $J(A + XB[X]) = J + XB[X]$ and $(I + XB[X])(J + XB[X]) = IJ + XB[X]$. Now $a \in IJ \backslash \{0\}$ means $a \in IJ + XB[X] = (I + XB[X])(J + XB[X])$. This means $a = f_1 f_2$ where $f_1 \in (I + XB[X])$ and $f_2 \in (J + XB[X])$, because $A + XB[X]$ is a condensed domain. Now $f_1 = r + Xg_1(X)$ and $f_2 = s + g_2(X)$ where $r \in I$ and $s \in J$. Thus $a = (r + Xg_1(X))(s + Xg_2(X)) = (rs + X(rg_2(X) + sg_1(X)) + X^2 g_1(X)g_2(X))$. Comparing coefficients, $a = rs$ where $r \in I$ and $s \in J$. □

Note that $(A + XB[[X]] : B[[X]]) \supseteq XB[[X]]$ and ideals of $A + XB[[X]]$ are of the form $I + XB[[X]]$ where I is an ideal of D or of the form $X^r JXL[[X]]$ where J is a D-submodule of L (see e.g. Proposition 2.6 of [2]). With reference to Corollary 3 we have the following Corollary.

Corollary 6. *Let $A \subseteq B$ be an extension of domains. If $A + XB[[X]]$ is a condensed domain, then B is a field and A is a condensed domain.*

For the converse of Proposition 6 we need to digress a little and recall Proposition 3 of [16].

Proposition 7. *Let D be an integral domain and let L be an extension field of the field of fractions K of D. Then each nonzero ideal F of $R = D + XL[X]$ is of the form $f(X)JR = f(X)(J + XL[X])$, where J is a D-submodule of L and $f(X) \in R$ such that $f(0)J \subseteq D$. If F is finitely generated, J is a finitely generated D-submodule of L.*

Using the tail-end part of the proof of the above proposition, we can conclude that if F is a two generated ideal of R, then $F = f(X)(J + XL[X])$ where J is a two generated D-submodule of L and $f(X) \in R$. The following special cases apply:

- (a) If $f(X) = 1$, J is an ideal of D and
- (b) If $f(X)$ is non constant with $f(0) \neq (0)$, J is still a fractional ideal of D. By replacing f by $\frac{1}{d}f$ we can assume that J is an ideal of D (as in that case $f(0) = 1$). Because $f(0) = 1$ we have $f(X) \in R$ and J is an ideal (since $f(0)J \subseteq D$) and so the case (b) reduces to case (a). This leaves the case of
- (c) for $f(0) = 0$. If $f(0) = 0$, then $f(X) = X^r g(X)$ where $r > 0$ and $g(0) = 1$. (We can assume that because if $g(0) = l \in L\backslash\{0\}$, we can replace the generators j_i of J by j_i/l). Now suppose that D is condensed and we want to show that $R = D + XL[X]$ is condensed. By Theorem 1 of [5], we need to show that the product of any pair A, B of nonzero 2-generated (or finitely generated) ideals of R is condensed. But the general $D + XL[X]$ case may be hard, as indicated in [2]. So, let's take care of the simpler cases before attacking the harder one(s). The first of the simpler cases is tackled in the following Lemma.

Lemma 2.1. *If $A = X^r g(X)L[X]$, where $g(0) = 1$ then the pair A, B is condensed for any ideal B of $R = D + XL[X]$.*

Proof. Indeed if Aa, B is a condensed pair, where $a \in D\backslash 0\}$, then so is A, B. This is because if $x \in (Aa)B$ implies $x = rs$ where $r \in Aa$ and $s \in B$, then $y \in AB$ implies $ya \in (Aa)B$, forcing $ya = r \in Aa$ and $s \in B$ and thus $y = (r/a)s$. That A, B being condensed implying Aa, B being condensed is direct. Consequently we can take $A = XL[X]$. The other ideal could be (a) $A_1 = XL[X]$ or (b) $B = \mathfrak{B} + XL[X]$, where $\mathfrak{B}$ is a nonzero ideal of D or (c) $C = X(\mathfrak{C} + XL[X]$, where $\mathfrak{C}$ is a nonzero D-submodule of K. In case (a,a) we have $AA_1 = X^2L[X]$ and $x \in AA_1$ implies $x = X^2h(X)$ and we can set $x = (X)(Xh(X))$. For the case (a,b) we have $AB = (XL[X])(\mathfrak{B} + XL[X]) = XL[X]$ and $x \in AB$ implies $x = Xh(X)$ where $h(X) \in K[X]$ and we can find $d \in D\backslash\{0\}$ such that $dh(X) \in R$. In this case $x = (X/d)(dXh(X))$ will do, as $X/d \in XL[X]$ and $dXh(X) \in \mathfrak{B} + XL[X]$. Finally, in case (a,c) we have $AC = (XL[X])(X(\mathfrak{C} + XL[X]) = X^2L[X]$ and $x \in AC$ means $x = X^2h(X)$ where $h(X) \in L[X]$. We can find $l \in L\backslash\{0\}$ so that $lh(X) \in (\mathfrak{C} + XL[X]$ and set $x = (X/l)(lh(X)$.

