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Almost valuation property in bi-amalgamations and pairs of rings

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This paper examines the transfer of the almost valuation property to various constructions of ring extensions such as bi-amalgamations and pairs of rings. Namely, Sec. 2 studies the transfer of this property to bi-amalgamation rings. Our results cover previous known results on duplications and amalgamations, and provide the construction of various new and original examples satisfying this property. Section 3 investigates pairs of integral domains where all intermediate rings are almost valuation rings. As a consequence of our results, we provide necessary and sufficient conditions for a pair (R, T), where R arises from a (T, M, D) construction, to be an almost valuation pair. Furthermore, we study the class of maximal non-almost valuation subrings of their quotient field.

Keywords: Almost valuation; amalgamated algebra; bi-amalgamation; pair of rings; ring extension.

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1. Introduction

Throughout, all rings considered are commutative with identity. The concept of valuation domains play a central role in several fields of mathematics such as number theory and algebraic geometry. In [1], Anderson and Zafrullah enlarged the class of valuation domains in the following way: they called a domain R an almost valuation domain (AV-domain for short) if given any two elements $x, y \in R$, there is a positive integer n such that the ideals x^nR and y^nR are comparable, which is equivalent to: for every $x \in qf(R)$ (the quotient field of R), there exists an integer $n \geq 1$ such that

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$x^n \in R$ or $x^{-n} \in R$. Among other things, they proved that an integral domain R is an AV-domain if and only if the integral closure R' of R is a valuation domain and $R \subseteq R'$ is a root extension. Later, in [17], the authors extended this notion defined in [1], to class of rings with zero-divisors. In particular, they defined an almost valuation ring (AV-ring for short) and they investigated when this condition is satisfied by an amalgamated algebra and by an idealization (also called Nagata's ring). Recently, in [12], Jahani-Nezhad and Khoshayand extended the notion of pseudo-almost valuation domain, introduced by Badawi in [3], to pseudo-almost valuation ring with zero-divisors. In [13], the authors proved that the class of AV-rings is properly contained in the class of pseudo-almost valuation rings. They also proved that pseudo-almost valuation rings are precisely the pullbacks of AV-rings.

The following diagram of implications summarizes the relations between the main three notions involved in this paper:

$$\begin{gathered} R \text{ is a valuation ring} \\ \Downarrow \\ R \text{ is an almost valuation ring} \\ \Downarrow \\ R \text{ is a quasi-local ring with linearly ordered prime ideals.} \end{gathered}$$

Recall that the above implications are irreversible, in general, as shown by examples provided in [3, 12, 17].

In this paper, we examine when a bi-amalgamation is an AV-ring. Our results capitalize previous works on amalgamations and duplications as well as generate new original examples of AV-rings which are not valuation rings and quasi-local rings that are not AV-rings. Furthermore, we investigate pairs of integral domains where all intermediate rings are AV-domains. As a consequence of our results, we provide necessary and sufficient conditions for a pair (R, T), where R arises from a (T, M, D) construction, to be an AV-domain pair. We also study the class of maximal non-almost valuation subrings of their quotient field. In particular, we show in Theorem 3.8 that if the "bottom" ring is integrally closed, then the class of maximal non-almost valuation, maximal non-quasi-local, maximal non-pseudo valuation and maximal non-valuation, subrings of their quotient field coincide. In Theorem 3.9, we prove that if A is a non-quasi-local integral domain which is not integrally closed, then A is never a maximal non-almost valuation subring of its quotient field. In the case A is quasi-local non-integrally closed, we prove that A is a maximal non-almost valuation subring of its quotient field if and only if each proper overring of A is an AV-domain, A' is a valuation domain and $A \subseteq A'$ is not a root extension.

Section 2 is devoted to the study of AV-ring property in bi-amalgamated algebras. For this purpose, we recall the definition of bi-amalgamation of rings: Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two ideals of B and C, respectively, such that $I_o := f^{-1}(J) = g^{-1}(J')$. The *bi-amalgamation* (or *bi-amalgamated algebra*) of A with (B, C) along (J, J') with respect to (f, g) is the subring of $B \times C$ given by

$$A \bowtie^{f,g} (J, J') := \{(f(a) + j, g(a) + j') \mid a \in A, (j, j') \in J \times J'\}.$$

This construction was introduced in [15] as a natural generalization of duplications [7, 10, 11] and amalgamations [8, 9]. In [15], the authors provide original examples of bi-amalgamations and, in particular, show that Boisen–Sheldon's CPI-extensions [6] can be viewed as bi-amalgamations (notice that [8, Example 2.7] shows that CPI-extensions can be viewed as quotient rings of amalgamated algebras). They also show how every bi-amalgamation can arise as a natural pullback (or even as a conductor square) and then characterize pullbacks that can arise as bi-amalgamations. Then, the last two sections of [15] deal, respectively, with the transfer of some basic ring theoretic properties to bi-amalgamations and the study of their prime ideal structures. All their results recover known results on duplications and amalgamations. Recently, in [16], the authors established necessary and sufficient conditions for a bi-amalgamation to inherit the arithmetical property, with applications on the weak global dimension and transfer of the semihereditary property.

