Zafrullah's flat (overring) problem

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Dedicated to memories

ABSTRACT. Let D be an integral domain with quotient field K. Call R an *overring* of D if $D \subseteq R \subseteq K$ and call an overring R of D *t-linked* over D if, whenever for a finitely generated ideal A of D we have $A^{-1} = D$ we have $(AR)^{-1} = R$. In this note I review some results on PVMDs in terms of t-linked overrings and show that for D a locally GCD PVMD every flat overring of D is a quotient ring of D if and only if for each finitely generated nonzero ideal A of D there is an element a and a natural number n such that $A^n \subseteq (a) \subseteq (A^{-1})^{-1}$. This may serve as an analogue of the well known result that every overring of D is a quotient ring if and only if for every nonzero finitely generated ideal A of D there exist an element a and a natural number n such that $A^n \subseteq (a) \subseteq A$.

1. Introduction

Let D be an integral domain with quotient field K. Call R an *overring* of D if $D \subseteq R \subseteq K$ and an overring R of D *t-linked* over D if, whenever for a finitely generated ideal A of D we have $A^{-1} = D$ we end up with $(AR)^{-1} = R$. An overring R of D is a flat overring if R is a flat D module. It is well known that R is a flat overring of D if and only if $((y) : (x))R = R$ for all $x/y \in R$, see, e.g., [**22**, Proposition 4.10] and $x, y \in D\backslash\{0\}$. We also know that if R is D-flat and if $x_1, ..., x_r \in K\backslash\{0\}$, then $(\cap(x_i))R = \cap x_iR$ see, e.g., Theorem 1 of [**21**].

Call D an *f-qr domain* if *every flat overring* R of D is a *quotient ring* of D. Also call an integral domain D an almost GCD domain if for each pair $x, y \in D\backslash\{0\}$ there is a natural number $n = n(x, y)$ such that $x^nD \cap y^nD$ is principal. After having shown in [**30**] that if D were an integrally closed AGCD domain, every flat overring of D was a quotient ring of D I took a steep angle of flight and my article, [**30**], ended with some, apparently, unsupported and unguarded claims laced with a lot of wishful thinking and hopes that one could find for flat overrings of PVMDs and Krull domains results similar to those for overrings of Prufer domains and Dedekind domains whose overrings are quotient rings. That is I claimed that a Krull domain D had a torsion divisor class group if and only if every flat overring of D was a quotient ring and hoped that for PVMDs one could find a condition

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similar to the one found for Prufer domains, so that for a PVMD satisfying that condition every flat overring was a quotient ring. It turned out that I could not find a reference for my claim about Krull domains, nor a proof. Later, the notion of *t-linked overrings* presented itself as a more potent notion than the flat overrings, see [**10**] for details. The aim of this article is to report on what I did to verify my claims and to show that what I hoped for PVMDs can be easily pulled off for locally GCD PVMDs (or generalized GCD domains of [**1**]) and to indicate that perhaps t-linked overring approach is the way to go as far as PVMDs are concerned. (That is, a t-linked overring is a stronger alternative to a flat overring.) Theorem 2.17, Corollary 2.18, Remark 2.19 seem to indicate that when I proposed the study of t-linked overrings, I was hoping to find some way of making sure that I was perhaps right, in making the claims that I did in [**30**]. But alas, I could not pull that off, because of a counter example mentioned in Remark 2.19 of [**10**]. I have also included some simpler proofs of some results on t-linked overrings.

Since a flat overring R of D is t-linked over D as well [**10**], it seems pertinent to treat *t-qr* (every *t-linked overring is a quotient ring*) domains. Yet to see the proof, clearly, one needs to know the techniques involved in the proof and the terminology that one may not be familiar with, even before seeing the plan of the paper.

Let $F(D)$ be the set of nonzero fractional ideals of D. A star operation is a function $A \mapsto A^\star$ on $F(D)$ with the following properties:

If $A, B \in F(D)$ and $a \in K\backslash\{0\}$, then

(i) $(a)^\star = (a)$ and $(aA)^\star = aA^\star$.

(ii) $A \subseteq A^\star$ and if $A \subseteq B$, then $A^\star \subseteq B^\star$.

(iii) $(A^\star)^\star = A^\star$.

We may call $A^\star$ the *$\star$-image* (or *$\star$-envelope*) of A. An ideal A is said to be a *$\star$-ideal* if $A^\star = A$. Thus $A^\star$ is a $\star$-ideal (by (iii)). Moreover (by (i)) every principal fractional ideal, including $D = (1)$, is a $\star$- ideal for any star operation $\star$.

For all $A, B \in F(D)$ and for each star operation $\star$, we can show that $(AB)^\star = (A^\star B)^\star = (A^\star B^\star)^\star$. These equations define what is called *$\star$-multiplication (or $\star$-product).* Associated with each star operation $\star$ is a star operation $\star_f$ defined by $A^{\star_f} = \bigcup\{J^\star |\ 0 \neq J$ is a finitely generated subideal of $A\}$, for each $A \in F(D)$. We say that a star operation $\star$ is of finite type or of finite character if $\star = \star_f$, i.e., $A^\star = A^{\star_f}$ for each $A \in F(D)$.

