On v-domains: a survey

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Abstract An integral domain D is a v–domain if, for every finitely generated nonzero (fractional) ideal F of D, we have $(FF^{-1})^{-1} = D$. The v–domains generalize Prüfer and Krull domains and have appeared in the literature with different names. This paper is the result of an effort to put together information on this useful class of integral domains. In this survey, we present old, recent and new characterizations of v–domains along with some historical remarks. We also discuss the relationship of v–domains with their various specializations and generalizations, giving suitable examples.

1 Preliminaries and Introduction

Let D be an integral domain with quotient field K. Let $\overline{\boldsymbol{F}}(D)$ be the set of all nonzero D–submodules of K and let $\boldsymbol{F}(D)$ be the set of all nonzero fractional ideals of D, i.e., $A \in \boldsymbol{F}(D)$ if $A \in \overline{\boldsymbol{F}}(D)$ and there exists an element $0 \neq d \in D$ with $dA \subseteq D$. Let $\boldsymbol{f}(D)$ be the set of all nonzero finitely generated D–submodules of K. Then, obviously $\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D) \subseteq \overline{\boldsymbol{F}}(D)$.

Recall that a *star operation* on D is a map $\ast : \boldsymbol{F}(D) \to \boldsymbol{F}(D),\ A \mapsto A^\ast$, such that the following properties hold for all $0 \neq x \in K$ and all $A, B \in \boldsymbol{F}(D)$:

$(\ast_1)$ $D = D^\ast$, $(xA)^\ast = xA^\ast$;
$(\ast_2)$ $A \subseteq B$ implies $A^\ast \subseteq B^\ast$;
$(\ast_3)$ $A \subseteq A^\ast$ and $A^{\ast\ast} := (A^\ast)^\ast = A^\ast$.

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(the reader may consult [51, Gilmer (1972), Sections 32 and 34] for a quick review of star operations).

In [105, Okabe-Matsuda (1994)], the authors introduced a useful generalization of the notion of a star operation: a *semistar operation* on D is a map $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D),\ E \mapsto E^\star$, such that the following properties hold for all $0 \neq x \in K$ and all $E, F \in \overline{\boldsymbol{F}}(D)$:

$(\star_1)$ $(xE)^\star = xE^\star$;
$(\star_2)$ $E \subseteq F$ implies $E^\star \subseteq F^\star$;
$(\star_3)$ $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

Clearly, a semistar operation $\star$ on D, restricted to $\boldsymbol{F}(D)$, determines a star operation if and only if $D = D^\star$.

If $*$ is a star operation on D, then we can consider the map $*_f : \boldsymbol{F}(D) \to \boldsymbol{F}(D)$ defined as follows:

$$A^{*_f} := \bigcup \{F^* \mid \ F \in \boldsymbol{f}(D) \text{ and } F \subseteq A\} \quad \text{ for all } A \in \boldsymbol{F}(D).$$

It is easy to see that $*_f$ is a star operation on D, called the *star operation of finite type associated to* $*$. Note that $F^* = F^{*_f}$ for all $F \in \boldsymbol{f}(D)$. A star operation $*$ is called a *star operation of finite type* (or a *star operation of finite character*) if $* = *_f$. It is easy to see that $(*_f)_f = *_f$ (i.e., $*_f$ is of finite type).

If $*_1$ and $*_2$ are two star operations on D, we say that $*_1 \leq *_2$ if $A^{*_1} \subseteq A^{*_2}$ for all $A \in \boldsymbol{F}(D)$. This is equivalent to saying that $(A^{*_1})^{*_2} = A^{*_2} = (A^{*_2})^{*_1}$ for all $A \in \boldsymbol{F}(D)$. Obviously, for any star operation $*$ on D, we have $*_f \leq *$, and if $*_1 \leq *_2$, then $(*_1)_f \leq (*_2)_f$.

Let $I \subseteq D$ be a nonzero ideal of D. We say that I is a $*$*–ideal* of D if $I^* = I$. We call a $*$–ideal of D a $*$*–prime ideal* of D if it is also a prime ideal and we call a maximal element in the set of all proper $*$–ideals of D a $*$*–maximal ideal* of D.

It is not hard to prove that a $*$–maximal ideal is a prime ideal and that each proper $*_f$–ideal is contained in a $*_f$–maximal ideal.

Let Δ be a set of prime ideals of an integral domain D and set

$$E^{\star_\Delta} := \bigcap \{ED_Q \mid Q \in \Delta\} \quad \text{ for all } E \in \overline{\boldsymbol{F}}(D).$$

The operation $\star_\Delta$ is a semistar operation on D called the *spectral semistar operation associated to* Δ. Clearly, it gives rise to a star operation on D if (and only if) $\bigcap \{D_Q \mid Q \in \Delta\} = D$.

Given a star operation $*$ on D, when Δ coincides with $\operatorname{Max}^{*_f}(D)$, the (nonempty) set of all $*_f$–maximal ideals of D, the operation $\widetilde{*}$ defined as follows:

$$A^{\widetilde{*}} := \bigcap \left\{AD_Q \mid Q \in \operatorname{Max}^{*_f}(D)\right\} \quad \text{ for all } A \in \boldsymbol{F}(D)$$

determines a star operation on D, called the *stable star operation of finite type associated to* $*$. It is not difficult to show that $\widetilde{*} \leq *_{_f} \leq *$.

It is easy to see that, *mutatis mutandis*, all the previous notions can be extended to the case of a semistar operation.

Let $A, B \in \boldsymbol{F}(D)$, set $(A : B) := \{z \in K \mid zB \subseteq A\}$, $(A :_D B) := (A : B) \cap D$, $A^{-1} := (D : A)$. As usual, we let v_D (or just v) denote the *v–operation* defined by $A^v := (D : (D : A)) = \left(A^{-1}\right)^{-1}$ for all $A \in \boldsymbol{F}(D)$. Moreover, we denote $(v_D)_{_f}$ by t_D (or just by t), the *t–operation* on D; and we denote the stable semistar operation of finite type associated to v_D (or, equivalently, to t_D) by w_D (or just by w), i.e., $w_D := \widetilde{v_D} = \widetilde{t_D}$.

Clearly $w_D \leq t_D \leq v_D$. Moreover, from [51, Gilmer (1972), Theorem 34.1(4)], we immediately deduce that $* \leq v_D$, and thus $\widetilde{*} \leq w_D$ and $*_{_f} \leq t_D$, for each star operation $*$ on D.

Integral ideals that are maximal with respect to being $*$–ideals, when $* = v$ or t or w are relevant in many situations. However, maximal v–ideals are not a common sight. There are integral domains, such as a nondiscrete rank one valuation domain, that do not have any maximal v–ideal [51, Gilmer (1972), Exercise 12, page 431]. Unlike maximal v–ideals, the maximal t–ideals are everywhere, in that every t–ideal is contained in at least one maximal t–ideal, which is always a prime ideal [78, Jaffard (1960), Corollaries 1 and 2, pages 30-31] (or, [91, Malik (1979), Proposition 3.1.2], in the integral domains setting). Note also that the set of maximal t–ideals coincides with the set of maximal w–ideals [9, D.D. Anderson-Cook (2000), Theorem 2.16].

We will denote simply by d_D (or just d) the *identity star operation* on D and clearly $d_D \leq *$, for each star operation $*$ on D. Another important star operation on an integrally closed domain D is the *b_D–operation* (or just *b–operation*) defined as follows:

$$A^{b_D} := \bigcap \{AV \mid V \text{ is a valuation overring of } D\} \quad \text{for all } A \in \boldsymbol{F}(D).$$

Given a star operation on D, for $A \in \boldsymbol{F}(D)$, we say that A is *$*$–finite* if there exists a $F \in \boldsymbol{f}(D)$ such that $F^* = A^*$. (Note that in the above definition, we do not require that $F \subseteq A$.) It is immediate to see that if $*_1 \leq *_2$ are star operations and A is $*_1$–finite, then A is $*_2$–finite. In particular, if A is $*_{_f}$–finite, then it is $*$–finite. The converse is not true in general, and one can prove that A is $*_{_f}$–finite if and only if there exists $F \in \boldsymbol{f}(D)$, $F \subseteq A$, such that $F^* = A^*$ [124, Zafrullah (1989), Theorem 1.1].

Given a star operation on D, for $A \in \boldsymbol{F}(D)$, we say that A is *$*$–invertible* if $(AA^{-1})^* = D$. From the fact that the set of maximal $\widetilde{*}$–ideals, $\mathrm{Max}^{\widetilde{*}}(D)$, coincides with the set of maximal $*_{_f}$–ideals, $\mathrm{Max}^{*_f}(D)$, [9, D.D. Anderson-Cook (2000), Theorem 2.16], it easily follows that a nonzero fractional ideal A is $\widetilde{*}$–invertible if and only if A is $*_{_f}$–invertible (note that if $*$ is a star operation of finite type, then $(AA^{-1})^* = D$ if and only if $AA^{-1} \not\subseteq Q$ for all $Q \in \mathrm{Max}^*(D)$).

An invertible ideal is a $*$–invertible $*$–ideal for any star operation $*$ and, in fact, it is easy to establish that, if $*_1$ and $*_2$ are two star operations on an integral domain D with $*_1 \leq *_2$, then any $*_1$–invertible ideal is also $*_2$–invertible.

A classical result due to Krull [78, Jaffard (1960), Théorème 8, Ch. I, §4] shows that for a star operation $*$ of finite type, $*$–invertibility implies $*$–finiteness. More precisely, for $A \in \boldsymbol{F}(D)$, we have that A is $*_f$–invertible if and only if A and A^{-1} are $*_f$–finite (hence, in particular, $*$–finite) and A is $*$–invertible (see [44, Fontana-Picozza (2005), Proposition 2.6] for the semistar operation case).

