D + M CONSTRUCTIONS WITH GENERAL OVERRINGS

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Suppose that T is a domain and K is a field that is a retract of T, that is, suppose T = K + M, where M is a maximal ideal of T. Each subring D of K determines a subring R = D + M of T. This construction has been studied extensively in two situations. The first systematic investigation of the properties of R is due to R. Gilmer [8, Appendix 2, p. 558] and Gilmer and W. Heinzer [9], who required that T be a valuation domain. More recently, a similar investigation has been conducted under the hypothesis that T = K[X], M = XK[X], and K is the quotient field of D [4]. The interest in this case arises because R is the symmetric algebra of the D-module K. In both cases the properties of R are related to those of D; in the case of a valuation domain, the relationship of D to K also plays an important role. In this paper, we investigate the construction described above, without placing any limitations on T. The authors find it remarkable that things proceed as well as in the special cases considered earlier. Of course, in the more general context the properties of T and M, or more often of T_M, also play a crucial role.

More specifically, we focus attention on four properties: we obtain necessary and sufficient conditions for R to be a coherent domain, a Prüfer domain, a Noetherian domain, and a GCD-domain. What is most satisfying is that the conditions are expressed solely in terms of the properties of the components of the construction. If K is the quotient field of D, it is also possible to describe the prime-ideal lattice of R and thus to compute the Krull dimension of R. If R is a Prüfer domain, so are D and T. Their ideal class groups are shown to be related by a short exact sequence. This yields conditions for R to be a Bézout domain. Unfortunately, if R is a Prüfer domain it has the n-generator property whenever D and T do. Thus, this construction casts no light on whether invertible ideals in Prüfer domains can require more than two generators. The paper concludes with a brief consideration of methods for obtaining domains T of the form K + M that satisfy the conditions of the theorems.

It is undoubtedly possible to characterize other properties. We have limited ourselves to these four because they have received attention in the special contexts investigated earlier, and because they seem adequate to demonstrate that such problems can often be handled in more generality than had previously seemed feasible. It seems quite likely that at least some of the results of this paper can be extended to a somewhat more general situation. As Gilmer [8] noted, the assumption that K is a retract of T is often not essential. Instead, it can be assumed that K = T/M, in which case R is replaced by the pullback of T and D. However, we have chosen to follow Gilmer's lead in this regard, and for the sake of clarity and simplicity we limit ourselves to the case of a retract.

Our interest in this problem was kindled by the recent paper of D. Dobbs and I. Papick [5], which gives necessary and sufficient conditions for R to be coherent when T is a valuation domain. We have benefited from possessing a preprint of that paper. We have also been helped by access to Doug Costa's thesis, which contains results concerning the case where T = K[X], and by correspondence with him concerning this problem. Finally, we must thank our colleagues Ray Heitmann and

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Dave Lantz, both of whom have made valuable contributions to this paper. In addition to their general assistance, Ray did most of the work in solving the n-generator problem in the case where R is a Prüfer domain, and Dave contributed the proof used to show that R is coherent in the case where K is the quotient field of D.

The letters T, K, M, D, and R will retain throughout the paper the meanings assigned to them in the opening paragraph. It being necessary to exclude the situations wherein the construction degenerates, we also assume without further mention that $M \neq 0$ and $D \neq K$.

LEMMA 1. *If there exists a nonzero ideal* A *of* T *that is finitely generated as an* R*-module, then* D *is a field and* $[K:D] < \infty$.

Proof. Clearly, A is finitely generated over T, and hence $MA \neq A$. For otherwise, $MT_M \cdot AT_M = AT_M$ and therefore $AT_M = 0$, by Nakayama's lemma. This is impossible, since $0 \neq A \subseteq AT_M$. It follows that A/MA is a nonzero $(T/M = K)$-module that is finitely generated as an $(R/M = D)$-module. Since K is a field, A/MA can be written as a direct sum of copies of K. Thus, K is a finitely generated D-module. But then D is a field, since the field K is integral over D and obviously $[K:D] < \infty$.