Define $A^{-1} = \{x \in K | xA \subseteq D\}$, for $A \in F(D)$. Thus $A^{-1} = \cap_{a \in A\backslash\{0\}}(\frac{1}{a})$. Also define $A_v = (A^{-1})^{-1}$ and $A_t = A_{v_f} = \bigcup\{J_v |\ 0 \neq J$ is a finitely generated subideal of $A\}$. By the definition $A_t = A_v$ for each finitely generated nonzero ideal of D. The functions $A \mapsto A_v$ and $A \mapsto A_t$ on $F(D)$ are more familiar examples of star operations defined on an integral domain. A fractional ideal $A \in F(D)$ is *$\star$-invertible* if $(AA^{-1})^\star = D$. An invertible ideal is a $\star$-invertible $\star$-ideal for each $\star$-operation $\star$ and so is a v-ideal. A v-ideal is better known as a divisorial ideal and using the definition it can be shown that $A_v = \cap_{\substack{x \in K\backslash\{0\} \\ A \subseteq xD}} xD$. The identity function d on $F(D)$, defined by $A \mapsto A$ is another example of a star operation. Indeed a "d-invertible" ideal is the usual invertible ideal. There are of course many more star operations that can be defined on an integral domain D. But for any star operation $\star$ and for any $A \in F(D)$, $A^\star \subseteq A_v$. Some other useful relations are: For any $A \in F(D)$, $(A^{-1})^\star = A^{-1} = (A^\star)^{-1}$ and so, $(A_v)^\star = A_v = (A^\star)_v$. Using the definition of the t-operation one can show that an ideal that is maximal w.r.t. being a proper integral t-ideal is a prime ideal of D, each nonzero ideal A of

D with $A_t \neq D$ is contained in a maximal t-ideal of D and $D = \cap D_M$, where M ranges over maximal t-ideals of D. The set of maximal t-ideals of D is denoted by t-$Max(D)$. An integral domain D is said to be a *Prufer v-multiplication domain (PVMD)* if every nonzero finitely generated ideal of D is t-invertible. For more on v- and t-operations the reader may consult sections 32 and 34 of Gilmer [**16**]. I plan to study PVMDs in connection with t-linked overrings in Section 2, indicating also that t-linked has more "teeth" than flat. In Section 3, I study cases where flat coincides with t-linked.

2. t-linked overrings and PVMDs

As I have said above I was looking for ways to undo my mistake. I had been going over several ways to get to flat overrings or something close to them. The idea of t-linked overrings sounded somewhat preposterous, but apparently that was the only thing on my mind when Evan Houston told me that David Dobbs was coming to UNC Charlotte, NC, and he was fast thinking of some problems to work on, with him. I said, "Why don't we work on extensions $D \subseteq R$ such that if $A^{-1} = D$ implies $(AR)^{-1} = R$ for all nonzero finitely generated ideals A of D?"
He looked at me quizzically for a bit, but being a sharp man that he is, soon grasped the idea. So, the next day when Dobbs came he (Evan) opened the session with, "Muhammad here thinks that we should study ...". The idea of "study" was sort of "collective brain storming". Now the resulting write up being written by David Dobbs, who in my opinion is at par with Carl Faith, at least in writing style, is bound to be in a scholarly style and a bit hard to understand. So I have chosen to include some of the results with proofs that I could understand. Personally I approach PVMDs via Prufer domains.

An integral domain D is a Prufer domain if every nonzero finitely generated ideal of D is invertible. The so called Prufer v-Multiplication Domains or PVMDs are a generalization of Prufer domains. They were introduced as a sort of curiosity by Dieudonne in [**9**], picked up by Bourbaki in [**4**] as Pseudo Pruferian and studied as a class of domains by Griffin [**19**] as *v-multiplication domains.* They got their current name in Gilmer's book [**16**] and were called PVMDs in [**29**]. After [**19**], they got studied as a generalization of Prufer domains in [**23**]. The trend of studying PVMDs as a generalization of Prufer domains continued with the study of t-linked overrings in [**10**]. So, as a prelude to our study of PVMDs I would look at Prufer domains. We call a a ring R containing D, as a subring, an extension of D. Moreover, R is a simple overring of D, if $R = D[u]$ for some $u \in K\backslash\{0\}$.

THEOREM 1. *(Theorem 6.13 of* [**22**]*) An integral domain D is a Prufer domain if and only if every overring of D is integrally closed.*

COROLLARY 1. *A domain D is a Prufer domain if and only if every simple overring of every overring of D is integrally closed.*

PROOF. If D is a Prufer domain, then it is well known that every overring R of D is integrally closed and so is every simple overring of every overring R of D. Conversely, suppose that every simple overring of every overring R of D is integrally closed. Let P be a maximal ideal of D. We shall show that D_P is a valuation ring. Let $u \neq 0$ belong to the quotient field K of D. By hypothesis, $D_P[u^2]$ is integrally closed, and since u is integral over $D_P[u^2]$, we conclude that