We recall now some notions and properties of monoid theory needed later. A nonempty set with a binary associative and commutative law of composition " $\cdot$ " is called a *semigroup*. A *monoid* $\mathcal{H}$ is a semigroup that contains an identity element $\mathfrak{1}$ (i.e., an element such that, for all $x \in \mathcal{H}$, $\mathfrak{1} \cdot x = x \cdot \mathfrak{1} = x$). If there is an element $\mathfrak{0}$ in $\mathcal{H}$ such that, for all $x \in \mathcal{H}$, $\mathfrak{0} \cdot x = x \cdot \mathfrak{0} = \mathfrak{0}$, we say that $\mathcal{H}$ has a *zero element*. Finally if, for all a, x, y in a monoid $\mathcal{H}$ with $a \neq \mathfrak{0}$, $a \cdot x = a \cdot y$ implies that $x = y$ we say that $\mathcal{H}$ is a *cancellative monoid*. In what follows we shall be working with commutative and cancellative monoids with or without zero. Note that, if D is an integral domain then D can be considered as a monoid under multiplication and, more precisely, D is a cancellative monoid with zero element 0.

Given a monoid $\mathcal{H}$, we can consider the set of invertible elements in $\mathcal{H}$, denoted by $\mathcal{U}(\mathcal{H})$ (or, by $\mathcal{H}^{\times}$) and the set $\mathcal{H}^{\bullet} := \mathcal{H} \setminus \{\mathfrak{0}\}$. Clearly, $\mathcal{U}(\mathcal{H})$ is a subgroup of (the monoid) $\mathcal{H}^{\bullet}$ and the monoid $\mathcal{H}$ is called a *groupoid* if $\mathcal{U}(\mathcal{H}) = \mathcal{H}^{\bullet}$. A monoid with a unique invertible element is called *reduced*. The monoid $\mathcal{H}/\mathcal{U}(\mathcal{H})$ is reduced. A monoid shall mean a reduced monoid unless specifically stated.

Given a monoid $\mathcal{H}$, we can easily develop a divisibility theory and we can introduce a GCD. A *GCD–monoid* is a monoid having a uniquely determined GCD for each finite set of elements. In a monoid $\mathcal{H}$ an element, distinct from the unit element $\mathfrak{1}$ and the zero element $\mathfrak{0}$, is called *irreducible* (or, *atomic*) if it is divisible only by itself and $\mathfrak{1}$. A monoid $\mathcal{H}$ is called *atomic* if every nonzero noninvertible element of $\mathcal{H}$ is a product of finitely many atoms of $\mathcal{H}$. A nonzero noninvertible element $p \in \mathcal{H}$ with the property that $p \mid a \cdot b$, with $a, b \in \mathcal{H}$ implies $p \mid a$ or $p \mid b$ is called a *prime* element. It is easy to see that in a GCD–monoid, irreducible and prime elements coincide.

Given a monoid $\mathcal{H}$, we can also form the monoids of fractions of $\mathcal{H}$ and, when $\mathcal{H}$ is cancellative, the groupoid of fractions $\boldsymbol{q}(\mathcal{H})$ of $\mathcal{H}$ in the same manner, avoiding the zero element $\mathfrak{0}$ in the denominator, as in the constructions of the rings of fractions and the field of fractions of an integral domain D.

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This survey paper is the result of an effort to put together information on the important class of integral domains called v–domains, i.e., integral domains in which every finitely generated nonzero (fractional) ideal is v–invertible. In the present work, we will use a ring theoretic approach. However, because in multiplicative ideal theory we are mainly interested in the multiplicative structure of the integral domains, the study of monoids came into multiplicative ideal theory at an early stage. For instance, as we shall indicate in the sequel, v–domains came out of a study of monoids. During the second half of the 20th century, essentially due to the work of Griffin [55], and due to Gilmer's books [51, Gilmer (1968)] and [52, Gilmer (1984)], multiplicative ideal theory from a ring theoretic point of view became a hot topic for the ring theorists. However, things appear to be changing. Halter-Koch has put together in [57, Halter-Koch (1998)], in the language of monoids, essentially all that was available at that time and essentially all that could be translated to the language of monoids. On the other hand, more recently, Matsuda, under the influence of [52, Gilmer (1984)], is keen on converting into the language of additive monoids and semistar operations all that is available and permits conversion [93, Matsuda (2002)].

Since translation of results often depends upon the interest, motivation and imagination of the "translator", it is a difficult task to indicate what (and in which way) can be translated into the language of monoids, multiplicative or additive, or to the language of semistar operations. But, one thing is certain, as we generalize, we gain a larger playground but, at the same time, we lose the clarity and simplicity that we had become so accustomed to.

With these remarks in mind, we indicate below some of the results that may or may not carry over to the monoid treatment, and we outline some general problems that can arise when looking for generalizations, without presuming to be exhaustive. The first and foremost is any result to do with polynomial ring extensions may not carry over to the language of monoids even though some of the concepts translated to monoids do get used in the study of semigroup rings. The other trouble-spot is the results on integral domains that use the identity (d–)operation. As soon as, one considers the multiplicative monoid of an integral domain, with or without zero, somethings get lost. For instance, the multiplicative monoid $R\backslash\{0\}$ of a PID R, with more than one maximal ideals, is no longer a principal ideal monoid, because a monoid has only one maximal ideal, which in this case is not principal. All you can recover is that $R\backslash\{0\}$ is a unique factorization monoid; similarly, from a Bézout domain you can recover a GCD-monoid. Similar comments can be made for Dedeking and Prüfer domains. On the other hand, if the v–operation is involved then nearly every result, other than the ones involving polynomial ring extensions, can be translated to the language of monoids. So, a majority of old ring theoretic results on v–domains and their specializations can be found in [57, Halter-Koch (1998)] and some in [93, Matsuda (2002)], in one form or another. We will mention or we will provide precise references only for those results on monoids that caught our fancy for one reason or another, as indicated in the sequel.

The case of semistar operations and the possibility of generalizing results on v–domains, and their specializations, in this setting is somewhat difficult in that the area of research has only recently opened up [105, Okabe-Matsuda (1994)]. Moreover, a number of results involving semistar invertibility are now available, showing a more complex situation for the invertibility in the semistar operation setting see for instance [107, Picozza (2005)], [44, Fontana-Picozza (2005)] and [108, Picozza (2008)]. However, in studying semistar operations, in connection with v–domains, we often gain deeper insight, as recent work indicates, see [13, D.F. Anderson-Fontana-Zafrullah (2008)], [5, Anderson-Anderson-Fontana-Zafrullah (2008)].

2 When and in what context did the v-domains show up?

2.a The genesis. The *v–domains* are precisely the integral domains D for which the v–operation is an "endlich arithmetisch brauchbar" operation, cf. [50, Gilmer (1968), page 391]. Recall that a star operation $*$ on an integral domain D is *endlich arithmetisch brauchbar* (for short, *e.a.b.*) (respectively, *arithmetisch brauchbar* (for short, *a.b.*)) if for all $F, G, H \in \boldsymbol{f}(D)$ (respectively, $F \in \boldsymbol{f}(D)$ and $G, H \in \boldsymbol{F}(D)$) $(FG)^* \subseteq (FH)^*$ implies that $G^* \subseteq H^*$.

In [88, Krull (1936)], the author only considered the concept of "a.b. $*$–operation" (more precisely, Krull's original notation was " $'$–Operation", instead of "$*$–operation"). He did not consider the (weaker) concept of "e.a.b. $*$–operation".

The e.a.b. concept stems from the original version of Gilmer's book [50, Gilmer (1968)]. The results of Section 26 in [50, Gilmer (1968)] show that this (presumably) weaker concept is all that one needs to develop a complete theory of Kronecker function rings. Robert Gilmer explained to us saying that ≪ I believe I was influenced to recognize this because during the 1966 calendar year in our graduate algebra seminar (Bill Heinzer, Jimmy Arnold, and Jim Brewer, among others, were in that seminar) we had covered Bourbaki's Chapitres 5 and 7 of *Algèbre Commutative*, and the development in Chapter 7 on the v–operation indicated that e.a.b. would be sufficient. ≫

Apparently there are no examples in literature of e.a.b. not a.b. star operations. A forthcoming paper [43, Fontana-Loper (2008)] will contain an explicit example for showing that the Krull's a.b. condition is really stronger than the Gilmer's e.a.b. condition.

We asked Robert Gilmer and Joe Mott about the origins of v–domains. They had the following to say: ≪ We believe that Prüfer's paper [109, Prüfer (1932)] is the first to discuss the concept in complete generality, though we still do not know who came up with the name of "v–domain". ≫

However, the basic notion of v–ideal appeared around 1929. More precisely, the notion of quasi-equality of ideals (where, for $A, B \in \boldsymbol{F}(D)$, A is *quasi-equal* to B, if $A^{-1} = B^{-1}$), special cases of v–ideals and the observation that the classes of quasi-equal ideals of a Noetherian integrally closed domain form a group first appeared in [117, van der Waerden (1929)] (cf. also [87, Krull (1935), page 121]), but this material was put into a more polished form by E. Artin and in this form was published for the first time by Bartel Leendert van der Waerden in "Modern Algebra" [118, van der Waerden (1931)]. This book originated from notes taken by the author from E. Artin's lectures and it includes research of E. Noether and her students. Note that the "v" of a v–ideal (or a v–operation) comes from the German "Vielfachenideale" or "V–Ideale" ("ideal of multiples"), terminology used in [109, Prüfer (1932), §7]. It is important to recall also the papers [15, Arnold (1929)] and [89, Lorenzen (1939)] that introduce the study of v–ideals and t–ideals in semigroups.

The paper [30, Dieudonné (1941)] provides a clue to where v–domains came out as a separate class of rings, though they were not called v–domains there. Note that [30, Dieudonné (1941)] has been cited in [78, Jaffard (1960), page 23] and, later, in [57, Halter-Koch (1998), page 216], where it is mentioned that J. Dieudonné gives an example of a v–domain that is not a *Prüfer v–multiplication domain* (for short, *PvMD*, i.e., an integral domain D in which every $F \in \boldsymbol{f}(D)$ is t–invertible).