Alternatively, let $x \in (XL[X])B$. Then $x = \sum Xf_ib_i$. Since $f_i \in L[X]$ we can find $l \in L\backslash\{0\}$ such that $lf_i \in R$. But then $x = (X/l)(\sum lf_ib_i)$. Now as $lf_i \in R$ we have $\sum lf_ib_i \in B$. But then we have an expression for x in the required form.

As an application of Lemma 2.1 when considering condensedness of two nonzero ideals I, J of R, we can avoid the cases where one of the ideals if of the form $A = X^r g(X)L[X]$. The following result can be proved as a corollary of a latter result, but we prove it separately for the sake of clarity.

Theorem 2.2. *Let D be a domain, K the quotient field of D and let X be an indeterminate over K. Then D is condensed if and only if $D+XK[X]$ is condensed.*

For a start let us display below the types of ideals that we may expect in our study, with reference to Proposition 4.12 of [7].

- (a) When $f(X) = 1$ we have $A = (\mathfrak{A} + XK[X])$ where $\mathfrak{A}$ is a 2 generated ideal of D and $\mathfrak{A} \neq (0)$, by Lemma 2.1
- (b) When $f(X)$ is such that $f(0) = 1$ we have $B = f(X)(\mathfrak{B} + XK[X])$, where $\mathfrak{B}$ is a nonzero two generated ideal of $D, \mathfrak{B} \neq (0)$, by Lemma 2.1. Since f belongs to R, case (b) reduces to case (a).
- (c) When $f(X) = X^r g(X)$, with $g(0) = 1$, where r is a positive integer and $\mathfrak{C}$ is a nonzero 2-generated fractional ideal of D. But as $X^{r-1}g \in R$ we get $C = X(\mathfrak{C} + XK[X])$

Depending on the types of the 2-generated ideals we need to study the following three cases (a,a), (a,c) (c,c).