$u \in D_P[u^2]$. Then there are elements v_0 , . . . , $v_n \in D_P$ such that $u = v_0 + v_1 u^2 + ... + v_n u^{2n}$...(I). If we multiply the previous equation ((I)) by v_0^{2n-1}/u^{2n}, we obtain $(v_0/u)^{2n} - (v_0/u)^{2n-1} + v_1 v_0 (v_0/u)^{2n-2} + ... + v_n v_0^{2n-1} = 0$. Thus v_0/u is integral over D_P, hence $v_0/u \in D_P$. If v_0/u is a unit in D_P, then $u \in D_P$. If v_0/u is not a unit in D_P, then $1 - (v_0/u)$ is a unit in D_P. If we multiply the equation expressing u in terms of powers of u^2, that is (I), throughout by $1/u^{2n}$, we get $(1 - v_0/u)(1/u)^{2n-1} - v_1(1/u)^{2n-2} - ... - v_n = 0$. Since $(1 - v_0/u)$ is a unit in D_P we conclude that $1/a$ is integral over D_P and so is in D_P. Now, for any $u \in K\backslash\{0\}$ we have $u \in D_P$ or $u^{-1} \in D_P$. Thus D_P is a valuation domain. Since P was an arbitrarily chosen maximal ideal, we conclude that D is a Prufer domain. □

Remark 1. *The above corollary has been included so that we can "lift" the proof of Theorem 6.13 of* [**22**] *as an example of a simple and direct proof. The following lemma was provided to me by Evan Houston.*

Lemma 1. *Let D be an integral domain and let P be a prime t-ideal of D. Then for every $u \in K\backslash\{0\}$ the ring $D_P[u]$ is t-linked over D.*

Proof. Let $T = D[u]$ and let P be a prime t-ideal of D. Then by Propotion 2.9 of [**10**], $T_{D\backslash P} = D[u]_{D\backslash P}$ is t-linked over D. But $D[u]_{D\backslash P} = D_P[u]$. This follows because if we set $S = D\backslash P$, then S is a multiplicative set of $D[u]$ and $D \subseteq D[u]$. So $D_S \subseteq D[u]_S$ and as u belongs to the RHS we have $D_S[u] \subseteq D[u]_S$. On the other hand $x \in D[u]_S$ implies that $x = f(u)/s$ for some $s \in S$ and $f(u) \in D[u]$. But $f(u) \in D_S[u]$ and s is a unit in $D_S[u]$, forcing $x = f(u)/s$ in $D_S[u]$. Thus $D_S[u] = D[u]_S$ or substituting for $S = D\backslash P$ we have $D_P[u] = D[u]_{D\backslash P}$. □

Theorem 2. *An integral domain D is a PVMD if and only if every t-linked overring of D is integrally closed.*

Proof. Let D be a PVMD and let T be any t-linked overring of D. Then by Proposition 2.13 of [**10**] $T = \cap T_{D\backslash P}$ where P varies over prime t-ideals of D. Now for each prime t-ideal P of D, $T_{D\backslash P}$ is an overring of D_P, a valuation ring, and so $T_{D\backslash P} = D_\wp$ for some prime ideal $\wp \subseteq P$. Since $D_\wp$, being an overring of D_P is a valuation domain, $\wp$ must be a prime t-ideal [**29**]. Thus T is what was termed as a subintersection in [**23**] of the PVMD D. Being an intersection of valuation domains T is integrally closed. Conversely suppose that every t-linked overring of D is integrally closed. Let $u \in K\backslash\{0\}$ and let P be a maximal t-ideal of D. Then, by Lemma 1 $D_P[u^2]$ is t-linked over D and so by the condition $D_P[u^2]$ is integrally closed. But then u being integral over $D_P[u^2]$ we conclude that $u \in D_P[u^2]$. Taking steps as in the proof of Corollary 1 we show that D_P is a valuation domain for each maximal t-ideal, a characteristic property of PVMDs. □

Just for keeps sake we keep the following corollary.

Corollary 2. *An integral domain D is a PVMD if and only if every simple extension of every t-linked overring of D is integrally closed.*

Given a nonzero ideal A of D we can define the *ideal transform* $T(A) = \{x \in K | xA^n \subseteq D$, for some natural number $n\}$. The ideal transform (A-*transform*) was introduced by Nagata in [**25**] and [**26**] and has been studied by a number of authors of which Gilmer and Huckaba [**17**] are my favorites. As indicated in [**25**] $T(A)$ is always an overring of D. Brewer [**7**] has shown that if A is a finitely generated ideal and if $F = \{P_\alpha\}$ is the family of prime ideals which do not contain A, then

$T(A)=\cap_{P\in F}D_{P_\alpha}$. Also if k is a positive integer such that $A^k\subseteq B$, then $T(A)\supseteq T(B)$, [**17**, Proposition 1]. Let's add to these results the following observations.

LEMMA 2. *Let A be a v-ideal of finite type and let $F=\{P_\alpha\}$ be a family of prime t-ideals not containing A. Then $T(A)=\cap_{P\in F}D_{P_\alpha}$.*

PROOF. Let's first note that, by the definition of an ideal transform, if A is a nonzero finitely generated ideal then $T(A)=T(A_v)$. (Since $A\subseteq A_v$ we have $T(A)\supseteq T(A_v)$. Now let $x\in T(A)$, then $xA^n\subseteq D$ which means $x(A^n)_v\subseteq D$. But $(A^n)_v=((A_v)^n)_v$ and this forces $x(A_v)^n\subseteq D$ forcing $x\in T(A_v)$.) Next $T(A_v)=\{x\in K|x(A_v)^n\subseteq D\}$ implies that $T(A_v)\subseteq D_P$ for any prime t-ideal P not containing A. Thus $T(A_v)\subseteq\cap_{P\in F}D_P$. Finally let $y\in\cap_{P\in F}D_P$. Then $y=\frac{u}{v}$where v does not belong to Q. So $(v):(u)$ is not contained in Q. So A is contained in $\sqrt{((v):(u))}$. Since A is finitely generated we have $A^n\subseteq(v):(u)$. for some n. But then $(u/v)A^n\subseteq(u/v)((v):(u))\subseteq D$, forcing $y\in T(A)$. □