2.b Prüfer domains and v–domains. The v–domains generalize the *Prüfer domains* (i.e., the integral domains D such that D_M is a valuation domain for all $M \in \mathrm{Max}(D)$), since an integral domain D is a Prüfer domain if and only if every $F \in \boldsymbol{f}(D)$ is invertible [51, Gilmer (1972), Theorem 22.1]. Clearly, an invertible ideal is $*$–invertible for all star operations $*$. In particular, a Prüfer domain is a *Prüfer $*$–multiplication domain* (for short, *P$*$MD*, i.e., an integral domain D such that, for each $F \in \boldsymbol{f}(D)$, F is $*_f$–invertible [73, Houston-Malik-Mott (1984), page 48]). It is clear from the definitions that a P$*$MD is a PvMD (since $* \leq v$ for all star operations $*$, cf. [51, Gilmer (1972), Theorem 34.1]) and a PvMD is a v–domain.

The picture can be refined. M. Griffin, a student of Ribenboim's, showed that D is a PvMD if and only if D_M is a valuation domain for each maximal t–ideal M of D [55, Griffin (1967), Theorem 5]. A generalization of this result is given in [73, Houston-Malik-Mott (1984), Theorem 1.1] by showing that D is a P$*$MD if and only if D_Q is a valuation domain for each maximal $*_f$–ideal Q of D.

Call a *valuation overring* V of D *essential* if $V = D_P$ for some prime ideal P of D (which is invariably the center of V over D) and call D an *essential domain* if D is expressible as an intersection of its essential valuation overrings. Clearly, a Prüfer domain is essential and so it is a P$*$MD and, in particular, a PvMD (because, in the first case, $D = \bigcap D_Q$ where Q varies over maximal $*_f$–ideals of D and D_Q is a valuation domain; in the second case, $D = \bigcap D_M$ where M varies over maximal t–ideals of D and D_M is a valuation domain; see [55, Griffin (1967), Proposition 4] and [82, Kang (1989), Proposition 2.9]).

From a local point of view, it is easy to see from the definitions that every integral domain D that is locally essential is essential. The converse is not true and the first example of an essential domain having a prime ideal P such that D_P is not essential was given in [65, Heinzer (1981)].

Now add to this information the following well known result [83, Kang (1989), Lemma 3.1] that shows that the essential domains are sitting in between PvMD's and v–domains.

Proposition 2.1. *An essential domain is a v-domain.*

Proof. Let Δ be a subset of Spec(D) such that $D = \bigcap\{D_P \mid P \in \Delta\}$, where each D_P is a valuation domain with center $P \in \Delta$, let F be a nonzero finitely generated ideal of D, and let $*_\Delta$ be the star operation induced by the family of (flat) overrings $\{D_P \mid P \in \Delta\}$ on D. Then

$$\begin{aligned}(FF^{-1})^{*_\Delta} &= \bigcap\{(FF^{-1})D_P \mid P \in \Delta\} = \bigcap\{FD_PF^{-1}D_P \mid P \in \Delta\} \\ &= \bigcap\{FD_P(FD_P)^{-1} \mid P \in \Delta\} \quad \text{(because } F \text{ is f.g.)} \\ &= \bigcap\{D_P \mid P \in \Delta\} \quad \text{(because } D_P \text{ is a valuation domain).}\end{aligned}$$

Therefore $(FF^{-1})^{*_\Delta} = D$ and so $(FF^{-1})^v = D$ (since $*_\Delta \leq v$ [51, Gilmer (1972), Theorem 34.1]).

For an alternate implicit proof of Proposition 2.1, and much more, the reader may consult [122, Zafrullah (1987), Theorem 3.1 and Corollary 3.2].

Remark 2.2. (a) Note that Proposition 2.1 follows also from a general result for essential monoids [57, Halter-Koch (1998), Exercise 21.6 (i), page 244], but the result as stated above (for essential domains) was already known for instance as an application of [123, Zafrullah (1988), Lemma 4.5].

If we closely look at [57, Halter-Koch (1998), Exercise 21.6, page 244], we note that part (ii) was already known for the special case of integral domains (i.e., an essential domain is a PvMD if and only if the intersection of two principal ideals is a v–finite v–ideal, [120, Zafrullah (1978), Lemma 8]) and part (iii) is related to the following fact concerning integral domains: for $F \in \boldsymbol{f}(D)$, F is t–invertible if and only if $(F^{-1} : F^{-1}) = D$ and F^{-1} is v–finite. The previous property follows immediately from the following statements:

(a.1) *let $F \in \boldsymbol{f}(D)$, then F is t–invertible if and only if F is v–invertible and F^{-1} is v–finite;*
(a.2) *let $A \in \boldsymbol{F}(D)$, then A is v–invertible if and only if $(A^{-1} : A^{-1}) = D$.*

The statement (a.1) can be found in [125, Zafrullah (2000)] and (a.2) is posted in [126, Zafrullah (2008)]. For reader's convenience, we next give their proofs.

For the "only if part" of (a.1), if $F \in \boldsymbol{f}(D)$ is t–invertible, then F is clearly v–invertible and F^{-1} is also t–invertible. Hence F^{-1} is t–finite and thus v–finite.

For a "semistar version" of (a.1), see for instance [44, Fontana-Picozza (2005), Lemma 2.5].

For the "if part" of (a.2), note that $AA^{-1} \subseteq D$ and so $(AA^{-1})^{-1} \supseteq D$. Let $x \in (AA^{-1})^{-1}$, hence $xAA^{-1} \subseteq D$ and so $xA^{-1} \subseteq A^{-1}$, i.e., $x \in (A^{-1} : A^{-1}) = D$. For the "only if part", note that in general $D \subseteq (A^{-1} : A^{-1})$. For the reverse inclusion, let $x \in (A^{-1} : A^{-1})$, hence $xA^{-1} \subseteq A^{-1}$. Multiplying both sides by A and applying the v–operation, we have $xD = x(AA^{-1})^v \subseteq (AA^{-1})^v = D$, i.e., $x \in D$ and so $D \subseteq (A^{-1} : A^{-1})$. A simple proof of (a.2) can also be deduced from [57, Halter-Koch (1998), Theorem 13.4].

It is indeed remarkable that all those results known for integral domains can be interpreted and extended to monoids.

(b) We have observed in (a) that a PvMD is an essential domain such that the intersection of two principal ideals is a v–finite v–ideal. It can be also shown that D is a PvMD if and only if $(a) \cap (b)$ is t–invertible in D, for all nonzero $a, b \in D$ [92, Malik-Mott-Zafrullah (1988), Corollary 1.8].

For v–domains we have the following "v–version" of the previous characterization for PvMD's:

D is a v–domain $\Leftrightarrow$ $(a) \cap (b)$ is v–invertible in D, for all nonzero $a, b \in D$.

The idea of proof is simple and goes along the same lines as those of PvMD's. Recall that every $F \in \boldsymbol{f}(D)$ is invertible (respectively, v–invertible; t–invertible) if and only if every nonzero two generated ideal of D is invertible (respectively, v–invertible; t–invertible) [109, Prüfer (1932), page 7] or [51, Gilmer (1972), Theorem 22.1] (respectively, for the "v–invertibility case", [97, Mott-Nashier-Zafrullah (1990), Lemma 2.6]; for the "t–invertibility case", [92, Malik-Mott-Zafrullah (1988), Lemma 1.7]); for the general case of star operations, see the following Remark 2.5 (c). Moreover, for all nonzero $a, b \in D$, we have:

$$(a,b)^{-1} = \tfrac{1}{a}D \cap \tfrac{1}{b}D = \tfrac{1}{ab}(aD \cap bD)\,,$$
$$(a,b)(a,b)^{-1} = \tfrac{1}{ab}(a,b)(aD \cap bD)\,.$$

Therefore, in particular, the fractional ideal $(a,b)^{-1}$ (or, equivalently, (a,b)) is v–invertible if and only if the ideal $aD \cap bD$ is v–invertible.

(c) Note that, by the observations contained in the previous point (b), if D is a Prüfer domain then $(a) \cap (b)$ is invertible in D, for all nonzero $a, b \in D$. However, the converse is not true, as we will see in Sections 2.c and 2.e (Irreversibility of $\Rightarrow_7$). The reason for this is that $aD \cap bD$ invertible allows only that the ideal $\frac{(a,b)^v}{ab}$ (or, equivalently, $(a,b)^v$) is invertible and not necessarily the ideal (a,b).

Call a *P-domain* an integral domain such that every ring of fractions is essential (or, equivalently, *a locally essential domain*, i.e., an integral domain

D such that D_P is essential, for each prime ideal P of D) [98, Mott-Zafrullah (1981), Proposition 1.1]. Note that every ring of fractions of a PvMD is still a PvMD (see Section 3 for more details), in particular, since a PvMD is essential, a locally PvMD is a P-domain. Examples of P-domains include Krull domains. As a matter of fact, by using Griffin's characterization of PvMD's [55, Griffin (1967), Theorem 5], a Krull domain is a PvMD, since in a Krull domain D the maximal t–ideals (= maximal v–ideals) coincide with the height 1 prime ideals [51, Gilmer (1972), Corollary 44.3 and 44.8] and $D = \bigcap\{D_P \mid P$ is an height 1 prime ideal of $D\}$, where D_P is a discrete valuation domain for all height 1 prime ideals P of D [51, Gilmer (1972), (43.1)]. Furthermore, it is well known that every ring of fractions of a Krull domain is still a Krull domain [23, BAC, Ch. 7, §1, N. 4, Proposition 6].

With these observations at hand, we have the following picture:

$$\begin{array}{rl} \text{Krull domain} & \Rightarrow_0 \text{P}v\text{MD}\,; \\ \text{Prüfer domain} & \Rightarrow_1 \text{P}v\text{MD} \;\Rightarrow_2\; \text{locally P}v\text{MD} \\ & \Rightarrow_3 \text{P-domain} \;\Rightarrow_4\; \text{essential domain} \\ & \Rightarrow_5 v\text{–domain}\,. \end{array}$$

Remark 2.3. Note that P-domains were originally defined as the integral domains D such that D_Q is a valuation domain for every *associated prime ideal Q of a principal ideal of D* (i.e., for every prime ideal which is minimal over an ideal of the type $(aD : bD)$ for some $a \in D$ and $b \in D \setminus aD$) [98, Mott-Zafrullah (1981), page 2]. The P-domains were characterized in a somewhat special way in [106, Papick (1983), Corollary 2.3]: D is a P-domain if and only if D is integrally closed and, for each $u \in K$, $D \subseteq D[u]$ satisfies INC at every associated prime ideal Q of a principal ideal of D.