Section 3 is devoted to the study of pairs of AV-domains and maximal non-almost valuation subrings. Recall that if each intermediate ring T between R and S (that is for each $T \in [R, S]$) satisfies a ring theoretic property P, then (R, S) is said to be a P-pair. Let $R \subset S$ be an extension of integral domains such that R does not satisfy a ring theoretic property P and each proper S-overring of R satisfies P. Then R is called a maximal non-P subring of S. The notions of P-pairs and maximal non-P subrings were studied for different properties P (for instance $P :=$ Noetherian, valuation, quasi-local and pseudo-valuation, see [5, 14, 19]). To complete this circle of ideas, we study the property $P :=$ almost valuation. Throughout, for an integral domain R, we denote by $qf(R)$ (respectively, R') the quotient field of R (respectively, the integral closure of R). For a ring extension $R \subset S$, we denote by $[R, S]$ (respectively, $]R, S]$) the set of all rings T such that $R \subseteq T \subseteq S$ (respectively, $R \subset T \subseteq S$). We shall call a ring T in $[R, S]$ an S-overring of R. Such a ring is said to be a proper S-overring of R if $T \neq S$. When $S = qf(R)$, then each ring $T \in [R, qf(R)]$ is called an overring of R. We denote by $\mathrm{Nil}(R)$, the set of all nilpotent elements of R, and by $\mathrm{Jac}(R)$, the Jacobson radical of R. Recall that an extension of integral domains $R \subseteq S$ is said to be a root extension if for each $x \in S$, there exists a positive integer n such that $x^n \in R$.

2. Transfer of AV-Ring Property to Bi-Amalgamated Algebras

The main idea of this section is to produce new examples of rings satisfying AV-ring property.

Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J')$. All along this section, $A \bowtie^{f,g} (J, J')$ will denote the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g).

Our first result investigates the transfer of AV-ring property to bi-amalgamation $A \bowtie^{f,g} (J, J')$ in case J and J' have trivial nilpotent elements.

Theorem 2.1. *Assume that $J \cap \mathrm{Nil}(B) = 0$ and $J' \cap \mathrm{Nil}(C) = 0$. Then the following statements are equivalent*:

(1) *$A \bowtie^{f,g} (J, J')$ is an* AV*-ring.*
(2) *$f(A) + J$ and $g(A) + J'$ are* AV*-rings and $J = 0$ or $J' = 0$.*

Proof. (1) $\Rightarrow$ (2) Assume that $A \bowtie^{f,g} (J, J')$ is an AV-ring. Using the fact that the AV-ring property is stable under factor ring and in view of the isomorphisms:

$$\frac{A \bowtie^{f,g} (J, J')}{0 \times J'} \cong f(A) + J$$

and

$$\frac{A \bowtie^{f,g} (J, J')}{J \times 0} \cong g(A) + J'$$

given by [15, Proposition 4.1(2)]. It follows that $f(A)+J$ and $g(A)+J'$ are AV-rings. Next, we claim that $J = 0$ or $J' = 0$; otherwise, for nonzero elements $j \in J$ and $j' \in J'$, we would have $(0, j')^n \in (j, 0)^n(A \bowtie^{f,g} (J, J'))$ or $(j, 0)^n \in (0, j')^n(A \bowtie^{f,g} (J, J'))$ for some integer $n \geq 1$. It follows that $j'^n = 0$ or $j^n = 0$ which yields a contradiction since $J' \cap \mathrm{Nil}(C) = 0$ and $J \cap \mathrm{Nil}(B) = 0$. The converse is straight via [15, Proposition 4.1(2)]. $\square$

Notice that in Theorem 2.1, the conditions "$J \cap \mathrm{Nil}(B) = 0$" and "$J' \cap \mathrm{Nil}(C) = 0$" are necessary to have the equivalence of assertions (1) and (2). A counter-example for the special case of duplication rings is given in [17, Example 2.7].

Theorem 2.1 allows us to construct new original class of AV-domains which are not valuation domains.