Note that if $\Delta\subseteq t\text{-}Spec(D)$, for an arbitrary domain D, then $R=\cap_{P\in\Delta}D_P$ was called a subintersection in Section 5 of [**23**] and it was shown that a subintersection of a PVMD is a PVMD . This was in analogy with " Every subintersection of a Krull domain is a Krull domain", [**15**].

LEMMA 3. *Let D be a PVMD with quotient field K and let $x\in K$. Then $T(D:x)$ is a PVMD.*

PROOF. Let $x=b/a$. Then, since D is a PVMD, $(D:x)=(a):_D(b)=A_v$ where A is a finitely generated ideal. But then by Lemma 2 $T(D:x)=T((a):_D(b))=T(A_v)=T(A)=\cap D_{P_\alpha}$ where P_α ranges over prime t-ideals not containing A. By Proposition 5.1 of [**23**], $T(A)=T(A_v)$ is a PVMD. □

LEMMA 4. *(Proposition 2.16 of [**10**]) Let D be a PVMD, let $A\in F(D)$ be finitely generated and let H be a subintersection of D. Then $(A_vH)_{v_H}=(AH)_{v_H}$, where v_H denotes the v-operation on the fractional ideals of H.*

The following result was stated in [**11**], with a proof that seems a little hard to me.

PROPOSITION 1. *An integral domain D is a t-qr domain if and only if D is a PVMD such that for each nonzero finitely generated ideal A of D there is an element $a\in A_v$ such that $A^n\subseteq(a)$ for some positive integer n.*

PROOF. Suppose that D has t-qr. Let P be an associated prime of a principal ideal of D and consider D_P. Since D_P is t-local, every overring R of D_P is t-linked over D_P and hence over D. Since D has t-qr, R is a quotient ring of D and hence of D_P. But then D_P has the qr property and so must be a valuation domain. This makes D an essential domain with the t-qr property. So every t-linked overring of D is integrally closed and this forces D to be a PVMD, Theorem 2. Next let A be a nonzero finitely generated ideal of D. Set $T=T(A)=T(A_v)$. By Lemma 2 T is t-linked over D and so $T=\cap D_{P_\alpha}$, where P_α ranges over prime t-ideals not containing A. By the condition, T has to be a quotient ring and hence there is a multiplicative set S of D such that $T=D_S$. On the other hand as $T=T(A_v)$ we must have $A_vT=T$. But that means $A_vD_S=D_S$, which in turn means $A_v\cap S\neq\phi$. Let $a\in A_v\cap S$, then $1/a\in T=T(A_v)=T(A)$. So $(1/a)A^n\subseteq D$. Or $A^n\subseteq(a)$ and thus $A^n\subseteq(a)\subseteq A_v$.

Conversely, let D be a PVMD satisfying the given condition, let H be a t-linked overring of D and put $S = U(H) \cap D$ where $U(H)$ denotes the set of units of H. We need to show that $H = D_S$. We note that $x \in T(D:x)$. Because D is a PVMD, there is a nonzero finitely generated integral ideal A such that $(D:x) = A_v$, Theorem 3.3 of [**23**]. On the other hand since D is a PVMD, H is t-linked over D if H is a subintersection and by Lemma 4 $((D:x)H)_{v_H} = H$, giving $((D:x)^n H)_{v_H} = H$, for all natural numbers n. Since $A^n \subseteq (a) \subseteq A_v$ we have $((D:x)^n H)_t \subseteq aH$, forcing $H \subseteq aH$ and making a a unit of H. But then $a \in S$. Also since $A^n \subseteq (a) \subseteq A_v$, by Proposition 1 of [**17**] we have $T(A) \supseteq T(a) \supseteq T(A_v)$, forcing $T(A) = T(a) = T(A_v)$, because $T(A) = T(A_v)$. Next $x \in T(A) = D[1/a]$ forces $x = r/a^n$, where $r \in D$. Hence $x \in T(D:x)$ implies that $x \in D_S$. Thus for each $x \in H$ we have $x \in D_S$, so $H \subseteq D_S$. Finally $D \subseteq H$ and S being a set of units of H, we have $D_S \subseteq H$. □

Corollary 3. *If D has the t-qr property, then D has the f-qr property. The reason is that a flat overring is t-linked. So if every t-linked overring is a quotient ring, then so is every flat overring.*

Corollary 4. *Every t-linked overring of a GCD domain D is a quotient ring of D and hence a GCD domain. The reason is $(d_1, ..., d_n) \subseteq (d) = (d_1, ..., d_n)_v$. Consequently, every flat overring of a GCD domain D is a quotient ring of D and hence a GCD domain.*

Call an element $a \in D\backslash\{0\}$ *primal* if for all $b, c \in D\backslash\{0\}$ $a|bc$ implies that $a = rs$ where $r|b$ and $s|c$. A domain all of whose nonzero elements are primal is called a *pre-Schreier* domain and an integrally closed pre-Schreier domain was called a Schreier domain in [**8**]. Note that if D is pre-Schreier then $Cl_t(D)$ is trivial, [**5**]. Call a nonzero element p of D *completely primal* if every factor of p is again primal. A prime element is an example of a primal element. According to Cohn [**8**], if S is a set multiplicatively generated by completely primal elements of an integrally closed domain D such that D_S is a Schreier domain, then D is a Schreier domain. This Theorem is usually referred to as: Cohn's Nagata type Theorem for Schreier domains.