2.c Bézout-type domains and v–domains. Recall that an integral domain D is a *Bézout domain* if every finitely generated ideal of D is principal and D is a *GCD domain* if, for all nonzero $a, b \in D$, a greatest common divisor of a and b, $\text{GCD}(a, b)$, exists and is in D. Among the characterizations of the GCD domains we have that D is a GCD domain if and only if, for every $F \in \boldsymbol{f}(D)$, F^v is principal or, equivalently, if and only if the intersection of two (integral) principal ideals of D is still principal (see, for instance, [2, D.D. Anderson (2000), Theorem 4.1] and also Remark 2.2 (b)). From Remark 2.2 (b), we deduce immediately that a GCD domain is a v–domain.

However, in between GCD domains and v–domains are sitting several other distinguished classes of integral domains. An important generalization of the notion of GCD domain was introduced in [3, Anderson-Anderson (1979)] where an integral domain D is called a *Generalized GCD* (for short, *GGCD*) *domain* if the intersection of two (integral) invertible ideals of D is invertible D. It is well known that D is a GGCD domain if and only if, for each $F \in \boldsymbol{f}(D)$, F^v is invertible [3, Anderson-Anderson (1979), Theorem 1]. In particular, a Prüfer domain is a GGCD domain. From the fact that an invertible ideal in a local domain is principal [84, Kaplansky (1970), Theorem

59], we easily deduce that a GGCD domain is locally a GCD domain. On the other hand, from the definition of PvMD, we easily deduce that a GCD domain is a PvMD (see also [2, D.D. Anderson (2000), Section 3]). Therefore, we have the following addition to the existing picture:

$$\begin{aligned} \text{Bézout domain } &\Rightarrow_6 \text{ GCD domain } \Rightarrow_7 \text{ GGCD domain} \\ &\Rightarrow_8 \text{ locally GCD domain } \Rightarrow_9 \text{ locally P}v\text{MD} \\ &\Rightarrow_3 \text{ ..... } \Rightarrow_4 \text{ ..... } \Rightarrow_5 v\text{–domain}\,. \end{aligned}$$

2.d Integral closures and v–domains. Recall that an integral domain D with quotient field K is called a *completely integrally closed* (for short, *CIC*) *domain* if $D = \{z \in K \mid \text{ for all } n \geq 0,\ az^n \in D \text{ for some nonzero } a \in D\}$. It is well known that *the following statements are equivalent.*

(i) *D is CIC;*
(ii) *for all $A \in \boldsymbol{F}(D)$, $(A^v : A^v) = D$;*
(ii′) *for all $A \in \boldsymbol{F}(D)$, $(A : A) = D$;*
(ii″) *for all $A \in \boldsymbol{F}(D)$, $(A^{-1} : A^{-1}) = D$;*
(iii) *for all $A \in \boldsymbol{F}(D)$, $(AA^{-1})^v = D$;*

(see [51, Gilmer (1972), Theorem 34.3] and Remark 2.2 (a.2); for a general monoid version of this characterization, see [57, Halter-Koch (1998), page 156]).

In Bourbaki [23, BAC, Ch. 7, §1, Exercice 30] an integral domain D is called *regularly integrally closed* if, for all $F \in \boldsymbol{f}(D)$, F^v is regular with respect to the v–multiplication (i.e., if $(FG)^v = (FH)^v$ for $G, H \in \boldsymbol{f}(D)$ then $G^v = H^v$).

Theorem 2.4. ([51, Gilmer (1972), Theorem 34.6] and [23, BAC, Ch. 7, §1, Exercice 30 (b)]) *Let D be an integral domain, then the following are equivalent.*

(i) *D is a regularly integrally closed domain.*
(ii$_{\boldsymbol{f}}$) *For all $F \in \boldsymbol{f}(D)$, $(F^v : F^v) = D$.*
(iii$_{\boldsymbol{f}}$) *For all $F \in \boldsymbol{f}(D)$ $(FF^{-1})^{-1} = D$ (or, equivalently, $(FF^{-1})^v = D$).*
(iv) *D is a v–domain.*

The original version of Theorem 2.4 appeared in [89, Lorenzen (1939), page 538] (see also [30, Dieudonné (1941), page 139] and [77, Jaffard (1951), Théorème 13]). A general monoid version of the previous characterization is given in [57, Halter-Koch (1998), Theorem 19.2].

Remark 2.5. (a) Note that the condition
(ii′$_{\boldsymbol{f}}$) *for all $F \in \boldsymbol{f}(D)$, $(F : F) = D$*
is equivalent to say that D is integrally closed [51, Gilmer (1972), Proposition 34.7] and so it is weaker than condition (ii$_{\boldsymbol{f}}$) of the previous Theorem 2.4, since $(F^v : F^v) = (F^v : F) \supseteq (F : F)$.

On the other hand, by Remark 2.2 (a.2), the condition

(ii$''_{\boldsymbol{f}}$) *for all* $F \in \boldsymbol{f}(D)$, $(F^{-1} : F^{-1}) = D$
is equivalent to the other statements of Theorem 2.4.

(b) By [97, Mott-Nashier-Zafrullah (1990), Lemma 2.6], condition (iii$_{\boldsymbol{f}}$) of the previous theorem is equivalent to
(iii$_{\mathbf{2}}$) *Every nonzero fractional ideal with two generators is v–invertible.*

This characterization is a variation of the Prüfer's classical result that an integral domain is Prüfer if and only if each nonzero ideal with two generators is invertible (Remark 2.2 (b)) and of the characterization of PvMD's also recalled in that remark.

(c) Note that several classes of Prüfer-like domains can be studied in a unified frame by using star and semistar operations. For instance Prüfer star-multiplication domains were introduced in [73, Houston-Malik-Mott (1984)]. Later, in [38, Fontana-Jara-Santos (2003)], the authors studied Prüfer semistar-multiplication domains and gave several characterizations of these domains, that are new also for the classical case of PvMD's. Other important contributions, in general settings, were given recently in [108, Picozza (2008)] and [61, Halter-Koch (2008)].
In [5, Anderson-Anderson-Fontana-Zafrullah (2008), Section 2], given a star operation $*$ on an integral domain D, the authors call D a $*$*–Prüfer domain* if every nonzero finitely generated ideal of D is $*$–invertible (i.e., $(FF^{-1})^* = D$ for all $F \in \boldsymbol{f}(D)$). (Note that $*$–Prüfer domains were previously introduced in the case of semistar operations $\star$ under the name of $\star$–domains [45, Fontana-Picozza (2006), Section 2].) Since a $*$–invertible ideal is always v–invertible, a $*$–Prüfer domain is always a v–domain. More precisely, d–Prüfer (respectively, t–Prüfer; v–Prüfer) domains coincide with Prüfer (respectively, Prüfer v–multiplication; v–) domains.

Note that, in [5, Anderson-Anderson-Fontana-Zafrullah (2008), Theorem 2.2], the authors show that a star operation version of (iii$_{\mathbf{2}}$) considered in point (b) characterizes $*$–Prüfer domains, i.e., D is a $*$–Prüfer domain if and only if every nonzero two generated ideal of D is $*$-invertible. An analogous result, in the general setting of monoids, can be found in [57, Halter-Koch (1998), Lemma 17.2].

(d) Let $\boldsymbol{f}^v(D) := \{F^v \mid F \in \boldsymbol{f}(D)\}$ be the set of all divisorial ideals of finite type of an integral domain D (in [30, Dieudonné (1941)], this set is denoted by $\mathfrak{M}_f$). By Theorem 2.4, we have that a v–domain is an integral domain D such that each element of $F^v \in \boldsymbol{f}^v(D)$ is v–invertible, but F^{-1} $(= (F^v)^{-1})$ does not necessarily belong to $\boldsymbol{f}^v(D)$. When (and only when), in a v–domain D, $F^{-1} \in \boldsymbol{f}^v(D)$, D is a PvMD (Remark 2.2 (a.1)).

The "regular" teminology for the elements of $\boldsymbol{f}^v(D)$ used by [30, Dieudonné (1941), page 139] (see the above definition of F^v regular with respect to the v–multiplication) is totally different from the notion of "von Neumann regular", usually considered for elements of a ring or of a semigroup. However, it may be instructive to record some observations showing that, in the present situation, the two notions are somehow related.

Recall that, by a *Clifford semigroup*, we mean a multiplicative commutative semigroup $\mathcal{H}$, containing a unit element, such that each element a of $\mathcal{H}$ is *von Neumann regular* (this means that there is $b \in \mathcal{H}$ such that $a^2b = a$).

(α) Let $\mathcal{H}$ be a commutative and cancellative monoid. If $\mathcal{H}$ is a Clifford semigroup, then a is invertible in $\mathcal{H}$ (and conversely); in other words, $\mathcal{H}$ is a group.

(β) Let D be a v–domain. If $A \in \boldsymbol{f}^v(D)$ is von Neumann regular in the monoid $\boldsymbol{f}^v(D)$ under v–multiplication, then A is t–invertible (or, equivalently, $A^{-1} \in \boldsymbol{f}^v(D)$). Consequently, an integral domain D is a PvMD if and only if D is a v–domain and the monoid $\boldsymbol{f}^v(D)$ (under v–multiplication) is Clifford regular.

The proofs of (α) and (β) are straightforward, after recalling that $\boldsymbol{f}^v(D)$ under v–multiplication is a commutative monoid and, by definition, it is cancellative if D is a v–domain.