- (aa) $A = (\mathfrak{A} + XK[X])$, $A_1 = (\mathfrak{A}_1 + XK[X])$, $AA_1 = (\mathfrak{A}\mathfrak{A}_1 + XK[X]$. Now $x \in AA_1$ implies $x = ij + Xh(X)$, where $i \in \mathfrak{A}$ and $j \in \mathfrak{A}_1$ because the product $\mathfrak{A}\mathfrak{A}_1$ is condensed. We can write $a = i \in \mathfrak{A} + XK[X]$, where $i \in \mathfrak{A}$ and $b = (j + (X/i)h(X)) \in \mathfrak{A}_1 + XK[X]$, here $Xh(X) \in R$ and so $(X/i)h(X) \in XK[X]$.
- (ac) $A = (\mathfrak{A}+XK[X])$, $C = X(\mathfrak{C}+XK[X])$. Then $AC = X(\mathfrak{A}\mathfrak{C}+XK[X])$. Let $x \in AC$. Then $x = X(\gamma + Xh(X))$ where $\gamma \in \mathfrak{A}\mathfrak{C}$. Since $\mathfrak{A}$ is an ideal and $\mathfrak{C} = \mathfrak{I}/\mathfrak{d}$ where $\mathfrak{d}$ is a nonzero element of D, $\mathfrak{A}, \mathfrak{C}$ is a condensed pair and we can write $\gamma = \alpha\beta$ where $\alpha \in \mathfrak{A}$ and $\beta \in \mathfrak{C} = \mathfrak{I}/\mathfrak{d}$. Set $a = \alpha$ and $c = (X(\beta + (X/\alpha)h(X))$. Since $(X/\alpha)h(X) \in XK[X]$ we have $\beta + (X/\alpha)h(X) \in \mathfrak{C} + XK[X]$ and so $X(\beta + (X/\alpha)h(X)) \in X(\mathfrak{C} + XK[X])$. Thus $x = ac = \alpha(X(\beta + (X/\alpha)h(X)))$.
- (cc) $C_1 = X(\mathfrak{C}_1 + XK[X])$, $C_2 = X(\mathfrak{C}_2 + XL[X])$, $C_1C_2 = X^2(\mathfrak{C}_1\mathfrak{C}_2 + XK[X])$. Let $x \in C_1C_2$ and let $\gamma \in \mathfrak{C}_1\mathfrak{C}_2 \backslash \{0\}$. Then $x = X^2(\gamma + Xh(X))$. Here too we must find $\gamma_1 \in \mathfrak{C}_1$ and $\gamma_2 \in \mathfrak{C}_2$ such that $\gamma = \gamma_1\gamma_2$. But this is easy in this case because by Proposition 4.12 of [7], and our assumption that $g(0) = 1$, $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are both fractional ideals of D. So $\mathfrak{C}_i = \frac{I_i}{d_i}$ where I_i are ideals of D and $d_i \in D\backslash\{0\}$. Thus $\gamma = \frac{y}{d_1 d_2}$ and as D is condensed, $y = y_1 y_2$ where $y_i \in I_i$ and so $\gamma = (\gamma_1)(\gamma_2)$ where $\gamma_i = \frac{y_i}{d_i} \in \mathfrak{C}_i$. Set $c_1 = X\gamma_1 \in C_1$ and set $c_2 = X(\gamma_2 + \frac{X}{\gamma_1}h(X))$. Now $X\gamma_2 \in C_2$ patently because $\gamma_2 \in \mathfrak{C}_2$ and $X(\frac{X}{\gamma_1}h(X)) \in C_2$ because $X(\frac{X}{\gamma_1}h(X)) \in X(XL[X])$. Since both belong to $X(\mathfrak{C}_2 + XK[X]$ their sum must do the same. Now check that $c_1c_2 = c = (X^r g_1(X)\gamma_1)(\ X^s g_2(X)(\gamma_2 + \frac{X}{\gamma_1}h(X))$. That D is condensed if $D + XK[X]$ is condensed follows from Proposition 6.

Another simple case is that of when D is a field, though here we shall consider the ring $K + XL[X]$ where L is an extension of K. Let us first write another version of Proposition 3 of [16]: Let K be a field, L an extension field of K and let X be an indeterminate over L. Then each nonzero ideal F of $R = K + XL[X]$ is of the form $F = f(X)JR = f(X)(J + XL[X])$,where J is a K-subspace of L and $f(X) \in R$ such that $f(0)J \subseteq K$. If F is finitely generated, J is a finitely generated K-subspace of L.

Now in this case $f(X) = 1$, as $f \in R$, gives F as either $F = K + XL[X] = R$ (if $F \cap K \neq (0)$) or $F = XL[X]$ (if $F \cap D = (0)$). (We could have had $F = f(X)X^r L[X]$, but the considerations like the ones in the proof of Lemma 2.1 would have whittled it down to the current form.) Next for f such that $f(0) = 1$ we have $F = f(X)R$ (when $J \neq 0$) and $F = f(X)XL[X]$ (when $J = (0)$). In the $f(0) = 0$ case we have $F = X^r g(X)JR$ where J is a K-submodule of L.

Of these $f(X)R$, being principal, will produce a condensed pair with any ideal J of R. So will $f(X)XL[X]$.

So, essentially, we have two types of ideals that need to be considered (a) $A = X^r g(X)XL[X]$ (or $A = XL[X]$ as $X^r g(X) \in R$ and so can be cancelled.) and (b) $B = F = X^s g(X)JR$ where J is a K-submodule of L.

Lemma 2.3. *$XL[X]$, A is a condensed pair for every ideal A of R.*

The proof works as in Lemma 2.1. Let $x \in XL[X]A$. Then $x = \sum X f_i(X)a_i(X)$. Since $f_i \in L[X]$ we have $(f_1, ..., f_n) = f$, because $L[X]$ is a PID. So $f_i = h_i(X)f(X)$ and $Xf_i = l_i X h_i(X)f(X)$ where $h_i(X) \in R$. But then $\sum l_i X f = lXf \in XL[X]$ (because L is a field) and $\sum h_i a_i(X) \in A$, because $h_i \in R$. Thus $x = g(X)a(X)$ where $g \in XL[X]$ and $a(X) \in A$.