Corollary 5. *Let D be a pre-Schreier domain. Then D has the t-qr property if and only if D is a GCD domain.*

Proof. Let D be pre-Schreier. The t-qr property makes D a PVMD and a pre-Schreier PVMD is a GCD domain, [**5**]. Corollary 4 may be used for the converse. □

Call D an *AGCD domain* if for each pair a, b of nonzero elements there is a positive integer n such that $(a^n, b^n)_v$ is principal. It was shown in [**2**] that D is an AGCD domain if and only if for each finite set of nonzero elements $a_1, a_2, ..., a_r$ there is a positive integer n such that $(a_1^n, a_2^n, ..., a_r^n)_v$ is principal. It had already been shown in [**30**, Theorem 3.9] that an integrally closed domain D is an AGCD domain if and only if for each finite set of nonzero elements $a_1, a_2, ..., a_r$ there is a positive integer n such that $((a_1, a_2, ..., a_r)^n)_v$ is principal. Let $T(D)$ denote the set of t-invertible t-ideals of D and let $P(D)$ be the set of nonzero principal fractional ideals of D. Then $T(D)$ is a group under t-multiplication and $P(D)$ a subgroup of $T(D)$. The quotient group $Cl_t(D) = T(D)/P(D)$ is called the (t-) class group. It was shown in [**30**], that if D is an integrally closed AGCD domain then $Cl_t(D)$ is a torsion group. This result was later generalized in [**2**] to: If D is an AGCD domain then $Cl_t(D)$ is torsion. It was also shown in [**30**] that if D is an integrally closed AGCD domain, then every flat overring of D is a quotient ring of D and hence an integrally closed AGCD domain. This result too was generalized in Theorem 3.5 of [**2**] to: If D is an AGCD domain, then every flat overring of D is a localization.

Corollary 6. *Let D be an AGCD domain. Then D has the t-qr property if and only if D is an integrally closed AGCD domain.*

Proof. Let D be an AGCD domain. Suppose that D has the t-qr property. Then D is a PVMD by Proposition 1 and a PVMD is integrally closed, being an intersection of valuation domains. Conversely an integrally closed AGCD domain is a PVMD with torsion t-class group, see, e.g., [**30**] and that translates to $(A^n)_v = aD$ for each finitely generated nonzero ideal A. □

Corollary 6 seems to indicate the difference between the t-linked overring approach and the flat overring approach. For, given an AGCD domain that is not, even, integrally closed every flat overring is a quotient ring [**2**, Theorem 3.5]. On the other hand for every t-linked overring to be a quotient ring, the domain has to be a PVMD satisfying a certain condition, detailed in Proposition 1. However, if D satisfies Proposition 1, there seems to be no guarantee that D is an AGCD domain.

In [**18**, Corollary 2.6], Gilmer and Ohm show that a Noetherian domain D has the qr (every overring is a quotient ring) property if and only if D is a Dedekind domain such that every ideal of D has a power that is principal. We prepare to give below a "t-analogue" of this result. Recall that D is called a *Mori domain* if D satisfies ACC on its integral divisorial ideals. Indeed as a Mori domain can be regarded as one each of whose t-ideals is t-finitely generated we can regard a Mori domain as a t-analogue of a Noetherian domain.

Corollary 7. *If D is a Mori domain that has the t-qr property, then D is a Krull domain such that for every nonzero ideal A of D there is a positive integer n making the ideal $(A^n)_t$ principal. Conversely if a domain D has the property that for every nonzero ideal A of D there is a positive integer n making the ideal $(A^n)_t$ principal, then D is a Mori (actually Krull) domain with the t-qr property.*

Remark 2. *The analogy fits as the Krull domains are often called t-Dedekind domains.*