Example 2.2. Let K be a field, $H = K(X)$ be the quotient field of $K[X]$. Consider $A := H$, $B := H + Y^3H[[Y]]$, $f : A \hookrightarrow B$ be the natural injection and $J := Y^3H[[Y]]$ be the maximal ideal of B. Let $C := A \propto A$ be the trivial ring extension of A by A, $g : A \hookrightarrow C$ be the natural injection and $J' := 0$ be the trivial ideal of C. Clearly, $J \cap \mathrm{Nil}(B) = 0$, $J' \cap \mathrm{Nil}(C) = 0$ and $f^{-1}(J) = g^{-1}(J') = 0$. Observe that $f(A) + J = H + Y^3H[[Y]]$ is an AV-domain by [1, Theorem 5.6] (as the integral closure $(f(A)+J)' = H+YH[[Y]]$ is a valuation domain and $(f(A)+J) \subset (f(A) + J)'$ is a root extension) and $g(A) + J' = A \propto 0 + 0 \simeq A$ which is a field and so is an AV-domain. Then:

(1) $A \bowtie^{f,g} (J, J')$ is an AV-domain.
(2) $A \bowtie^{f,g} (J, J')$ is not a valuation domain.

Proof. (1) By Theorem 2.1, $A \bowtie^{f,g} (J, J')$ is an AV-domain since $f(A) + J$ and $g(A) + J'$ are AV-domains and $J' = 0$.

(2) Since $f(A) + J = H + Y^3 H[[Y]]$ is not integrally closed, then it is not a valuation domain. Hence, by [16, Theorem 2.1(1)], $A \bowtie^{f,g} (J, J')$ is not a valuation domain. □

Recall that the amalgamation of A with B along J with respect to f is given by

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}.$$

Clearly, every amalgamation can be viewed as a special bi-amalgamation, since $A \bowtie^f J = A \bowtie^{\mathrm{id}_A, f} (f^{-1}(J), J)$.

The following result is an immediate consequence of Theorem 2.1.

Corollary 2.3. *Under the above notations, assume that* $f^{-1}(J) \cap \mathrm{Nil}(A) = 0$ *and* $J \cap \mathrm{Nil}(B) = 0$. *Then* $A \bowtie^f J$ *is an* AV-*ring if and only if both* A *and* $f(A) + J$ *are* AV-*rings and* $J = 0$ *or* $f^{-1}(J) = 0$.

Remark 2.4. We recall here the following result proved by Mahdou *et al.*: "Let A and B be a pair of rings and $f : A \to B$ be a ring homomorphism. If J is a nonzero proper ideal of B having no nontrivial nilpotent elements and A is reduced, then $A \bowtie^f J$ is an AV-ring if and only if f is injective, $f(A) + J$ is an AV-ring and $f(A) \cap J = 0$" [17, Theorem 2.4]. Corollary 2.3 recovers the above result. Indeed, in the case J is a nonzero proper ideal of B having no nontrivial nilpotent elements and A is reduced, the assumptions $f^{-1}(J) \cap \mathrm{Nil}(A) = 0$ and $J \cap \mathrm{Nil}(B) = 0$ hold. Moreover, we show that the following statements are equivalent:

(1) "A and $f(A) + J$ are AV-rings and $f^{-1}(J) = 0$".
(2) "f is injective, $f(A) + J$ is an AV-ring and $f(A) \cap J = 0$".

Indeed, (1) $\Rightarrow$ (2) Notice that $\ker(f) \subseteq f^{-1}(J) = 0$. So f is injective, therefore the conclusion is trivial.

(2) $\Rightarrow$ (1) Since $f(A) + J$ is an AV-ring and $f(A) \simeq \frac{f(A)+J}{J}$, then $f(A)$ is an AV-ring (as a factor ring of an AV-ring) and since f is injective, then $A(\simeq f(A))$ is also an AV-ring. Using the fact that $f(A) \cap J = 0$, then $f^{-1}(J) = f^{-1}(\{0\}) = \ker(f) = 0$ since f is injective.

Let I be a proper ideal of A. The (amalgamated) duplication of A along I is a special amalgamation given by

$$A \bowtie I := A \bowtie^{\mathrm{id}_A} I = \{(a, a + i) \mid a \in A, i \in I\}.$$

The next corollary is a consequence of Corollary 2.3 on the transfer of AV-ring property to duplications and is [17, Corollary 2.6].

Corollary 2.5. *Let* A *be a reduced ring. Then* $A \bowtie I$ *is an* AV-*ring if and only if so is* A *and* $I = 0$.

The following proposition is a partial result about when a bi-amalgamation is an AV-ring, in case J and J' have nontrivial nilpotent elements.