Next note that, in $F = X^s g(X)JR$, or in $F = XJR$, F is 2-generated if and only if J is a 2-generated K-subspace of L. of r.

Let $K \subseteq L$ be an extension of fields. In the second section of [2] Anderson and Dumitrescu introduce the notion of $K \subseteq L$ being vs-closed as follows. Let V, W be two K-subspaces of L. Let $P(V, W) = \{vw | v \in V \text{ and } w \in W\}$ and let VW denote the K-subspace of L generated by $P(V, W)$. Call $K \subseteq L$ vs-closed if for each pair V, W of K-subspaces of L we have $VW = P(V, W)$. According to Proposition 2.6 of [2], $K \subseteq L$ is vs-closed if and only if for every $\alpha, \beta \in L$, $1+\alpha\beta = (a+b\alpha)(c+d\beta)$ for some $a, b, c, d \in K$. Using the fact that if $[L : K] \geq 4$ then L affords a pair of elements α, β such that $1, \alpha, \beta, \alpha\beta$ are linearly independent over K the authors of [2] concluded that when $K \subseteq L$ is vs-closed $[L : K] \leq 3$.

Lemma 2.4. *The ring $R = K + XL[X]$ is condensed if and only if for every pair of distinct ideals of the form $C = X(J + XL[X])$ where J is a strictly two generated nonzero K-subspace of L, is condensed.*

Proof. Indeed the assertion holds if R is condensed. For the converse we note, using the observations prior to Lemma 2.3, that R has proper ideals of the following types: (a) $A = XL[X]$, and this covers the case of $f(X)XL[X]$, where $J = 0$ and $f(0) = 1$. (Because if $f \in R\backslash\{0\}$, (fA, B) is a condensed pair if and only if (A, B) is a condensed pair), (b) $B = f(X)R$, but this is principal and will form a condensed pair with every other ideal, as we have already observed.) This leaves ideals of the type (c) $C = X^r g(X)(J + XL[X])$ where J is a strictly two generated nonzero K subspace of L. Now with reference to the proof of Theorem 2.2 the cases of (a,a), (a,b) and (a,c) have been settled in Lemma 2.3. The cases of (b,b) and (b,c) are settled because B is nonzero principal. That leaves the case of (c,c) and that establishes the lemma. □

Proposition 8. *Let $K \subseteq L$ be an extension of fields, let X be an indeterminate over L and let $R = K + XL[X]$. Then R is condensed if and only if $K \subseteq L$ is vs-closed. Moreover if $[L : K] \geq 4$, $R = K + XL[X]$ is not condensed.*