For the purpose of casting the next result in generality sufficient to cover all the situations that arise, we use the following terminology from [14]. A domain S with quotient field L is called a *finite conductor* domain if for each pair $x, y \in L$, $xS \cap yS$ is a finitely generated S-module. Every coherent domain is a finite conductor domain, as is every GCD-domain [3, Theorem 2.2] and [8, Theorem B, p. 605].

PROPOSITION 2. *If* R *is a finite conductor domain, then exactly one of the following conditions holds:*

(i) D *is a field,* $[K:D] < \infty$, *and* M *is a finitely generated ideal of* T.

(ii) K *is the quotient field of* D *and* T_M *is a valuation domain.*

Proof. If K is not the quotient field of D, then there is an x $(0 \neq x \in K)$ such that $xD \cap D = (0)$. Clearly, $xR = xD + xM = xD + M$, since x is a unit in T and M is an ideal of T. Now R is a finite conductor domain, and xR and R are principal fractional ideals of R. Therefore, $xR \cap R$ is a finitely generated R-module. But

$$xR \cap R = (xD + M) \cap (D + M) = (xD \cap D) + (M \cap M) = M.$$

Hence, M is a finitely generated ideal of T, and by Lemma 1, (i) holds.

If K is the quotient field of D, let a and b be nonzero elements of T. Now $aR \cap bR \supseteq aM \cap bM$, and the latter is a nonzero ideal of T. Moreover, since R is a finite conductor domain and K is the quotient field of D, it follows from Lemma 1 that

$$aR \cap bR \neq aM \cap bM.$$

Choose $x \in (aR \cap bR) \setminus (aM \cap bM)$. Write $x = (d_1 + m_1)a = (d_2 + m_2)b$ with $d_1, d_2 \in D$ and $m_1, m_2 \in M$. One of the elements d_1 and d_2 is nonzero, say $d_1 \neq 0$. Since $d_1 + m_1 \notin M$, $d_1 + m_1$ is a unit in T_M. Therefore,

$$a = (d_1 + m_1)^{-1}(d_2 + m_2)b \in bT_M,$$

and thus $aT_M \subseteq bT_M$. It follows that T_M is a valuation domain.

This is a convenient juncture for recording some observations that we shall use frequently.

R and T have the same quotient field. This is a general remark about integral domains that have a nonzero ideal in common.

If T is integrally closed, the integral closure of R is J + M, where J is the integral closure of D in K. This follows easily from the fact that R and T have the same quotient field.

If D is a field and $[K:D] < \infty$, then T is a finite R-module. Indeed, if $\{1, b_2, \cdots, b_n\}$ is a field basis for K/D, then $\{1, b_2, \cdots, b_n\}$ is an R-module generating set for T.

If K is the quotient field of D, then $T = R_{D\setminus\{0\}}$ is a localization of R. Moreover, R is a faithfully flat D-module. That no maximal ideal of D blows up in R is obvious; moreover, as a D-module, R is the direct sum of D and a D-module, namely M, which is a direct sum of copies of K, a flat D-module.

We come now to our first theorem. Recall that a domain S is *coherent* if direct products of flat S-modules are flat. Other characterizations include "finitely generated ideals are finitely presented" and "any two finitely generated ideals of S have finite intersection" [3, Theorems 2.1 and 2.2]. Thus, Noetherian domains and Prüfer domains are coherent.

THEOREM 3. R *is coherent if and only if* T *is coherent and exactly one of the following holds:*

(i) M *is a finitely generated ideal of* T, D *is a field, and* $[K:D] < \infty$.

(ii) K *is the quotient field of* D, D *is coherent, and* T_M *is a valuation domain.*

Proof (⇒). By Proposition 2, two cases arise.

If D is a field, $[K:D] < \infty$, and M is a finitely generated ideal of T, then T is a finite R-module. It follows from [11, Corollary 1.5, p. 476] that T is coherent.

If K is the quotient field of D, then T, being a localization of R, is coherent, and by Proposition 2, T_M is a valuation domain. To see that D is coherent, one can show directly that the intersection of two finitely generated ideals is finitely generated, or, given a finitely generated ideal I of D, one can use the faithful flatness of R over D to descend the finite presentation of $IR = I \otimes_D R$ to a finite presentation of I.