We recall an important characterization of a local Gaussian ring A. Namely, for any two elements a and b in the ring A, we have $\langle a,b\rangle^2 = \langle a^2\rangle$ or $\langle b^2\rangle$; moreover if $ab = 0$ and $\langle a,b\rangle^2 = \langle a^2\rangle$, then $b^2 = 0$ (see, [4, Theorem 2.2]).

Proposition 2.6. *Assume that (A, m) is a local Gaussian ring, $J \times J' \subsetneq \mathrm{Jac}(B \times C)$, $J^2 = 0$, $J'^2 = 0$, and $f(a)J = f(a)^2J$, $g(a)J' = g(a)^2J'$, for all $a \in \mathfrak{m}$. Then $A \bowtie^{f,g} (J, J')$ is an* AV*-ring.*

Proof. Assume that (A, m) is local Gaussian, $J^2 = 0$, $J'^2 = 0$ and $f(a)J = f(a)^2J$, $g(a)J' = g(a)^2J'$, for all $a \in \mathfrak{m}$. By [15, (b) Proposition 5.3], $A \bowtie^{f,g} (J, J')$ is local since A is local and $J \times J' \subsetneq \mathrm{Jac}(B \times C)$. Let $(f(a)+i, g(a)+i')$ and $(f(b)+j, g(b)+j')$ be elements of $A \bowtie^{f,g} (J, J')$. Two cases are possible:

Case 1: a or $b \notin m$. Then a or b is invertible in A. Assume without loss of generality that a is invertible, then $(f(a)+i, g(a)+i')(f(a^{-1}) - f(a^{-1})^2 i, g(a^{-1}) - g(a^{-1})^2 i') = (1, 1)$. So $(f(a)+i, g(a)+i')$ is invertible in $A \bowtie^{f,g} (J, J')$. Therefore, $(f(b)+j, g(b)+j') \in (f(a)+i, g(a)+i')A \bowtie^{f,g} (J, J')$. Hence, it follows that there exists an integer $n = 1$ such that $(f(a)+i, g(a)+i')^n$ and $(f(b)+j, g(b)+j')^n$ are comparable, as desired.

Case 2: a and $b \in m$. Using the fact that A is local Gaussian, then $\langle a,b\rangle^2 = \langle a^2\rangle$ or $\langle b^2\rangle$. We may assume that $\langle a,b\rangle^2 = \langle a^2\rangle$. So we have, $b^2 = a^2x$ for some $x \in A$, and so $f(b)^2 = f(a)^2 f(x)$, $g(b)^2 = g(a)^2 g(x)$. By assumption, $2f(b)j \in f(b)^2J$ and $2f(a)if(x) \in f(a)^2J$. Therefore, there exist $j_1, i_1 \in J$ such that $2f(b)j = f(a)^2 f(x)j_1$, $2f(a)if(x) = f(a)^2 i_1$, and similarly, there exist $j'_1, i'_1 \in J'$ such that $2g(b)j' = g(a)^2 g(x)j'_1$, $2g(a)i'g(x) = g(a)^2 i'_1$. In view of the fact that $J^2 = 0$ and $J'^2 = 0$, one can easily check that $(f(b)+j, g(b)+j')^2 = (f(a)+i, g(a)+i')^2(f(x) + f(x)j_1 - i_1, g(x) + g(x)j'_1 - i'_1)$. Hence, there exists an integer $n = 2$ such that $(f(a)+i, g(a)+i')^n$ and $(f(b)+j, g(b)+j')^n$ are comparable, as desired. Thus, $A \bowtie^{f,g} (J, J')$ is an AV-ring. □

Proposition 2.6 recovers the special case of amalgamated algebras, as recorded in the next corollary.

Corollary 2.7. *Let $f : A \to B$ be an injective ring homomorphism and J be an ideal of B such that $J \subsetneq \mathrm{Jac}(B)$. If (A, m) is local Gaussian, $J^2 = 0$ and $f(a)J = f(a)^2J$, $af^{-1}(J) = a^2 f^{-1}(J)$ for all $a \in \mathfrak{m}$, then $A \bowtie^f J$ is an* AV*-ring.*

Proof. It is easy to show that $(f^{-1}(J))^2 \subseteq f^{-1}(J^2)$. Since f is injective and $J^2 = 0$, it follows that $(f^{-1}(J))^2 \subseteq f^{-1}(\{0\}) = 0$. Therefore, by using Proposition 2.6, $A \bowtie^f J = A \bowtie^{\mathrm{id}_A, f} (f^{-1}(J), J)$ is an AV-ring. □

The following corollary is an immediate consequence of Corollary 2.7, which gives the special case of duplication rings.