Note that, in the "if part“ of (β), the assumption that D is a v–domain is essential. As a matter of fact, it is not true that an integral domain D, such that every member of the monoid $\boldsymbol{f}^v(D)$ under the v–operation is von Neumann regular, is a v–domain. For instance, in [127, Zanardo-Zannier (1994), Theorem 11] (see also [29, Dade-Taussky-Zassenhaus (1962)]), the authors show that for every quadratic order D, each nonzero ideal I of D satisfies $I^2J = cI$, i.e., $I^2J(1/c) = I$, for some (nonzero) ideal J of D and some nonzero $c \in D$. So, in particular, in this situation $\boldsymbol{f}(D) = \boldsymbol{F}(D)$ and every element of the monoid $\boldsymbol{f}^v(D)$ is von Neumann regular (we do not even need to apply the v–operation in this case), however not all quadratic orders are integrally closed (e.g., $D := \mathbb{Z}[\sqrt{5}]$) and so, in general, not all elements of $\boldsymbol{f}^v(D)$ are regular with respect to the v–operation (i.e., D is not a v–domain).

Clifford regularity for class and t–class semigroups of ideals in various types of integral domains was investigated, for instance, in [20 and 21, Bazzoni (1996), (2001)] [47, Fossum (1973)], [71,72 and 73, Kabbaj-Mimouni, (2003), (2007), (2008)], [114, Sega (2007)], and [54 and 55, Halter-Koch (2007), (2008)]. In particular, in the last paper, Halter-Koch proves a stronger and much deeper version of (β), that is, a v–domain having the t–class semigroups of ideals Clifford regular is a domain of Krull-type (i.e., a PvMD with finite t–character). This result generalizes [80, Kabbaj-Mimouni (2007), Theorem 3.2] on Prüfer v–multiplication domains.

(e) In the situation of point (d, β), the condition that every v–finite v–ideal is regular, in the sense of von Neumann, in the larger monoid $\boldsymbol{F}^v(D) := \{A^v \mid A \in \boldsymbol{F}(D)\}$ of all v–ideals of D (under v–multiplication) is too weak for concluding that D is a PvMD.

As a matter of fact, if we assume that D is a v–domain, then every $A \in \boldsymbol{f}^v(D)$ is v–invertible in the (larger) monoid $\boldsymbol{F}^v(D)$. Therefore, A is von Neumann regular in $\boldsymbol{F}^v(D)$, since $(AB)^v = D$ for some $B \in \boldsymbol{F}^v(D)$ and thus, multiplying both sides by A and applying the v–operation, we get $(A^2B)^v = A$.

Remark 2.6. Regularly integrally closed integral domains make their appearance with a different terminology in the study of a weaker form of integrality, introduced in the paper [14, D.F. Anderson-Houston-Zafrullah (1991)]. Recall that, given an integral domain D with quotient field K, an element $z \in K$ is called *pseudo-integral over* D if $z \in (F^v : F^v)$ for some $F \in \boldsymbol{f}(D)$. The terms *pseudo-integral closure* (i.e., $\widetilde{D} := \bigcup\{(F^v : F^v)) \mid F \in \boldsymbol{f}(D)\}$ and *pseudo-integrally closed domain* (i.e., $D = \widetilde{D}$) are coined in the obvious fashion and it is clear from the definition that pseudo-integrally closed coincides with regularly integrally closed.

From the previous observations, we have the following addition to the existing picture:

$$\text{CIC domain} \ \Rightarrow_{10} \ \ v\text{–domain} \ \Rightarrow_{11} \ \text{integrally closed domain.}$$

Note that in the Noetherian case, the previous three classes of domains coincide (see the following Proposition 2.8 (2) or [51, Gilmer (1972), Theorem 34.3 and Proposition 34.7]). Recall also that Krull domains can be characterized by the property that, for all $A \in \boldsymbol{F}(D)$, A is t–invertible [83, Kang (1989), Theorem 3.6]. This property is clearly stronger than the condition (iii$_{\boldsymbol{f}}$) of previous Theorem 2.4 and, more precisely, it is strictly stronger than (iii$_{\boldsymbol{f}}$), since a Krull domain is CIC (by condition (iii) of the above characterizations of CIC domains, see also [23, BAC, Ch.7, §1, N. 3, Théorème 2]) and a CIC domain is a v–domain, but the converse does not hold, as we will see in the following Section 2.e.

Remark 2.7. Note that Okabe and Matsuda [104, Okabe-Matsuda (1992)] generalized pseudo-integral closure to the star operation setting. Given a star operation $*$ on an integral domain D, they call the *$*$–integral closure of* D its overring $\bigcup\{(F^* : F^*) \mid F \in \boldsymbol{f}(D)\}$ denoted by $\mathtt{cl}^*(D)$ in [56, Halter-Koch (1997)]. Note that, in view of this notation, $\widetilde{D} = \mathtt{cl}^v(D)$ (Remark 2.6) and the integral closure $\overline{D}$ of D coincides with $\mathtt{cl}^d(D)$ [51, Gilmer (1972), Proposition 34.7]. Clearly, if $*_1$ and $*_2$ are two star operations on D and $*_1 \leq *_2$, then $\mathtt{cl}^{*_1}(D) \subseteq \mathtt{cl}^{*_2}(D)$. In particular, for each star operation $*$ on D, we have $\overline{D} \subseteq \mathtt{cl}^*(D) \subseteq \widetilde{D}$.

It is not hard to see that $\mathtt{cl}^*(D)$ is integrally closed [104, Okabe-Matsuda (1992), Theorem 2.8] and it is contained in the complete integral closure of D, which coincides with $\bigcup\{(A : A) \mid A \in \boldsymbol{F}(D)\}$ [51, Gilmer (1972), Theorem 34.3].

Recall also that, in [57, Halter-Koch (1998), Section 3], the author introduces a star operation of finite type on the integral domain $\mathtt{cl}^*(D)$, that we denote here by $\mathtt{cl}(*)$, defined as follows, for all $G \in \boldsymbol{f}(\mathtt{cl}^*(D))$:

$$G^{\mathtt{cl}(*)} := \bigcup\{((F^* : F^*)G)^* \mid F \in \boldsymbol{f}(\mathtt{cl}^*(D))\}\,.$$

Clearly, $D^{\mathrm{cl}(*)} = \mathtt{cl}^*(D)$. Call an integral domain D *$*$–integrally closed* when $D = \mathtt{cl}^*(D)$. Then, from the fact that $\mathtt{cl}(*)$ is a star operation on $\mathtt{cl}^*(D)$, it follows that $\mathtt{cl}^*(D)$ is $\mathtt{cl}(*)$–integrally closed. In general, if D is not necessarily $*$–integrally closed, then $\mathtt{cl}(*)$, defined on $\boldsymbol{f}(D)$, gives rise naturally to a semistar operation (of finite type) on D [40, Fontana-Loper (2001), Definition 4.2].

Note that the domain $\widetilde{D}$ $(= \mathtt{cl}^v(D))$, even if it is $\mathtt{cl}(v)$–integrally closed, in general is not $v_{\widetilde{D}}$–integrally closed; a counterexample is given in [14, D.F. Anderson-Houston-Zafrullah (1991), Example 2.1] by using a construction due to [53, Gilmer-Heinzer (1966)]. On the other hand, since an integral domain D is a v–domain if and only if $D = \mathtt{cl}^v(D)$ (Theorem 2.4), from the previous observation we deduce that, in general, $\widetilde{D}$ is not a v–domain. On the other hand, using a particular "$D{+}M$ construction", in [104, Okabe-Matsuda (1992), Example 3.4], the authors construct an example of a non–v–domain D such that $\widetilde{D}$ is a v–domain, i.e., $D \subsetneq \widetilde{D} = \mathtt{cl}^{v_{\widetilde{D}}}(\widetilde{D})$.

2.e Irreversibility of the implications "$\Rightarrow_n$". We start by observing that, under standard finiteness assumptions, several classes of domains considered above coincide. Recall that an integral domain D is called *v–coherent* if a finite intersection of v–finite v–ideals is a v–finite v–ideal or, equivalently, if F^{-1} is v–finite for all $F \in \boldsymbol{f}(D)$ [34, Fontana-Gabelli (1996), Proposition 3.6] and it is called a *v–finite conductor domain* if the intersection of two principal ideals is v–finite [32, Dumitrescu-Zafrullah (2008)]. From the definitions, it follows that a v–coherent domain is a v–finite conductor domain. From Remark 2.2 (a.1), we deduce immediately that

D is a PvMD $\Leftrightarrow$ D is a v–coherent v–domain.

In case of a v–domain, the notions of v–finite conductor domain and v–coherent domain coincide. As a matter of fact, as we have observed in Remark 2.5 (c), a PvMD is exactly a t–Prüfer domain and an integral domain D is t–Prüfer if and only if every nonzero two generated ideal is t–invertible. Which translates to D is a PvMD if and only if (a, b) is v–invertible and $(a) \cap (b)$ is v–finite, for all $a, b \in D$ (see also Remark 2.5 (b)). In other words,

D is a PvMD $\Leftrightarrow$ D is a v–finite conductor v–domain.

Recall that an integral domain D is a GGCD domain if and only if D is a PvMD that is a locally GCD domain [3, Anderson-Anderson (1979), Corollary 1 and page 218] or [122, Zafrullah (1987), Corollary 3.4]. On the other hand, we have already observed that a locally GCD domain is essential and it is known that an essential v–finite conductor domain is a PvMD [120, Zafrullah (1978), Lemma 8]. The situation is summarized in the following:

Proposition 2.8. *Let D be an integral domain.*

(1) *Assume that D is a v–finite conductor (e.g., Noetherian) domain. Then, the following classes of domains coincide:*

(a) *PvMD's;*
(b) *locally PvMD's;*
(c) *P–domains;*
(d) *essential domains.*
(e) *locally v–domains;*
(f) *v–domains.*

(2) *Assume that D is a Noetherian domain. Then, the previous classes of domains* (a)–(f) *coincide also with the following:*

(g) *Krull domains;*
(h) *CIC domains;*
(i) *integrally closed domains.*

(3) *Assume that D is a v–finite conductor (e.g., Noetherian) domain. Then, the following classes of domains coincide:*

(j) *GGCD domains;*
(k) *locally GCD domains.*

Since the notion of Noetherian Bézout (respectively, Noetherian GCD) domain coincides with the notion of PID or principal ideal domain (respectively, of Noetherian UFD (= unique factorization domain) [51, Gilmer (1972), Proposition 16.4]), in the Noetherian case the picture of all classes considered above reduces to the following:

$$\begin{array}{l}\text{Dedekind domain} \Rightarrow_{1,2,3,4,5} \quad v\text{–domain} \\ \qquad\qquad\qquad \text{PID} \Rightarrow_{6} \quad \text{UFD} \;\Rightarrow_{7,8}\; \text{locally UFD} \;\Rightarrow_{9,3,4,5} v\text{–domain.}\end{array}$$

In general, of the implications $\Rightarrow_n$ (with $0 \leq n \leq 11$) discussed above all, except $\Rightarrow_3$, are known to be irreversible. We leave the case of irreversibility of $\Rightarrow_3$ as an open question and proceed to give examples to show that all the other implications are irreversible.