(⇐). Suppose the conditions of (i) hold. We shall need the following general remark. Let $\{1, b_2, \cdots, b_n\}$ be a field basis for K/D. If ϕ is the R-homomorphism from R^n to T given by $\phi(r_1, \cdots, r_n) = \sum r_i b_i$, then ϕ is surjective and the kernel of ϕ is isomorphic to M^{n-1}. The only statement that needs justification is the one about the kernel. Write $r_i = d_i + m_i$. Then $\phi(r_1, \cdots, r_n) = 0$ if and only if $\sum (d_i + m_i)b_i = 0$, which implies $\sum d_i b_i = 0$; thus $d_1 = \cdots = d_n = 0$, since $1, b_2, \cdots, b_n$ are D-linearly independent. Also, $\sum m_i b_i = 0$, which entails $m_1 = -\sum_2^n m_i b_i$. The isomorphism from M^{n-1} onto the kernel of ϕ is given by

$$(m_2, \cdots, m_n) \mapsto \left(-\sum_2^n m_i b_i, m_2, \cdots, m_n\right).$$

Therefore,

$$0 \to M^{n-1} \to R^n \to T \to 0$$

is a presentation of T as an R-module, and since M is finitely generated, T is a finitely presented R-module. To show that R is coherent, we shall argue that direct products of flat R-modules are flat. Thus, let $\{E_\alpha\}$ be a collection of flat R-modules. For each α, the T-module $E_\alpha \otimes_R T$ is T-flat, and since T is coherent, $\prod_\alpha (E_\alpha \otimes_R T)$ is T-flat. But since T is finitely presented,

$$\prod_\alpha (E_\alpha \otimes_R T) \simeq \left(\prod_\alpha E_\alpha\right) \otimes_R T$$

[1, Exercise 9, p. 43]. By the descent lemma of D. Ferrand [7, p. 946], $\prod_\alpha E_\alpha$ is R-flat.

Suppose the conditions of (ii) hold. Since T_M is a valuation domain, C/MC is a K-vector space of dimension at most 1 for each ideal C of T. Indeed, let $a, b \in C \setminus MC$. Then either a/b or b/a lies in T_M, say

$$a/b = t/(k+m) \qquad (t \in T,\ m \in M,\ k \in K \setminus (0)).$$

Then $(k+m)a = bt$ or, what is the same thing, $a = k^{-1}tb - k^{-1}ma$. Thus $a \equiv (k^{-1}t)b \pmod{MC}$.

Now, let A and B be nonzero, finitely generated ideals of R. Then $AT \cap BT = (A \cap B)T$ is finitely generated, say by $c_1, \cdots, c_n \in A \cap B$. This is possible since T is a localization of R. Since

$$R(c_1, \cdots, c_n) \supseteq MR(c_1, \cdots, c_n) = MT(c_1, \cdots, c_n) = MT(A \cap B) = M(A \cap B),$$

if we can show that $(A \cap B)/M(A \cap B)$ is finitely generated over R, it will follow that $A \cap B$ is finitely generated. It is clearly sufficient to prove that $(A \cap B)/M(A \cap B)$ is finitely generated over D. Therefore, consider the exact sequence

$$0 \to A \cap B \to A \oplus B \to A + B \to 0.$$

Tensoring with $R/M = D$, we obtain the exact sequence

$$(A \cap B)/M(A \cap B) \xrightarrow{\alpha} (A/MA) \oplus (B/MB) \xrightarrow{\beta} (A+B)/M(A+B) \to 0.$$

We claim α is monic. Tensoring

$$(A/MA) \oplus (B/MB) \xrightarrow{\beta} (A+B)/M(A+B) \to 0$$

with T or K, we see that the sequence

$$(TA/MA) \oplus (TB/MB) \xrightarrow{T \otimes \beta} T(A+B)/M(A+B) \to 0$$

is exact. Since A and B are finitely generated, the K-dimension of TC/MC is 1 for $C = A$, B, or $A + B$. Therefore $T \otimes \beta$ is not monic, and therefore β is not monic. Now the kernel of β is a nonzero submodule of the torsion-free D-module $(A/MA) \oplus (B/MB)$, each factor being embeddable in K, the quotient field of D. But

α maps onto the kernel of β, and $(A \cap B)/M(A \cap B)$ is embedded in one copy of K. This proves that α is monic.