Corollary 2.8. *Let (A, m) be a local Gaussian ring and I be an ideal of A. If $I^2 = 0$ and $aI = a^2I$ for all $a \in m$, then $A \bowtie I$ is an* AV*-ring.*

Proposition 2.6 enriches the literature with new original class of AV-rings that are not valuation rings.

Example 2.9. Let (A, m) be a valuation domain, $K := qf(A)$. Let E be a K-vector space such that $\dim_K(E) \geq 2$ and $B := A \propto E$ be the trivial ring extension of A by E. Consider $f : A \hookrightarrow B$ be the natural injection (defined by $f(a) = (a, 0)$). Let $F := A/m$, $C := A \propto F$ be the trivial ring extension of A by F and $g : A \hookrightarrow C$ be the canonical embedding of A in C. Let $J = 0 \propto E$ be the ideal of B and $J' = 0 \propto F$ be the ideal of C. Clearly, $f^{-1}(J) = g^{-1}(J') = 0$, $J^2 = J'^2 = 0$, and $f(a)J = f(a)^2J = 0$, $g(a)J' = g(a)^2J' = 0$ for all $a \in m$. Then:

(1) $A \bowtie^{f,g} (J, J')$ is an AV-ring.
(2) $A \bowtie^{f,g} (J, J')$ is not a valuation ring.

Proof. (1) By using Proposition 2.6, $A \bowtie^{f,g} (J, J')$ is an AV-ring.

(2) Since $f(A) + J = A \propto 0 + 0 \propto E = A \propto E$ which is not a valuation ring by [17, Corollary 2.2], then by [16, Theorem 2.1(1)], $A \bowtie^{f,g} (J, J')$ is not a valuation ring. □

Now, we consider the case where J and J' are regular ideals.

Proposition 2.10. *Assume that J (respectively, J') is a nonzero proper regular ideal of B (respectively, C). Then $A \bowtie^{f,g} (J, J')$ is never an* AV*-ring.*

Proof. Assume by the way of contradiction that $A \bowtie^{f,g} (J, J')$ is an AV-ring. Using the fact that $J \times J'$ is a regular ideal, then there is a regular element (j, j') in $J \times J'$ such that $0 \neq j$ and $0 \neq j'$. Obviously, (j, j') and $(0, j')$ are elements of $A \bowtie^{f,g} (J, J')$ which is an AV-ring. And so there is a natural integer $n \geq 1$ such that $(0, j')^n \in (j, j')^n A \bowtie^{f,g} (J, J')$ or $(j, j')^n \in (0, j')^n A \bowtie^{f,g} (J, J')$. Two cases are possible:

Case 1: $(0, j')^n \in (j, j')^n A \bowtie^{f,g} (J, J')$.

So, $(0, j')^n = (0, j'^n) = (j, j')^n(f(a) + i, g(a) + k)$ for some $(f(a) + i, g(a) + k) \in A \bowtie^{f,g} (J, J')$. Therefore, $j^n(f(a) + i) = 0$ and $j'^n(1 - g(a) - k) = 0$. It follows that $f(a) = -i \in J$ and $g(a) = 1 - k$. And so $a \in f^{-1}(J) = g^{-1}(J')$. Therefore, $g(a) \in J'$ and $g(a) + k = 1 \in J'$. Consequently, $J' = C$ and $g^{-1}(J') = A = f^{-1}(J)$ and so $J = B$, which is a contradiction.

Case 2: $(j, j')^n \in (0, j')^n A \bowtie^{f,g} (J, J')$.

Then $(j^n, j'^n) = (0, j'^n)(f(a) + s, g(a) + l)$ for some $(f(a) + s, g(a) + l) \in A \bowtie^{f,g} (J, J')$. It follows that $j^n = 0$, which is a contradiction again. Hence, $A \bowtie^{f,g} (J, J')$ is never an AV-ring, as desired. □

Corollary 2.11. *Let $f : A \to B$ be a ring homomorphism. Assume that $f^{-1}(J)$ (respectively, J) is a nonzero proper regular ideal of A (respectively, B). Then $A \bowtie^f J$ is never an* AV*-ring.*

Proof. Recall that $A \bowtie^f J = A \bowtie^{\mathrm{id}_A, f} (f^{-1}(J), J)$. By Proposition 2.10, we deduce the result. □

Corollary 2.11 capitalizes the following result for duplication rings.

Corollary 2.12. *Let A be a ring and I be a nonzero proper regular ideal of A. Then $A \bowtie I$ is never an* AV*-ring.*

Proposition 2.10 provides the construction of new class of quasi-local rings which are not AV-rings.