Proof. Suppose that $K \subseteq L$ is vs-closed. By Lemma 2.4, all we have to do is study the case (c,c) of pairs of two generated ideals of the form $C = X(\mathfrak{C} + XL[X])$. That is $C_1 = X(\mathfrak{C}_1 + XL[X])$ and $C_2 = X(\mathfrak{C}_2 + XL[X])$. $C_1C_2 = X^2(\mathfrak{C}_1\mathfrak{C}_2 + XL[X])$. Let $x \in C_1C_2$ and let $\gamma \in \mathfrak{C}_1\mathfrak{C}_2\{0\}$. Then $x = X^2(\gamma + Xh(X))$. But $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are both K-subspaces of L. So $\mathfrak{C}_i = (l_{i1}, l_{i2})$ where l_{ij} are elements of L. As $K \subseteq L$ is vs-closed $\gamma = (\gamma_1)(\gamma_2)$ where $\gamma_i = (k_{i1}l_{i1} + k_{i1}l_{i2}) \in \mathfrak{C}_i$. Set $c_1 = X\gamma_1 \in C_1$ and set $c_2 = X(\gamma_2 + \frac{X}{\gamma_1}h(X))$. Now $X\gamma_2 \in C_2$ patently because $\gamma_2 \in \mathfrak{C}_2$ and $X(\frac{X}{\gamma_1}h(X)) \in C_2$ because $X(\frac{X}{\gamma_1}h(X)) \in X(XL[X])$. Since both belong to $X^s g_2(X)(\mathfrak{C}_2 + XL[X)$ their sum must do the same. Now check that $c_1c_2 = c = (X\gamma_1)\ X(\gamma_2 + \frac{X}{\gamma_1}h(X))$. The converse can be proved as follows. Suppose that $K + XL[X]$ is condensed, then for each pair of two generated nonzero ideals $C_1 = X^r g(X)(\mathfrak{C}_1 + XL[X])$ and $C_2 = X^r g(X)(\mathfrak{C}_2 + XL[X])$. That is $C_1' = X(\mathfrak{C}_1 + XL[X])$ and $C_2' = X(\mathfrak{C}_2 + XL[X])$ is a condensed pair. That is, for $c = X^2(\gamma + Xh(X))$ we must find $c_1 = X(\gamma_1 + Xf(X))$ and $c_2 = X(\gamma_2 + Xg(X))$ to get $c_1c_2 = X^2(\gamma_1 + Xf(X))(\gamma_2 + Xg(X)) = X^2(\gamma_1\gamma_2 + \gamma_1 Xg(X) + \gamma_2 Xf(X) + X^2 f(X)g(X)) = X^2(\gamma + Xh(X)) = c$. Comparing the coefficients of X^2 we must have $\gamma = \gamma_1\gamma_2$ where $\gamma_i \in \mathfrak{C}_i$, as desired. (This leaves the case of $Xh(X)$ not being a product, as indicated. The situation can be resolved by taking $c_1 = X\gamma_1$ and $c_2 = X(\gamma_2 + (X/\gamma_1)h(X))$. For the moreover part, observe that as we have noted $[L : K] \geq 4$ implies that $K \subseteq L$ is not vs-closed. □

Now one can go mimicking the vs-closed idea of [2] by letting M, N D-submodules of L and letting $P(M, N) = \{mn | m \in M, n \in N\}$, letting MN be the module generated by $P(M, N)$, and calling $D \subseteq L$ sm-closed (submodule closed), if for every pair of two generated submodules M, N one has $P(M, N) = MN$. Repeating the steps taken in the proofs of Theorem 2.2 and Proposition 8 one can prove the following theorem.

Corollary 7. *Let D be a domain, K the quotient field of D, let L be an extension of L and let X be an indeterminate over L. Then the following hold. (1) $D + XL[X]$ is condensed if and only if D is condensed and $D \subseteq L$ is sm-closed. (2) If $D + XL[X]$ is condensed,* $[L : K] \leq 3$.

Proof. We leave (1) for an interested reader and for (2) we note that if $R = D + XL[X]$ is condensed and if $S = D \backslash \{0\}$, then so is $R_S = K + XL[X]$ and this forces $[K : L] \leq 3$. □

This study may give us a number of examples and indirect results such as the following. The go to reference for the following examples is [3].

Example 2.5. (1) Let $K \subseteq L$ be an extension of fields with $K = Q$ the field of rational numbers and L a quadratic extension of Q. Then $K + XL[X]$ is atomic, and condensed and hence cannot be a $*$-domain, nor a pre-Schreier domain.

(2) With Q and L as above, $Q + XL[X]$ is atomic, and condensed, with the property that every overring is atomic. This is because the integral closure of $Q + XL[X]$ is $L[X]$ [11]. Of course if $[L : K] < \infty$, every overring of $K + XL[X]$ would still be atomic, but in most cases the ring is not condensed.

(3) Let $K \subseteq L$ be an extension of fields with $D = K + XL[X]$ condensed. Then the following are equivalent. (a) D is a $*$-domain, (b) D is a PID, (c) D is pre-Schreier, (d) D is integrally closed and (e) $K = L$. (a) $\Rightarrow$ (b) (A condensed star domain is pre-Schreier), a pre-Schreier atomic domain is a UFD and a condense

UFD is a PID, (b) ⇒ (c) Obvious (c) ⇒ (a) a pre-Schreier domain is a $*$-domain, (b) ⇒ (d) a PID is integrally closed (d) ⇒ (b) An integrally closed condensed domain is Bezout and an atomic Bezout domain is a PID. Finally, the equivalence of (d) and (e) is obvious.

Theorem 2.2 can be used to prove that if D is condesed and if K is a quotient field of D, then $D + XK[X]$ is condensed.

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