Now the D-modules at both ends of β are finitely generated submodules of direct sums of copies of K, and consequently they are finitely presented [3, Theorem 2.1]. It follows that the kernel $(A \cap B)/M(A \cap B)$ of α is finitely generated [1, Lemma 9, p. 21].

The Noetherian case is much easier to handle.

THEOREM 4. R *is Noetherian if and only if* T *is Noetherian and* D *is a field with* $[K:D] < \infty$.

Proof ($\Rightarrow$). Since M is a finitely generated ideal of R, it follows from Lemma 1 that D is a field and $[K:D] < \infty$. Thus, T is module-finite over the Noetherian ring R.

($\Leftarrow$). T is a Noetherian ring and module-finite over the subring R. This is the situation covered by P. M. Eakin's Theorem [6].

The Prüfer-domain case also presents little difficulty.

THEOREM 5. R *is a Prüfer domain if and only if* T *is a Prüfer domain,* K *is the quotient field of* D, *and* D *is a Prüfer domain.*

Proof ($\Rightarrow$). Since T is a localization of R, T is a Prüfer domain. Moreover, since Prüfer domains are integrally closed, K is the quotient field of D, by Proposition 2. That finitely generated ideals of D are invertible may be seen directly, or one can argue this, using the fact that the faithfully flat D-module R is a Prüfer domain.

($\Leftarrow$). Given a finitely generated nonzero ideal I of R, we must show that I is a projective R-module [2, Proposition 3.2, p. 132]. What comes to the same thing, since R is a domain [12, Corollary 3.2, p. 108], is to show that I is a flat R-module. Now $IT = I \otimes_R T$ is T-projective, since T is a Prüfer domain. Moreover,

$$0 \neq I/MI \subseteq IT/MTI = IT/MI \simeq IT \otimes_T (T/M) \simeq IT \otimes_T K,$$

a K-vector space. In particular, I/MI is a torsion-free D-module, and D is a Prüfer domain. Consequently, I/MI is D-flat, and it follows from the descent lemma of Ferrand that I is R-flat [7, p. 946].

Recall that the class group $\mathscr{C}(S)$ of a Prüfer domain S is the multiplicative group of invertible fractional ideals of S modulo the subgroup of nonzero principal fractional ideals. The class group may also be regarded as the multiplicative group of isomorphism classes of invertible fractional ideals of S. In the construction of this paper, the class groups of the Prüfer domains R, D, and T are nicely related.

PROPOSITION 6. *If* R *is a Prüfer domain, there exists an exact sequence*

$$1 \longrightarrow \mathscr{C}(D) \xrightarrow{\alpha} \mathscr{C}(R) \xrightarrow{\beta} \mathscr{C}(T) \longrightarrow 1,$$

where $\alpha[J] = [JR]$ *and* $\beta[I] = [IT]$ *for all finitely generated fractional ideals* J *of* D *and* I *of* R. *Here,* [I] *denotes the isomorphism class of the ideal* I.

Proof. Clearly, α and β are well-defined homomorphisms. Also, since the quotient field K of D is contained in T, $\beta\alpha[J] = [JRT] = [JT] = [JKT] = [T]$ for each fractional ideal J of D. Thus $\alpha(\mathscr{C}(D)) \subseteq \operatorname{Ker} \beta$. Suppose $\alpha[I] \in \operatorname{Ker} \beta$. We may assume I is an integral ideal of R, since [I] has such a representative. Therefore IT = xT, with $x \in T$. Since $T = R_{D\setminus(0)}$, we may choose $x \in I$, say $x = d + m$ with $d \in D$ and $m \in M$. Suppose $I = R(d_1 + m_1) + \cdots + R(d_t + m_t)$ with $d_i \in D$ and $m_i \in M$. Then