Example 2.13. Let (A, m) be a quasi-local domain, $K := A/m$ and E be a nonzero K-vector space. Consider the injective ring homomorphism $f : A \hookrightarrow A \propto E$ given by $f(a) = (a, 0)$. Let I be a nonzero prime ideal of A such that $I \subset m$ (for instance $(A, m) := (K[[X_1, \ldots, X_n]], \langle X_1, \ldots, X_n \rangle)$ be the formal power series ring in n variables, $I := \langle X_1 \rangle$ and $E := K^q$, where n and q are two positive integers). Consider the canonical surjective ring homomorphism $g : A \to A/I$. Let $J := m \propto E$ and $J' := m/I$. Clearly, $f^{-1}(J) = m = g^{-1}(J')$. Then:

(1) $A \bowtie^{f,g} (J, J')$ is a quasi-local ring.
(2) $A \bowtie^{f,g} (J, J')$ is not an AV-ring.

Proof. (1) Since A is a quasi-local ring, then by [2, Theorem 3.1], $f(A) + J = A \propto 0 + m \propto E = A \propto E$ is a quasi-local ring and $g(A) + J' = A/I + m/I = A/I$ is also a quasi-local ring. Hence, $A \bowtie^{f,g} (J, J')$ is a quasi-local ring by [15, Proposition 5.4(1)].

(2) Let $(n, e) \in m \propto E$ such that $0 \neq n$ and $0 \neq e$ and let $a \in m - I$. Since I is a prime ideal of A, then $\overline{a}$ is a regular element in J' and so $((n, e), \overline{a})$ is a regular element in $J \times J'$. It follows that $J \times J'$ is a regular ideal of $(A \propto E) \times A/I$. Therefore, by Proposition 2.10, $A \bowtie^{f,g} (J, J')$ is not an AV-ring. □

3. Pairs of Almost Valuation Domains and Maximal Non-Almost Valuation Subrings

In this section, we characterize pairs of almost valuation domains and maximal non-almost valuation subrings.

First, we give the definition of pairs of almost valuation domains.

Definition 3.1. Let $R \subset S$ be an extension of integral domains. We say that (R, S) is an almost valuation domain pair (for short AV-domain pair) if each ring $T \in [R, S]$ is an AV-domain.

As an initial step toward understanding what pairs of domains (R, S) are AV-domains pairs, we determine how the quotient fields of R and S must be related.

Proposition 3.2. *Let $R \subset S$ be an extension of integral domains. If each proper S-overring of R is an* AV*-domain, then the following statements hold*:

(1) *$R \subset S$ is an algebraic extension.*
(2) *If moreover R is not a field, then R and S have the same quotient field and for each $T \in [R, S], T \subseteq T^* \subseteq T'$ are root extensions, where T^* denotes the integral closure of T in S.*
(3) *(R, S) is a residually algebraic pair.*

Proof. (1) Assume that $R \subset S$ is not an algebraic extension. Then there exists $t \in S$ such that t is transcendental over R. Thus, the domain $T = R[t]$ is a proper S-overring of R and so is an AV-domain. Consequently, T is quasi-local, which is a contradiction. Hence, $R \subset S$ is an algebraic extension.

(2) Let K be the quotient field of R. Assume by the way of contradiction that there exists an element $b \in S$ such that $b \notin K$. Since $R \subset S$ is an algebraic extension then b is algebraic over R. Therefore, there is $a \in R$ such that ab is integral over R. Set $\alpha = ab$. Let $n = [K(\alpha) : K]$ be the dimension of $K(\alpha)$ as a K-vector space. Consider a nonzero non-unit element $\beta \in R$ (such element exists since R is not a field) and consider the domain $D = R + \alpha\beta R + \alpha^2\beta R + \cdots + \alpha^{n-1}\beta R$. Then D is an intermediate ring between R and S. So D is an AV-domain. Since α is integral over R, then α is integral over D. From [1, Theorem 5.6], it follows that $D \subseteq D'$ is a root extension, as D is an AV-domain. Using the fact that $\alpha \in D'$, there exist an integer $q \geq 1$ and $a_0, \ldots, a_{n-1} \in R$ such that

$$\alpha^q = a_0 + a_1\alpha\beta + \cdots + a_{n-1}\alpha^{n-1}\beta. \qquad (*)$$

Two cases are possible:

Case 1: $q \leq n - 1$. By identification of the coefficients of α^q, we obtain $a_q\beta = 1$, which is a contradiction, since β is a non-unit element.