• Irreversibility of $\Rightarrow_0$. Take any nondiscrete valuation domain or, more generally, a Prüfer non-Dedekind domain.

• Irreversibility of $\Rightarrow_1$ (even in the Noetherian case). Let D be a Prüfer domain that is not a field and let X be an indeterminate over D. Then, as $D[X]$ is a PvMD if and only if D is [11, D.D. Anderson-Kwak-Zafrullah (1995), Corollary 3.3] (see also the following Section 4), we conclude that $D[X]$ is a PvMD that is not Prüfer. An explicit example is $\mathbb{Z}[X]$, where $\mathbb{Z}$ is the ring of integers.

• Irreversibility of $\Rightarrow_2$. It is well known that every ring of fractions of a PvMD is again a PvMD [67, Heinzer-Ohm (1973), Proposition 1.8] (see also the following Section 3). The fact that $\Rightarrow_2$ is not reversible has been shown by producing examples of locally PvMD's that are not PvMD's. In [98, Mott-Zafrullah (1981), Example 2.1] an example of a non PvMD essential domain due to Heinzer and Ohm [67, Heinzer-Ohm (1973)] was shown to have the property that it was locally PvMD and hence a P-domain.

• Irreversibility of $\Rightarrow_3$: Open. However, as mentioned above, [98, Mott-Zafrullah (1981), Example 2.1] shows the existence of a P-domain which is not a PvMD. Note that [123, Zafrullah (1988), Section 2] gives a general method of constructing P-domains that are not PvMD's.

• Irreversibility of $\Rightarrow_4$. An example of an essential domain which is not a P-domain was constructed in [65, Heinzer (1981)]. Recently, in [39, Fontana-Kabbaj (2004), Example 2.3], the authors show the existence of n-dimensional essential domains which are not P-domains, for all $n \geq 2$.

• Irreversibility of $\Rightarrow_5$. Note that, by $\Rightarrow_{10}$, a CIC domain is a v–domain and Nagata, solving with a counterexample a famous conjecture stated by Krull in 1936, has produced an example of a one dimensional quasilocal CIC domain that is not a valuation ring (cf. [99, Nagata (1952)], [100, Nagata (1955)], and [112, Ribenboim (1956)]). This proves that a v–domain may not be essential. It would be desirable to have an example of a nonessential v–domain that is simpler than Nagata's example.

• Irreversibility of $\Rightarrow_6$ (even in the Noetherian case). This case can be handled in the same manner as that of $\Rightarrow_1$, since a polynomial domain over a GCD domain is still a GCD domain (cf. [84, Kaplansky (1970), Exercise 9, page 42]).

• Irreversibility of $\Rightarrow_7$ (even in the Noetherian case). Note that a Prüfer domain is a GGCD domain, since a GGCD domain is characterized by the fact that F^v is invertible for all $F \in \boldsymbol{f}(D)$ [3, Anderson-Anderson (1979), Theorem 1]. Moreover, a Prüfer domain D is a Bézout domain if and only if D is GCD. In fact, according to [27, Cohn (1968)] a Prüfer domain D is Bézout if and only if D is a generalization of GCD domains called a *Schreier domain* (i.e., an integrally closed integral domain whose group of divisibility is a *Riesz group*, that is a partially ordered directed group G having the following interpolation property: given $a_1, a_2, ..., a_m, b_1, b_2, ..., b_n \in G$ with $a_i \leq b_j$, there exists $c \in G$ with $a_i \leq c \leq b_j$ see [27, Cohn (1968)] and also [2, D.D. Anderson (2000), Section 3]). Therefore, a Prüfer non-Bézout domain (e.g., a Dedekind non principal ideal domain, like $\mathbb{Z}[i\sqrt{5}]$) shows the irreversibility of $\Rightarrow_7$.

• Irreversibility of $\Rightarrow_8$. From the characterization of GGCD domains recalled in the irreversibility of $\Rightarrow_7$ [3, Anderson-Anderson (1979), Theorem 1], it follows that a GGCD domain is a PvMD. More precisely, as we have already observed just before Proposition 2.8, an integral domain D is a GGCD domain if and only if D is a PvMD that is a locally GCD domain. Finally, as noted above, there are examples in [123, Zafrullah (1988)] of locally GCD domains that are not PvMD's. More explicitly, let $\boldsymbol{E}$ be the *ring of entire functions* (i.e., complex functions that are analytic in the whole plane). It is well known that $\boldsymbol{E}$ is a Bézout domain and every nonzero non unit $x \in \boldsymbol{E}$

is uniquely expressible as an associate of a "countable" product $x = \prod p_i^{e_i}$, where $e_i \geq 0$ and p_i is an irreducible function (i.e., a function having a unique root) [68, Helmer (1940), Theorems 6 and 9]. Let S be the multiplicative set of $\boldsymbol{E}$ generated by the irreducible functions and let X be an indeterminate over $\boldsymbol{E}$, then $\boldsymbol{E}+X\boldsymbol{E}_S[X]$ is a locally GCD domain that is not a PvMD [123, Zafrullah (1988), Example 2.6 and Proposition 4.1].

• Irreversibility of $\Rightarrow_9$ (even in the Noetherian case). This follows easily from the fact that there do exist examples of Krull domains (which we have already observed are locally PvMD's) that are not locally factorial (e.g., a non-UFD local Noetherian integrally closed domain, like the power series domain $D[\![X]\!]$ constructed in [113, Samuel (1961)], where D is a two dimensional local Noetherian UFD). As a matter of fact, a Krull domain which is a GCD domain is a UFD, since in a GCD domain, for all $F \in \boldsymbol{f}(D)$, F^v is principal and so the class group $\mathrm{Cl}(D) = 0$ [24, Bouvier-Zafrullah (1988), Section 2]; on the other hand, a Krull domain is factorial if and only if $\mathrm{Cl}(D) = 0$ [46, Fossum (1973), Proposition 6.1].

• Irreversibility of $\Rightarrow_{10}$. Let R be an integral domain with quotient field L and let X be an indeterminate over L. By [28, Costa-Mott-Zafrullah (1978), Theorem 4.42] $T := R + XL[X]$ is a v–domain if and only if R is a v–domain. Therefore, if R is not equal to L, then obviously T is an example of a v–domain that is not completely integrally closed (the complete integral closure of T is $L[X]$ [51, Gilmer (1972), Lemma 26.5]). This establishes that $\Rightarrow_{10}$ is not reversible.

Note that, in [34, Fontana-Gabelli (1996), Section 4] is studied the transfer in pullback diagrams of the PvMD property and related properties. A characterization of v–domains in pullbacks is proved in [48, Gabelli-Houston (1997), Theorem 4.15]. We summarize these results in the following:

Theorem 2.9. *Let R be an integral domain with quotient field k and let T be an integral domain with a maximal ideal M such that $L := T/M$ is a field extension of k. Let $\varphi : T \to L$ be the canonical projection and consider the following pullback diagram:*

$$\begin{array}{ccc} D := \varphi^{-1}(R) & \longrightarrow & R \\ \downarrow & & \downarrow \\ T_1 := \varphi^{-1}(k) & \longrightarrow & k \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & L \end{array}$$

Then, D is a v–domain (respectively, a PvMD) if and only if $k = L$, T_M is a valuation domain and R and T are v–domains (respectively, PvMD's).

Remark 2.10. Recently, bringing to, a sort of, a close a lot of efforts to restate results of [28, Costa-Mott-Zafrullah (1978)] in terms of very general pullbacks, in the paper [74, Houston-Taylor (2007)], the authors use some remarkable techniques to prove a generalization of the previous theorem. Although that paper is not about v–domains in particular, but it does have a few good results on v–domains. One of these results will be recalled in Proposition 3.6. Another one, with a pullback flavor, can be stated as follows: *Let I be a nonzero ideal of an integral domain D and set $T := (I : I)$. If D is a v–domain (respectively, a PvMD) then T is a v–domain (respectively, a PvMD)* [74, Houston-Taylor (2007), Proposition 2.5].

• Irreversibility of $\Rightarrow_{11}$. Recall that an integral domain D is called a *Mori domain* if D satisfies ACC on its integral divisorial ideals. According to [101, Nishimura (1963), Lemma 1] or [110, Querré (1971)], D is a Mori domain if and only if for every nonzero integral ideal I of D there is a finitely generated ideal $J \subseteq I$ such that $J^v = I^v$ (see also [19, Barucci (2000)] for an updated survey on Mori domains). Thus, if D is a Mori domain then D is CIC (i.e., every nonzero ideal is v–invertible) if and only if D is a v–domain (i.e., every nonzero finitely generated ideal is v–invertible). On the other hand, a completely integrally closed Mori domain is a Krull domain (see for example [46, Fossum (1973), Theorem 3.6]). More precisely, Mori v–domains coincide with Krull domains [102, Nishimura (1967), Theorem]. Therefore an integrally closed Mori non Krull domain provides an example of the irreversibility of $\Rightarrow_{11}$. An explicit example is given next.