$$d_i + m_i = (k_i + n_i)(d + m) = k_i(d + m) + n_i(d + m),$$

and therefore

$$k_i(d + m) = (d_i + m_i) - n_i(d + m) \in I.$$

Therefore, $I \supseteq R(Dk_1 + \cdots + Dk_t)(d + m)$. Since M is an ideal of T, we see that $k_1^{-1} M \subseteq M$ and hence $Rk_1 \supseteq k_1 M \supseteq M$. It follows that

$$R(Dk_1 + \cdots + Dk_t)(d + m) \supseteq M(d + m).$$

Hence, for $1 \leq i \leq t$,

$$d_i + m_i = k_i(d + m) + n_i(d + m) \in R(Dk_1 + \cdots + Dk_t)(d + m).$$

But these elements generate I. Therefore, $I = R(Dk_1 + \cdots + Dk_t)(d + m)$. Hence

$$[I] = [R(Dk_1 + \cdots + Dk_t)] = \alpha[Dk_1 + \cdots + Dk_t].$$

It follows that $\operatorname{Ker} \beta \subseteq \alpha(\mathscr{C}(D))$ and therefore that $\operatorname{Ker} \beta = \alpha(\mathscr{C}(D))$.

It follows immediately from the relation $T = R_{D\setminus(0)}$ that β is an epimorphism. It remains only to show that α is monic. Suppose that $\alpha[J] = [R]$. As before, we may assume that J is an integral ideal of D. Since JM = M as above, $JR = JD + JM = J + M$. Thus, $J + M = (D + M)(d + m) = Dd + M$, because $(D + M)(d + m) \supseteq M$. It follows that $J = Dd$, since both sums are direct. Therefore, $[J] = [D]$ and α is monic.

Remark. Our proof shows that when T is a Bézout domain, each finitely generated ideal I of R can be written in the form $I = (k_1, \cdots, k_n, 1) \cdot b \cdot R$ ($k_i \in K$, $b \in I$). In particular, I is R-isomorphic to the extension of a finitely generated ideal of D. This is often useful in cases where T is a valuation domain or K[X].

THEOREM 7. *R is a Bézout domain if and only if T is a Bézout domain, K is the quotient field of D, and D is a Bézout domain.*

Proof. Because Bézout domains are precisely the Prüfer domains having trivial class group, the result follows from Theorem 5 and Proposition 6.

We can easily describe the prime-ideal lattice of R in case K is the quotient field of D.

PROPOSITION 8. *Let K be the quotient field of D. If Q is a prime ideal of R, then either $Q = P \cap R$ for some prime ideal P of T, or $Q = P + M$ for some prime ideal P of D.*

Proof. Because $T = R_{D\setminus(0)}$, there exists a one-to-one correspondence between primes of R that miss $D \setminus (0)$ and primes of T. On the other hand, if $Q \cap D \neq (0)$, let $d \in Q \cap D$, $d \neq 0$. For $m \in M$, $d^{-1}m \in M \subseteq R$, and hence $m \in Q$. Therefore, $Q \supseteq M$. But $R/M = D$.

Consequently, the lattice of prime ideals of R looks like the lattice of prime ideals of D "pasted" at M to that of T. This gives the following result.

COROLLARY 9. *If* K *is the quotient field of* D, *and if* D *and* T *have finite Krull dimension, then* R *has finite Krull dimension equal to*

$$\max\{\text{height}_T(M) + \dim(D), \dim(T)\}.$$

A Prüfer domain S is said to have the *n-generator property* if each finitely generated ideal of S can be generated by n or fewer elements. It is an open question whether all Prüfer domains have the 2-generator property. As the following result shows, the construction of this paper fails to shed new light on this question.

THEOREM 10. R *is an* n-*generator Prüfer domain if and only if* T *and* D *are* n-*generator Prüfer domains.*

Proof ($\Rightarrow$). By Theorem 5, T and D are Prüfer domains. Moreover, T is a localization of R and D is a homomorphic image of R.