Proof. If D is Mori with t-qr property then D is a Krull domain with the t-qr property, because being a t-qr domain, D is a PVMD and a PVMD Mori is Krull. That means D is a Krull domain such that for each finitely generated nonzero ideal I there is a positive integer n and an $a \in I_v$ such that $I^n \subseteq (a)$. Take I to be such that I_v is a height one prime ideal P. (In a Krull domain, every nonzero ideal of D is t-invertible, [**24**, Theorem 2.5] and so a v-ideal of finite type.) Then $I^n \subseteq (a) \subseteq P$ leads to $(P^n)_v \subseteq (a) \subseteq P$. Since D is Krull P is t-invertible. Being a t-invertible prime t-ideal, P is a maximal t-ideal. Finally because $t = v$ in a Krull domain we can write the previous inequality as $(P^{n-1})_v \subseteq (aP^{-1}) \subseteq D$ and if $n > 1$, we must have $(P^{n-1})_v \subseteq (aP^{-1}) \subseteq P$ as P is a maximal t-ideal. Indeed for every $r < n$ we have $(P^{n-r})_v \subseteq (aP^{-r})_v \subseteq P$. This forces $(P^n)_v = (a)$. Thus if D is a Mori domain with the t-qr property, then D is a Krull domain for each of whose height one primes P there is a number n such that $(P^n)_t$ is principal. Now it is well known that for every nonzero ideal A of a Krull domain D we have $A_t = (P_1^{n_1} P_2^{n_2} ... P_r^{n_r})_t$ where P_i are height one prime ideals of D, [**3**, Theorem 8]. When, as in this case, the Krull domain D has the t-qr property, $(P^r)_t$ is principal for some r which may be called the order of P. Now if n is a positive integer divisible by all the orders s_i of P_i, then $(A^n)_t = ((A_t)^n)_t = (((P_1^{n_1}P_2^{n_2}...P_r^{n_r})_t)^n)_t = (P_1^{nn_1}P_2^{nn_2}...P_r^{n_r})_t = ((P_1^{s_1})_t)^{nn_1/s_1}...(P_r^{s_r})_t)^{nn_r/s_r})_t$. So $(A^n)_t$ is principal for all nonzero finitely generated ideals A of D.

Conversely if D is a domain that has the property that for each nonzero ideal A of D there is a positive integer n such that $(A^n)_t$ is principal then D is Krull [**20**] and Krull is Mori. Because $A^n \subseteq (A^n)_t \subseteq (A)_t$ and because over a Krull domain $t = v$ it becomes true that a domain some t-powers of whose nonzero ideals are principal also has the property that for each nonzero finitely generated ideal I there is an n such that for some $a \in I_v$ we have $I^n \subseteq (a)$ and by Proposition 1, D has t-qr. □

It may be noted that for a Krull domain D a t-linked overring is the same as a subintersection. Thus Theorem 1 of [**27**] (or Theorem 6.7 of Fossum [**15**]) is a special case of Corollary 7.

With these results in the bag, hopes are up for my wild claims at the end of [**30**] being true. Now there is no hope that if every flat overring of D is a quotient ring, D should be anything close to being a PVMD. For, say, if D has only finitely many nonzero prime ideals then every flat overring of D is a quotient ring of D and there is no guarantee of such a domain being a PVMD. (That flat overrings of domains with finitely many primes are quotient rings follows from the following statement from Wajnryb and Zaks [**27**]: For any family $\{P\} \cup \{P_\alpha\}_{\alpha \in I}$ of prime ideals of D, if $P \subseteq \cup P_\alpha$, then P is contained in some P_α.) So we must restrict to PVMDs.

3. When "flat" and "t-linked" coincide

Let's call D a *Special PVMD* (*SPVMD*) if every t-linked overring of D is flat. According to Theorem 2.17 of [**10**], D is an SPVMD if and only if for every finitely generated nonzero ideal A of D, A_vR is divisorial for each t-linked overring R of D.

Thus as a corollary to Proposition 1 we can state and prove the following result.

Proposition 2. *Let D be an SPVMD. Then D being an f-qr domain implies that for each nonzero finitely generated ideal A of D there is a natural number n such that $A^n \subseteq (a)$ for some $a \in A_v$. Conversely if D is a PVMD that satisfies $*$: for each finitely generated nonzero ideal A there is $x \in A_v$ and a positive integer n such that $A^n \subseteq (x) \subseteq A_v$, then D is a t-qr domain and hence an f-qr domain.*

Proof. Because in an SPVMD t-linked is flat we have f-qr = t-qr over an SPVMD. So if D is an SPVMD we have, by Proposition 1, for every nonzero finitely generated ideal A there exist a natural number n and an element a such that $A^n \subseteq (a) \subseteq A_v$. Conversely suppose the given condition holds. Let R be a flat overring of D and let S be the set of those elements of D that are units in R. Then obviously $D_S \subseteq R$. We show that for any $a, b \in D$ $((a) : (b))R = R$ implies $((a) : (b))D_S = D_S$.

Let $((a) : (b))R = R$ or $b/a \in R$, because R is flat. Because D is a PVMD $((a) : (b))$ is a v-ideal of finite type. Let A be a finitely generated ideal such that $((a) : (b)) = A_v$. By the condition $*$ on D we have a positive integer n such that $A^n \subseteq (x) \subseteq ((a) : (b))$ and this leads to $((a) : (b))^n \subseteq (x) \subseteq ((a) : (b))$. Extending to R we have $((a) : (b))^nR \subseteq (x)R \subseteq ((a) : (b))R$ forcing $xR = R$ and the conclusion that x is an element of D and a unit in R that lies also in $((a):(b))$. But then $((a):(b))D_S = D_S$ forcing $b/a \in D_S$. □

The characterization of SPVMDs does not seem to separate them visibly from general PVMDs. So, in the absence of a proof that each PVMD is an SPVMD we are reduced to using special cases. The easier to use special case that presents itself is the one of locally GCD PVMDs, also known as the generalized GCD domains. Recall from [**1**] that D is a G-GCD domain if for all $a, b \in D\backslash\{0\}$ the ideal $aD \cap bD$ is invertible. Indeed if D is a G-GCD domain, then for all $a, b \in D\backslash\{0\}$ we have $(a):(b)$ invertible. That a G-GCD domain is an SPVMD can be established as follows.