It can be shown that, if $k \subseteq L$ is an extension of fields and if X is an indeterminate over L, then $k + XL[X]$ is always a Mori domain (see, for example, [48, Gabelli-Houston (1997), Theorem 4.18] and references there to previous papers by V. Barucci and M. Roitman). It is easy to see that the complete integral closure of $k + XL[X]$ is precisely $L[X]$ [51, Gilmer (1972), Lemma 26.5]. Thus if $k \subsetneq L$ then $k + XL[X]$ is not completely integrally closed and, as an easy consequence of the definition of integrality, it is integrally closed if and only if k is algebraically closed in L. This shows that there do exist integrally closed Mori domains that are not Krull. A very explicit example is given by $\overline{\mathbb{Q}} + X\mathbb{R}[X]$, where $\mathbb{R}$ is the field of real numbers and $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{R}$.

3 v–domains and rings of fractions

We have already mentioned that, if S is a multiplicative set of a PvMD D, then D_S is still a PvMD [67, Heinzer-Ohm (1973), Proposition 1.8]. The easiest proof of this fact can be given noting that, given $F \in \boldsymbol{f}(D)$, if F is t–invertible in D then FD_S is t–invertible in D_S, where S is a multiplicative set of D [24, Bouvier-Zafrullah (1988), Lemma 2.6]. It is natural to ask if D_S is a v–domain when D is a v–domain.

The answer is no. As a matter of fact an example of an essential domain D with a prime ideal P such that D_P is not essential was given in [65, Heinzer

(1981)]. What is interesting is that an essential domain is a v–domain by Proposition 2.1 and that, in this example, D_P is a (non essential) overring of the type $k + XL[X]_{(X)} = (k + XL[X])_{XL[X]}$, where L is a field and k its subfield that is algebraically closed in L. Now, a domain of type $k+XL[X]_{(X)}$ is an integrally closed local Mori domain, see [48, Gabelli-Houston (1997), Theorem 4.18]. In the irreversibility of $\Rightarrow_{11}$, we have also observed that if a Mori domain is a v–domain then it must be CIC, i.e., a Krull domain, and hence, in particular, an essential domain. Therefore, Heinzer's construction provides an example of an essential (v–)domain D with a prime ideal P such that D_P is not a v–domain.

Note that a similar situation holds for CIC domains. If D is CIC then it may be that for some multiplicative set S of D the ring of fractions D_S is not a completely integrally closed domain. A well known example in this connection is the ring $\boldsymbol{E}$ of entire functions. For $\boldsymbol{E}$ is a completely integrally closed Bézout domain that is infinite dimensional (see [61 and 62, Henriksen (1952), (1953)], [51, Gilmer (1972), Examples 16-21, pages 146-148] and [37, Fontana-Huckaba-Papick (1997), Section 8.1]). Localizing $\boldsymbol{E}$ at one of its prime ideals of height greater than one would give a valuation domain of dimension greater than one, which is obviously not completely integrally closed [51, Gilmer (1972), Theorem 17.5]. For another example of a CIC domain that has non–CIC rings of fractions, look at the integral domain of integer-valued polynomials $\mathrm{Int}(\mathbb{Z})$ [6, Anderson-Anderson-Zafrullah (1991), Example 7.7 and the following paragraph at page 127]. (This is a non-Bézout Prüfer domain, being atomic and two-dimensional.)

Note that these examples, like other well known examples of CIC domains with some overring of fractions not CIC, are all such that their overrings of fractions are at least v–domains (hence, they do not provide further counterexamples to the transfer of the v–domain property to the overrings of fractions). As a matter of fact, the examples that we have in mind are CIC Bézout domains with Krull dimension ≥ 2 (and polynomial domains over them), constructed using Krull-Jaffard-Ohm-Heinzer Theorem (for the statement, a brief history and applications of this theorem see [51, Gilmer (1972), Theorem 18.6, page 214, page 136, Example 19.12]). Therefore, it would be instructive to find an example of a CIC domain whose overrings of fractions are not all v–domains. Slightly more generally, we have the following.

It is well known that if $\{D_\lambda \mid \lambda \in \Lambda\}$ is a family of overrings of D with $D = \bigcap_{\lambda\in\Lambda} D_\lambda$ and if each D_λ is a completely integrally closed (respectively, integrally closed) domain then so is D (for the completely integrally closed case see for instance [51, Gilmer (1972), Exercise 11, page 145]; the integrally closed case is a straightforward consequence of the definition). It is natural to ask if in the above statement "completely integrally closed/integrally closed domain" is replaced by "v–domain" the statement is still true.

The answer in general is no, because by Krull's theorem every integrally closed integral domain is expressible as an intersection of a family of its valuation overrings (see e.g. [51, Gilmer (1972), Theorem 19.8]) and of course a valuation domain is a v–domain. But, an integrally closed domain is not necessarily a v–domain (see the irreversibility of $\Rightarrow_{11}$). If however each of D_λ is a ring of fractions of D, then the answer is yes. A slightly more general statement is given next.

Proposition 3.1. *Let $\{D_\lambda \mid \lambda \in \Lambda\}$ be a family of flat overrings of D such that $D = \bigcap_{\lambda\in\Lambda} D_\lambda$. If each of D_λ is a v–domain then so is D.*

Proof. Let v_λ be the v–operation on D_λ and let $\ast := \wedge v_\lambda$, be the star operation on D defined by $A \mapsto A^\ast := \bigcap_\lambda (AD_\lambda)^{v_\lambda}$, for all $A \in \boldsymbol{F}(D)$ [1, D.D. Anderson (1988), Theorem 2]. To show that D is a v–domain it is sufficient to show that every nonzero finitely generated ideal is $\ast$–invertible (for $\ast \leq v$ and so, if $F \in \boldsymbol{f}(D)$ and $(FF^{-1})^\ast = D$, then applying the v–operation to both sides we get $(FF^{-1})^v = D$).

Now, we have

$$\begin{aligned}(FF^{-1})^\ast &= \textstyle\bigcap_\lambda ((FF^{-1})D_\lambda)^{v_\lambda} = \bigcap_\lambda ((FD_\lambda)(F^{-1}D_\lambda))^{v_\lambda} \\ &= \textstyle\bigcap_\lambda ((FD_\lambda)(FD_\lambda)^{-1})^{v_\lambda} \quad \text{(since } D_\lambda \text{ is } D\text{–flat and } F \text{ is f.g.)} \\ &= \textstyle\bigcap_\lambda D_\lambda \quad \text{(since } D_\lambda \text{ is a } v_\lambda\text{–domain)} \\ &= D\,.\end{aligned}$$

Corollary 3.2. *Let Δ be a nonempty family of prime ideals of D such that $D = \bigcap\{D_P \mid P \in \Delta\}$. If D_P is a v–domain for each $P \in \Delta$, then D is a v–domain. In particular, if D_M is a v–domain for all $M \in \mathrm{Max}(D)$ (for example, if D is locally a v–domain, i.e., D_P is a v–domain for all $P \in \mathrm{Spec}(D)$), then D is a v–domain.*

Note that the previous Proposition 3.1 and Corollary 3.2 generalize Proposition 2.1, which ensures that an essential domain is a v-domain. Corollary 3.2 in turn leads to an interesting conclusion concerning the overrings of fractions of a v–domain.

Corollary 3.3. *Let S be a multiplicative set in D. If D_P is a v–domain for all prime ideals P of D such that P is maximal with respect to being disjoint from S, then D_S is a v–domain.*

In Corollary 3.2 we have shown that, if D_M is a v–domain for all $M \in \mathrm{Max}(D)$, then D is a v–domain. However, if D_P is a v–domain for all $P \in \mathrm{Spec}(D)$, we get much more in return. To indicate this, we note that, if S is a multiplicative set of D, then $D_S = \bigcap\{D_Q \mid Q$ ranges over associated primes of principal ideals of D with $Q \cap S = \emptyset\}$ [25, Brewer-Heinzer (1974), Proposition 4] (the definition of associated primes of principal ideals was recalled in Remark 2.3). Indeed, if we let $S = \{1\}$, then we have $D = \bigcap D_Q \mid Q$ ranges over all associated primes of principal ideals of $D\}$ (see also [84, Kaplansky (1970), Theorem 53] for a "maximal-type" version of this property). Using this terminology and the information at hand, it is easy to prove the following result.

Proposition 3.4. *Let D be an integral domain. Then, the following are equivalent.*

(i) *D is a v-domain such that, for every multiplicative set S of D, D_S is a v–domain.*
(ii) *For every nonzero prime ideal P of D, D_P is a v-domain.*
(iii) *For every associated prime of principal ideals of D, Q, D_Q is a v–domain.*

From the previous considerations, we have the following addition to the existing picture:

$$\text{locally P}v\text{MD } \Rightarrow_{12} \text{locally } v\text{–domain } \Rightarrow_{13} v\text{–domain.}$$

The example discussed at the beginning of this section shows the irreversibility of $\Rightarrow_{13}$. Nagata's example (given for the irreversibility of $\Rightarrow_5$) of a one dimensional quasilocal CIC domain that is not a valuation ring shows also the irreversibility of $\Rightarrow_{12}$.

Remark 3.5. In the same spirit of Proposition 3.4, we can make the following statement for CIC domains: *Let D be an integral domain. Then, the following are equivalent:*

(i) *D is a CIC domain such that, for every multiplicative set S of D, D_S is CIC.*
(ii) *For every nonzero prime ideal P of D, D_P is CIC.*
(iii) *For every associated prime of principal ideals of D, Q, D_Q is CIC.*

At the beginning of this section, we have mentioned the existence of examples of v–domains (respectively, CIC domains) having some localization at prime ideals which is not a v–domain (respectively, a CIC domain). Therefore, the previous equivalent properties (like the equivalent properties of Proposition 3.4) are strictly stronger than the property of being a CIC domain (respectively, v–domain).

On the other hand, for the case of integrally closed domains, the fact that, for every nonzero prime ideal P of D, D_P is integrally closed (or, for every maximal ideal M of D, D_M is integrally closed) returns exactly the property that D is integrally closed (i.e., the "integrally closed property" is a local property; see, for example, [16, Atiyah-Macdonald (1969), Proposition 5.13]). Note that, more generally, the semistar integral closure is a local property (see for instance [58, Halter-Koch (2003), Theorem 4.11]).

We have just observed that a ring of fractions of a v–domain may not be a v–domain, however there are distinguished classes of overrings for which the ascent of the v–domain property is possible.