($\Leftarrow$). By Theorem 5, R is a Prüfer domain. Let $I = (a_1, \cdots, a_k)$ be a nonzero finitely generated ideal of R. Since R_M is a valuation domain, IR_M is principal generated by some a_j, which we may assume to be a_1. Then there exists $u \notin M$ such that for $2 \leq j \leq k$, $a_j = (r_j/u)a_1$ with $r_j \in R$. Thus, I is R-isomorphic to $I(u/a_1) = Ru + Rr_2 + \cdots + Rr_k$, which is an ideal of R not contained in M. It is therefore harmless to assume that $I \not\subseteq M$.

Thus, $(I + M)/M$ is a nonzero (R/M)-submodule of $R/M = D$. Since D is an n-generator Prüfer domain,

$$I + M = R(d_1 + m_1) + \cdots + R(d_n + m_n) + M$$

with $d_i \in D$, $d_i \neq 0$, and $m_i \in M$ for $1 \leq i \leq n$. Because T is also an n-generator Prüfer domain, $IT = T(k_1 + m_1') + \cdots + T(k_n + m_n')$ with $k_i \in K$ and $m_i' \in M$. Now, since $I \not\subseteq M$, some $k_i \neq 0$, say $k_1 \neq 0$. We may assume, in fact, that $k_i \neq 0$ for each i; for if this is not already the case, we can replace $k_i + m_i'$ with $(k_1 + m_1') + (k_i + m_i')$. But then

$$IT = T(d_1 + m_1'') + \cdots + T(d_n + m_n''),$$

where $m_i'' = k_i^{-1} d_i m_i'$, since $d_i + m_i'' = d_i k_i^{-1}(k_i + m_i')$ and $d_i k_i^{-1}$ is a unit in T.

We claim that $I = (d_1 + m_1'', \cdots, d_n + m_n'')R$. It suffices to verify that this equality holds locally at each maximal ideal P of R. By Proposition 8, there are two types of maximal ideals, those that contain M and those that have trivial intersection with D. If P is a maximal ideal of R with $P \cap D = (0)$, then $R_{PT} = T_{PT}$, and the desired equality certainly holds since it already holds when the ideals are extended to T. Now suppose that P is a maximal ideal of R with $P \supset M$. Before considering what happens in this case, note that if A is an ideal of R not contained in M, then $AR_P \supset MR_P$. This is because these ideals must be comparable, since R_P is a valuation domain. But if $MR_P \supseteq A$, then

$$A \subseteq AR_P \cap R \subseteq MR_P \cap R = M,$$

since M is prime in R. In particular, if $d \in D$, $d \neq 0$, and $m \in M$, then $(d + m)R_P \supseteq MR_P$ and $dR_P \supseteq MR_P$. Therefore, $(d + m)R_P = dR_P$. It is apparent from these observations that

$$\begin{aligned} IR_P = IR_P + MR_P &= (d_1 + m_1)R_P + \cdots + (d_n + m_n)R_P + MR_P \\ &= (d_1 + m_1)R_P + \cdots + (d_n + m_n)R_P \\ &= (d_1, \cdots, d_n)R_P = (d_1 + m_1'', \cdots, d_n + m_n'')R_P. \end{aligned}$$

Our next result concerns GCD-domains, and we refer the reader to [8, Appendix 4, p. 601] and [13, pp. 32-33] for the relevant facts. We remark that Bézout domains and unique-factorization domains afford the most common examples of GCD-domains.

We shall adopt the following notation. Let S be a domain, and suppose that B is a torsion-free S-module. If $0 \neq b_1, b_2 \in B$, we shall write $c = \inf_S(b_1, b_2)$ provided $c \in B$, $Sc \supseteq Sb_1 + Sb_2$, and Sc is the minimal principal S-submodule of B containing $Sb_1 + Sb_2$. When $\inf_S(b_1, b_2)$ exists, it is unique to within multiplication by a unit of S. It is easily verified that S is a GCD-domain if and only if $\inf_S(\ell_1, \ell_2)$ exists for all $0 \neq \ell_1, \ell_2$ belonging to the quotient field of S.