Proposition 3. *(cf Theorem 5 of* [**1**]*) Let D be a G-GCD domain. Then the following are equivalent for an overring R of D.*

(1) R is a flat overring,
(2) R is t-linked,
(3) R is a generalized transform,
(4) R is an invertible ideal transform,
Moreover a generalized transform of a G-GCD domain is a G-GCD domain and a G-GCD domain is an SPVMD.

Proof. The equivalence of (1) (3) and (4) has been established in [**1**]. We show that (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (2). This follows from Proposition 2.2 (c) of [**10**].

(2) $\Rightarrow$ (1). R being t-linked over D implies that $R = \cap R_{D\backslash P}$, where P ranges over prime t-ideals of D, [**10**, Propoisition 2.13]. Since $R_{D\backslash P} \supseteq D_{D\backslash P} = D_P$ which is a valuation ring, because D is a PVMD, we conclude that R is a subintersection of D. Now as D is a G-GCD domain, A_v is invertible for each nonzero finitely generated ideal of D. But then A_vR is invertible and hence divisorial. Now by Theorem 2.17 of [**10**] we have the conclusion that R is flat. □

Corollary 8. *Let D be a G-GCD domain. Then D is an f-qr domain if and only if for each nonzero finitely generated ideal A of D there is an element $a \in A_v$ such that $A^n \subseteq (a) \subseteq A_v$ for some positive integer n.*

Proof. The fact that a G-GCD domain is an SPVMD will suffice as a proof, via Proposition 2. □

For those who may want a full blown proof in action here's an ab-initio proof in the style of Theorem 5 of Pendleton's [**28**].)

Proposition 4. *Let D be a G-GCD domain. Then D is an f-qr domain if and only if for each nonzero finitely generated ideal A of D there is an element $a \in A_v$ such that $A^n \subseteq (a) \subseteq A_v$ for some positive integer n.*

Proof. Let D be an f-qr domain. Following Theorem 2.5 (g) of [**18**], let $A = (c_1, ..., c_n)_v$. Then, as D is a G-GCD domain, we have $A = (a_1, ..., a_r)$. being invertible and $A^{-1} = (b_1, ..., b_r)$ with $\sum a_ib_i = 1$. Let B be the ideal transform of A. By definition $B = \{x \in K | xA^n \subseteq D\} = \cup(A^{-1})^n = \cup(A^n)^{-1} = D[b_1, ..., b_r]$. Next, B is a ring of quotients of D with some multiplicative set S. Then there is $a \in S$ such that $b_i = c_i/a$, $1 \leq i \leq r$. Then $a = a(\sum a_ib_i) = \sum a_ic_i \in A$. Moreover

$a \in S$ implies $1/a \in B$. So, $(1/a)A^n \subseteq D$ for some n. But this means $A^n \subseteq (a)$ for some n.

Conversely suppose D satisfies the condition stated in the theorem for finitely generated nonzero ideals A: $A^n \subseteq (a) \subseteq A_v$ for some n and some $a \in D$. Now let H be a flat overring of D. We shall show that $H = D_S$ where $S = U(H) \cap D$. Let $y = u/v \in H$. Since H is flat, for $B = (v) : (u) = A_v$ where A is finitely generated, we have $((v) : (u))H = H$. That means $((v) : (u))^n H = H$ for all natural numbers n. By the condition $A^n \subseteq (a)$ and so $(A^n)_v \subseteq (a)$. Or $((v) : (u))^n \subseteq (a)$, Thus $H = ((v) : (u))^n H \subseteq aH$, making a a unit in H. Next, again, for $y \in H$ we have $y = u/v$ and $B = (v) : (u) = A_v$ we have $T(A) = T(a) = D[1/a]$ and so every element in $T(A) = T(A_v)$ is of the form r/a^n where a is a unit in H. Also as $a \in A_v$ we have $a \in D$ and so $a \in S$. But $y = u/v \in T(A) = T(A_v)$ because $(u/v)((u : (v)) \subseteq D$. This gives $y = r/s$ where $s \in S$. Thus for each $y \in H$ we have $y = r/s \in D_S$. Thus $H \subseteq D_S$. On the other hand, as $D, S \subseteq H$ we have D_S. □

Now still restricting to PVMDs we note that if D is a PVMD with torsion t-class group then every flat overring of D is a quotient ring as shown in [**30**]. Recall that when D is Krull $Cl_t(D)$ is the same as $Cl(D)$ the divisor class group of D. Thus when the divisor class group of the Krull domain D is torsion, D has the f-qr according to [**2**] and [**27**] and that leaves open the question: If a Krull domain D has f-qr, must D have torsion divisor class group?

So the next question is: If a PVMD D has the f-qr property, must D satisfy $*$?

I must stress that I would be very surprised if the answers to both turn out to be positive.

Here's a possible reason for my new position.

Let me first note the following auxiliary result.