Case 2: $q \geq n$. There exist $b_0, \ldots, b_{n-1}$ such that $\alpha^q = b_0 + b_1\alpha + \cdots + b_{n-1}\alpha^{n-1}$. Let j be the smallest integer for which $b_j \neq 0$. One can easily check that $B = \{1, \alpha, \ldots, \alpha^{j-1}, \alpha^q, \alpha^{j+1}, \ldots, \alpha^{n-1}\}$ is an other basis of $K(\alpha)$ as a K-vector space. Substituting α^j by its linear combination of elements of the basis B into equation $(*)$, we deduce that there exist $r_0, \ldots, r_n$ such that $\alpha^q = r_o + r_1\alpha\beta + \cdots + r_{j-1}\alpha^{j-1}\beta + r_j\alpha^q\beta + r_{j+1}\alpha^{j+1}\beta + \cdots + r_n\alpha^n\beta$. It follows that $1 = \beta r_j$, which is a contradiction again, since β is a non-unit element.

Hence, $S \subseteq K$ and so $qf(S) = K$, as desired.

Notice that $T \subseteq T^*$ and $T^* \subseteq T'$ are root extensions since $T \subseteq T'$ is a root extension by [1, Theorem 5.6], as T is an AV-domain.

(3) Let $T \in [R, S]$, for each prime ideal Q of T, setting $P = Q \cap R$. Let A be a ring between R/P and T/Q.

$$\begin{array}{ccc} R & \xrightarrow{\phi} & R/P \\ \downarrow & & \downarrow \\ R_1 & \xrightarrow{\phi} & A \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & T/Q. \end{array}$$

Let $R_1 = (T, Q, A)$. Since $A = \phi(R_1)$ is a homomorphic image of R_1 which is an AV-domain, then A is an AV-domain. Therefore, each proper T/Q-overring of R/P is an AV-domain and it follows from assertion (1) that $R/P \subset T/Q$ is an algebraic extension. Hence, (R, S) is a residually algebraic pair, as desired. □

We state now the following result about pairs of AV-domains.

Theorem 3.3. *Let $R \subset S$ be an extension of integral domains.*

(1) *If R is a field, then the following statements are equivalent*:
   (i) (R, S) *is an* AV*-domain pair.*
   (ii) S *is a field algebraic over* R.

(2) *If R is not a field, then the following statements are equivalent*:
   (i) (R, S) *is an* AV*-domain pair.*
   (ii) R *is an* AV*-domain and* S *is an overring of* R.

Proof. (1) (i) ⇒ (ii) By Proposition 3.2, $R \subset S$ is an algebraic extension. Since R is a field, then so is S.

(ii) ⇒ (i) If S is field algebraic over R, then each intermediate ring between R and S is a field and so is an AV-domain.

(2) (i) ⇒ (ii) Straightforward from assertion (2) of Proposition 3.2.

(ii) ⇒ (i) Let $T \in [R, S]$ and $x \in qf(T)$. By assertion (2) of Proposition 3.2, $qf(T) = qf(R) = qf(S)$, and so $x \in qf(R)$. Using the fact that R is an AV-domain, there exists an integer $n \geq 1$ such that x^n or $x^{-n} \in R \subseteq T$. Hence, T is an AV-domain. □

The following corollary is an immediate consequence of Theorem 3.3.

Corollary 3.4. *Let R be an integral domain that is not a field and K be a field containing R. Then the following statements hold*:

(i) $(R, qf(R))$ *is an* AV*-domain pair if and only if R is an* AV*-domain.*
(ii) (R, K) *is an* AV*-domain pair if and only if R is an* AV*-domain and* $K = qf(R)$.

Recall that $R := (T, M, D)$ is a pullback of canonical homomorphisms

$$\begin{tikzcd} R \arrow[r] \arrow[d] & D \arrow[d] \\ T \arrow[r, "\phi"] & T/M, \end{tikzcd}$$

where T is an integral domain, M is a maximal ideal of T, $\phi : T \to T/M$ is the natural projection, D is an integral domain contained in $K = T/M$ and $R = \phi^{-1}(D)$.

Now, we give necessary and sufficient conditions for (R, T) to be an AV-domain pair, when R arises from a (T, M, D) construction, where M is a maximal ideal of T.

Corollary 3.5. *If $R := (T, M, D)$, $k := qf(D)$ and $K := T/M$, then the following statements are equivalent*:

(i) *(R, T) is an* AV*-domain pair.*
(ii) *R is an* AV*-domain.*
(iii) *T and D are* AV*-domains and the extension $k \subseteq K$ is a root extension.*

Proof. (i) $\Leftrightarrow$ (ii) Combining [1, Theorem 5.6] and [1, Lemma 4.5], it follows that T is an AV-domain as an overring of R and so by using statement (2) of Theorem 3.3, the conclusion is trivial.

(ii) $\Leftrightarrow$ (iii) This follows readily from [18, Theorem 2.2]. $\square$

Now, we define the notion of maximal non-almost valuation subring.