THEOREM 11. R *is a* GCD-*domain if and only if* T *is a* GCD-*domain,* K *is the quotient field of* D, D *is a* GCD-*domain, and* T_M *is a valuation domain.*

Proof (⇒). By Proposition 2, since GCD-domains are integrally closed [13, Theorem 50, p. 33], K is the quotient field of D and T_M is a valuation domain. Moreover, T, being a localization of R, is a GCD-domain, and D, being a retract of R, is also a GCD-domain.

(⇐). We begin with two observations.

First, if S is an integral domain contained in a field L and if $0 \neq a, b, c \in L$ are such that $\inf_S(b, c)$ exists, then $\inf_S(ab, ac)$ exists and is equal to $a \cdot \inf_S(b, c)$. To see this, note that the map $L \xrightarrow{a} L$ is an S-homomorphism. Moreover, the map is monic, because L is S-torsion-free and epic since L is S-divisible. Thus, the map is an S-automorphism of L. Consequently, it induces a one-to-one inclusion-preserving correspondence between the S-submodules of L containing Sb and Sab and also between Sc and Sac. Since corresponding submodules are isomorphic, they require the same number of S-generators. Therefore, if there is a principal S-submodule of L minimal over Sb + Sc, then there is a principal S-submodule of L minimal over Sab + Sac and it is induced by multiplication by a.

Second, let $t_1 = k_1 + m_1$ and $t_2 = k_2 + m_2$ belong to T, with $m_i \in M$ and $k_i \in K$ and not both k_1, k_2 equal to zero. Because T is a GCD-domain, $\inf_T(t_1, t_2)$ exists and has the form $k_3 + m_3$, with $m_3 \in M$ and $k_3 \in K$, $k_3 \neq 0$, since $Tt_1 + Tt_2 \not\subseteq M$. Furthermore, $\inf_D(k_1, k_2) = k \neq 0$ exists because D is a GCD-domain with quotient field K. Because kk_3^{-1} is a unit in T, we may assume that

$$\inf_T(t_1, t_2) = k + m$$

with $m \in M$. For this choice, a straight-forward calculation shows that $k + m = \inf_R(t_1, t_2)$.

Now, let a and b be nonzero elements of R. Since T_M is a valuation domain, aT_M and bT_M are comparable, say $a = (t/u) \cdot b$ with $t \in T$ and $u \in T \setminus M$, both nonzero. Thus, ua = tb. By our first observation, if $\inf_R(ta, tb)$ exists, then $\inf_R(a, b)$ exists and is equal to $(1/t) \cdot \inf_R(ta, tb)$. But $\inf_R(ta, tb) = \inf_R(ta, ua)$. Again applying our first observation, we see that if $\inf_R(t, u)$ exists, then $\inf_R(at, au)$ exists and is equal to $a \cdot \inf_R(t, u)$. By our second observation,

$\inf_R(t, u)$ does exist since $u \notin M$, so that $u = m + k$, where $k \neq 0$. This completes the proof.

We conclude by describing two large classes of domains, different from those previously studied, admitting an arbitrary field K as retract, and to which the program of this paper can be applied.

Let S be an abelian, torsion-free, cancellative semigroup with 0, and K a field. The semigroup rings K[S] can be regarded as generalizations of polynomial rings. Conditions on S for K[S] to be a Prüfer domain, Bézout domain, GCD domain, and so forth are given in [10]. Moreover, K[S] contains a maximal ideal M, the so-called *augmentation ideal*, with the property that $K[S] = K + M$. For most choices of S, K[S] is neither Noetherian nor a polynomial ring, and K[S] is never a valuation domain.

Finally, let K be an algebraically closed field, and let L be a field of algebraic functions of a single variable over K having positive genus. It is well known that by intersecting all but one of the DVR's on L that contain K, one obtains a Dedekind domain S having infinite class group and the additional property that $S = K + M$ for each maximal ideal M of S.

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