Proposition 5. *Let R be a flat overring of D. Then R is a quotient ring of D if and only if for each pair $a, b \in D$, $((a) : (b))R = R$ implies that $((a) :_D (b))$ contains an element of D that is a unit in R.*

Proof. Let R be a quotient ring of D. Then $R = D_S$ where S is a multiplicative set of D. Then $((a) : (b))R = R$ implies $((a) : (b))D_S = D_S$ which means that $((a) : (b)) \cap S \neq \phi$. Conversely let S be the set of elements of D that are units in R and let $y/x \in R$. Then $((x) : (y))R = R$ and by the condition there is a unit s such that $sy = lx$. Or $y/x = l/s \in D_S$, forcing $R \subseteq D_S$. But already $D_S \subseteq R$. □

Call D a v-coherent domain if for each nonzero finitely generated ideal I of D we have that I^{-1} is of finite type, i.e., there is a nonzero finitely generated ideal B such that $I^{-1} = B_v$. Obviously in a v-coherent domain $(a) : (b)$ is of finite type too.

Lemma 5. *Let D be a v-coherent domain. If R is a flat overring of D, then for every nonzero finitely generated ideal A of D we have $(AR)_v = A_v R$.*

Proof. Mimick the proof of (c) of Proposition 2.2 of [**10**], use Proposition .06 of [**13**] or proceed as follows. Let A be a nonzero finitely generated ideal. Since D is v-coherent $A^{-1} = B_v = (b_1, ..., b_r)_v$. But then $A_v = \cap(1/b_i)$. Since R is flat we have $A_v R = (\cap(1/b_i))R = \cap(1/b_i)R$ a divisorial ideal, Theorem 1 if [**21**]. It is well known that if R is D-flat and A finitely generated, then $(AR)_v = (A_v R)_v$

(Proposition .06 of [**13**]). Now using the fact that A_vR is divisorial, we get the result. □

Proposition 6. *Let D be a v-coherent domain. If for each nonzero finitely generated ideal A of D there is a natural number n and an element $a \in A_v$ such that $A^n \subseteq (a) \subseteq A_v$, then every flat overring of D is a quotient ring.*

Proof. Suppose that for each nonzero finitely generated ideal A of D there is a natural number n and an element $a \in A_v$ such that $A^n \subseteq (a) \subseteq A_v$ and let R be a flat overring of D. If $x = u/v \in R$, then $((v) : (u))R = R$. Consequently $((v) : (u))^m R = R$ for every natural number m. Let's note that $A^n \subseteq (a) \subseteq A_v$ implies $A_v^n \subseteq (a) \subseteq A_v$. Thus taking $((v) : (u)) = A_v$ for finitely generated A and applying Lemma 5we have $R = A_v^n R \subseteq (a)R \subseteq A_vR$. Or $R \subseteq aR$, which makes $a \in D$ a unit of R. Now $a \in ((v) : (u))$ gives $au = rv$ or $u/v = r/a$. Thus $u/v \in D_S$ forcing $R \subseteq D_S$. But already $D_S \subseteq R$. □

Corollary 9. *Let D be a Mori domain. If for each finitely generated ideal A of D we have $a \in D$ and a positive integer n such that $A^n \subseteq (a) \subseteq A_v$, then every flat overring of D is a quotient ring.*

The following result is a kind of a stand-alone result that can also be proved as a corollary to Proposition 4.

Proposition 7. *Let D be a locally factorial Krull domain. If every flat overring of D is a quotient ring of D then the ideal class group of D is torsion. Conversely if the ideal class group of D is torsion then every flat overring of D is a quotient ring.*

Proof. Let P be a height one prime ideal of D. Then P is invertible because D is locally factorial. Also $P = (a) : (b) = (a) :_D (b)$, because P is a maximal t-ideal. The transform $R = T(P)$ is flat over D because P is invertible. So, $((a) : (b))R = R = PR$. But as R is a quotient ring there is $x \in (a) : (b)$ such that $x, 1/x \in R$. Since $1/x \in R$ there is a positive integer n such that $P^n(1/x) \subseteq D$. But then $P^n \subseteq (x) \in P$. This forces x to be P primary and so $(x) = P^{(m)} = (P^m)_v = P^m$ because P is invertible. But this forces the divisor class group to be torsion and the divisor class group in this case is the ideal class group. The converse follows from the fact that a locally factorial Krull domain with torsion class group is a PVMD with a torsion (t-)class group and so every flat overring is a quotient ring [**30**]. □

Now the trouble is that each t-linked overring being flat does not make a domain a G-GCD domain, as for every AGCD domain we have flat overrings all quotient rings. We also know that a domain in which every t-linked overring is flat is a PVMD such that each t-linked overring has the same property. But is there a PVMD with a t-linked overring that is not flat? Is there a Krull domain with a non-flat subintersection? Yes, look up Remark 2.19 of [**10**]. But of course the existence of such an example would not stop every flat overring being a quotient ring from making the class group torsion, in the Krull case. So the problem stays in the balance, for now. However, one may surmise that I would have been safe if instead of jumping the gun all the way to general Krull domains, I had landed on locally factorial Krull domains. Finally, let me end this reportage with an intersting indirect example.

Example 1. *Let D be an AGCD domain that is not integrally closed. Then D has at least one t-linked overring that is not flat.*

Proof. If every t-linked overring of D were flat then every t-linked overring would be a quotient ring, because over an AGCD domain, a flat overring is a quotient ring. But then D would be a PVMD a contradiction. □

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