Definition 3.6. Let $R \subset S$ be an extension of integral domains. We say that R is a maximal non-almost valuation subring of S if R is not an AV-domain and each proper S-overring of R is an AV-domain.

Next, we study the localization and homomorphic image of maximal non-almost valuation subring.

Proposition 3.7. *Let $R \subset S$ be an extension of integral domains. Assume that R is a maximal non-almost valuation subring of S, then the following statements hold*:

(i) *For each multiplicative subset U of R, either $(U^{-1}R, U^{-1}S)$ is an* AV*-domain pair or $U^{-1}R$ is a maximal non-almost valuation subring of $U^{-1}S$.*
(ii) *For each prime ideal Q of S, set $P = Q \cap R$, either $(R/P, S/Q)$ is an* AV*-domain pair or R/P is a maximal non-almost valuation subring of S/Q.*

Proof. Observe that each ring between the localizations of R and S is clearly the localization of a ring between R and S and that each ring T between R/P and S/Q is the homomorphic image of a ring between R and S (considering a pullback, namely the ring T_1 of the construction (S, Q, T)). Thus, in both cases, each intermediate ring is an AV-domain, with the possibility that $U^{-1}R$ itself, or R/P itself, is an AV-domain in which case, one clearly obtains an AV-domain pair. □

The following theorem establishes that the class of maximal non-almost valuation, the class of maximal non-quasi-local, the class of maximal non-pseudo-valuation, and the class of maximal non-valuation subrings of their quotient field coincide when the "bottom" ring R is integrally closed.

Theorem 3.8. *Let R be an integrally closed domain with quotient field K. Then the following statements are equivalent*:

(i) *R is a maximal non-almost valuation subring of K.*
(ii) *R is a maximal non-quasi-local subring of K.*
(iii) *R is a maximal non-pseudo-valuation subring of K.*
(iv) *R is a maximal non-valuation subring of K.*
(v) *R is a semilocal Prüfer domain with exactly two maximal ideals M and N such that $[(0), M[= [(0), N[$.*

Proof. (i) ⇒ (ii) Using the fact that every AV-domain is quasi-local, it follows that each proper overring of R is quasi-local. We claim that R is not quasi-local. Deny. (R, K) would be a pair of quasi-local rings and by [14, Lemma 1], R' is a prüfer domain. Since R is a quasi-local integrally closed domain, then R is quasi-local and is a Prüfer domain. Therefore, R is a valuation domain and so R is an AV-domain, which is absurd. Consequently, R is a maximal non-quasi-local ring.

(ii) ⇒ (i) Since R is not a quasi-local ring, then R is not an AV-domain. From [14, Corollary 1], it follows that each proper overring of R is a valuation domain. Therefore, each overring of R is an AV-domain, as desired.

(iii) ⇔ (ii) From [14, Theorem 3].

(ii) ⇔ (iv) ⇔ (v) Trivial from [14, Corollary 1]. □

Now, we treat the case where the bottom ring R is not integrally closed.

Theorem 3.9. *Let R be an integral domain with quotient field K. Assume R is not integrally closed. Then the following statements hold*:

(1) *If R is not a quasi-local ring, then R is never a maximal non-almost valuation subring of K.*
(2) *If R is a quasi-local ring, then the following statements are equivalent*:
   (i) *R is a maximal non-almost valuation subring of K.*
   (ii) *Each proper overring of R is an AV-domain, R' is a valuation domain and $R \subseteq R'$ is not a root extension.*

The proof of Theorem 3.9 requires the following lemma.

Lemma 3.10. *Let R be an integral domain with quotient field K. Assume that R is not a quasi-local ring. If R is a maximal non-almost valuation subring of K, then R is a maximal non-quasi-local subring of K.*

Proof. Straightforward. □

Proof of Theorem 3.9:

(1) From [14, Corollary 1], R is not a maximal non-quasi-local subring of K since R is not integrally closed. By Lemma 3.10, it follows that R is never a maximal non-almost valuation subring of K.

(2) (i) $\Rightarrow$ (ii) Since R is a maximal non-almost valuation subring of K, then each proper overring of R is an AV-domain and so is a quasi-local domain. Using the fact that R is a quasi-local domain, then (R, K) is a quasi-local pair. Therefore, by [14, Lemma 1(ii)], R' is a prüfer domain which is a quasi-local domain, making R' a valuation domain. By [1, Theorem 5.6], it follows that $R \subseteq R'$ is not a root extension (as R is not an AV-domain).

(ii) $\Rightarrow$ (i) Since $R \subseteq R'$ is not a root extension and R' is a valuation domain, then by [1, Theorem 5.6], R is not an AV-domain. Hence, the conclusion is trivial.

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