Given an extension of integral domains $D \subseteq T$ with the same field of quotients, T is called *v–linked* (respectively, *t–linked*) *over* D if whenever I

is a nonzero (respectively, finitely generated) ideal of D with $I^{-1} = D$ we have $(IT)^{-1} = T$. It is clear that v–linked implies t–linked and it is not hard to prove that flat overring implies t–linked [31, Dobbs-Houston-Lucas-Zafrullah (1989), Proposition 2.2]. Moreover, the complete integral closure and the pseudo-integral closure of an integral domain D are t–linked over D (see [31, Dobbs-Houston-Lucas-Zafrullah (1989), Proposition 2.2 and Corollary 2.3] or [56, Halter-Koch (1997), Corollary 2]). Examples of v–linked extensions can be constructed as follows: take any nonzero ideal I of an integral domain then the overring $T := (I^v : I^v)$ is a v–linked overring of D [74, Houston-Taylor (2007), Lemma 3.3].

The t–linked extensions were used in [31, Dobbs-Houston-Lucas-Zafrullah (1989)] to deepen the study of PvMD's. It is known that *an integral domain D is a PvMD if and only if each t–linked overring of D is a PvMD* (see [71, Houston (1986), Proposition 1.6], [82, Kang (1989), Theorem 3.8 and Corollary 3.9]). More generally, in [31, Dobbs-Houston-Lucas-Zafrullah (1989), Theorem 2.10], the authors prove that *an integral domain D is a PvMD if and only if each t–linked overring is integrally closed.* On the other hand, a ring of fractions of a v–domain may not be a v–domain, so a t–linked overring of a v–domain may not be a v–domain. However, when it comes to a v–linked overring we get a different story. The following result is proven in [74, Houston-Taylor (2007), Lemma 2.4].

Proposition 3.6. *If D is a v–domain and T is a v–linked overring of D, then T is a v-domain.*

Proof. Let $J := y_1T + y_2T + ... + y_nT$ be a nonzero finitely generated ideal of T and set $F := y_1D + y_2D + ... + y_nD \in \boldsymbol{f}(D)$. Since D is a v–domain, $(FF^{-1})^v = D$ and, since T is v–linked, we have $(JF^{-1}T)^v = (FF^{-1}T)^v = T$. We conclude easily that $(J(T : J))^v = T$.

4 v–domains and polynomial extensions

4.a The polynomial ring over a v–domain. As for the case of integrally closed domains and of completely integrally closed domains [51, Gilmer (1972), Corollary 10.8 and Theorem 13.9], we have observed in the proof of irreversibility of $\Rightarrow_1$ that, *given an integral domain D and an indeterminate X over D,*

$$D[X] \text{ is a P}v\text{MD} \Leftrightarrow D \text{ is a P}v\text{MD.}$$

A similar statement holds for v–domains. As a matter of fact, *the following statements are equivalent* (see part (4) of [11, D.D. Anderson-Kwak-Zafrullah (1995), Corollary 3.3]).

(i) *For every $F \in \boldsymbol{f}(D)$, F^v is v–invertible in D.*

(ii) *For every* $G \in \boldsymbol{f}(D[X])$*,* G^v *is* v*–invertible in* $D[X]$*.*

This equivalence is essentially based on a polynomial characterization of integrally closed domains given in [111, Querré (1980)], for which we need some introduction. Given an integral domain D with quotient field K, an indeterminate X over K and a polynomial $f \in K[X]$, we denote by $\boldsymbol{c}_D(f)$ the *content of* f, i.e., the (fractional) ideal of D generated by the coefficients of f. For every fractional ideal B of $D[X]$, set $\boldsymbol{c}_D(B) := (\boldsymbol{c}_D(f) \mid f \in B)$. The integrally closed domains are characterized by the following property: for each integral ideal J of $D[X]$ such that $J \cap D \neq (0)$, $J^v = (\boldsymbol{c}_D(J)[X])^v = \boldsymbol{c}_D(J)^v[X]$ (see [111, Querré (1980), Section 3] and [11, D.D. Anderson-Kwak-Zafrullah (1995), Theorem 3.1]). Moreover, an integrally closed domain is an *agreeable domain* (i.e., for each fractional ideal B of $D[X]$, with $B \subseteq K[X]$, there exists $0 \neq s \in D$ –depending on B– with $sB \subseteq D$) [11, D.D. Anderson-Kwak-Zafrullah (1995), Theorem 2.2]. (Note that agreeable domains were also studied in [63, Hamann-Houston-Johnson (1988)] under the name of almost principal ideal domains.)

The previous considerations show that, for an integrally closed domain D, there is a close relation between the divisorial ideals of $D[X]$ and those of D [111, Querré (1980), Théorème 1 and Remarque 1]. The equivalence (i)⇔(ii) will now follow easily from the fact that, given an agreeable domain, for every integral ideal J of $D[X]$, there exist an integral ideal J_1 of $D[X]$ with $J_1 \cap D \neq (0)$, a nonzero element $d \in D$ and a polynomial $f \in D[X]$ in such a way that $J = d^{-1} f J_1$ [11, D.D. Anderson-Kwak-Zafrullah (1995), Theorem 2.1].

On the other hand, using the definitions of v–invertibility and v–multiplication, one can easily show that for $A \in \boldsymbol{F}(D)$, A is v–invertible if and only if A^v is v–invertible. By the previous equivalence (i)⇔(ii), we conclude that every $F \in \boldsymbol{f}(D)$ is v–invertible if and only if every $G \in \boldsymbol{f}(D[X])$ is v–invertible and this proves the following:

Theorem 4.1. *Given an integral domain* D *and an indeterminate* X *over* D*,* D *is a* v*-domain if and only if* $D[X]$ *is a* v*–domain.*

Note that a much more interesting and general result was proved in terms of pseudo-integral closures in [14, D.F. Anderson-Houston-Zafrullah (1991), Theorem 1.5 and Corollary 1.6].

4.b v**–domains and rational functions.** Characterizations of v–domains can be also given in terms of rational functions, using properties of the content of polynomials.

Recall that Gauss' Lemma for the content of polynomials holds for Dedekind domains (or, more generally, for Prüfer domains). A more precise and general statement is given next.

Lemma 4.2. *Let* D *be an integral domain with quotient field* K *and let* X *be an indeterminate over* D*. The following are equivalent.*

(i) *D is an integrally closed domain (respectively, a PvMD; a Prüfer domain).*
(ii) *for all nonzero $f, g \in K[X]$, $\boldsymbol{c}_D(fg)^v = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^v$ (respectively, $\boldsymbol{c}_D(fg)^w = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^w$; $\boldsymbol{c}_D(fg) = \boldsymbol{c}_D(f)\boldsymbol{c}_D(g)$).*

For the "Prüfer domains part" of the previous lemma, see [51, Gilmer (1972), Corollary 28.5], [116, Tsang (1965)], and [49, Gilmer (1967)]; for the "integrally closed domains part", see [88, Krull (1936), page 557] and [111, Querré (1980), Lemme 1]; for the "PvMD's part", see [13, D.F. Anderson-Fontana-Zafrullah (2008), Corollary 1.6] and [26, Chang (2008), Corollary 3.8]. For more on the history of Gauss' Lemma, the reader may consult [66, Heinzer-Huneke (1998), page 1306] and [2, D.D. Anderson (2000), Section 8].

For general integral domains, we always have the inclusion of ideals $\boldsymbol{c}_D(fg) \subseteq \boldsymbol{c}_D(f)\boldsymbol{c}_D(g)$, and, more precisely, we have the following famous lemma due to Dedekind and Mertens (for the proof, see [103, Northcott (1959)] or [51, Gilmer (1972), Theorem 28.1] and, for some complementary information, see [2, D.D. Anderson (2000), Section 8]):

Lemma 4.3. *In the situation of Lemma 4.2, let $0 \neq f, g \in K[X]$ and let $m := \deg(g)$. Then*

$$\boldsymbol{c}_D(f)^m \boldsymbol{c}_D(fg) = \boldsymbol{c}_D(f)^{m+1}\boldsymbol{c}_D(g)\,.$$

A straightforward consequence of the previous lemma is the following:

Corollary 4.4. *In the situation of Lemma 4.2, assume that, for a nonzero polynomial $f \in K[X]$, $\boldsymbol{c}_D(f)$ is v–invertible (e.g., t–invertible). Then $\boldsymbol{c}_D(fg)^v = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^v$ (or, equivalently, $\boldsymbol{c}_D(fg)^t = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^t$), for all nonzero $g \in K[X]$.*

From Corollary 4.4 and from the "integrally closed domain part" of Lemma 4.2, we have the following result (see [97, Mott-Nashier-Zafrullah (1990), Theorem 2.4 and Section 3]):

Corollary 4.5. *In the situation of Lemma 4.2, set $\mathtt{V}_D := \{g \in D[X] \mid \boldsymbol{c}_D(g)$ is v–invertible$\}$ and $\mathtt{T}_D := \{g \in D[X] \mid \boldsymbol{c}_D(g)$ is t–invertible$\}$. Then, $\mathtt{T}_D$ and V_D are multiplicative sets of $D[X]$ with $\mathtt{T}_D \subseteq V_D$. Furthermore, $\mathtt{V}_D$ (or, equivalently, $\mathtt{T}_D$) is saturated if and only if D is integrally closed.*

It can be useful to observe that, from Remark 2.2 (a.1), we have

$$\mathtt{T}_D = \{g \in \mathtt{V}_D \mid \boldsymbol{c}_D(g)^{-1} \text{ is } t\text{–finite}\}.$$

We are now in a position to give a characterization of v–domains (and PvMD's) in terms of rational functions (see [97, Mott-Nashier-Zafrullah (1990), Theorem 2.5 and Theorem 3.1]).

Theorem 4.6. *Suppose that D is an integrally closed domain, then the following are equivalent:*

(i) *D is a v-domain (respectively, a PvMD).*
(ii) $\mathcal{V}_D = D[X]\setminus\{0\}$ *(respectively,* $\mathcal{T}_D = D[X]\setminus\{0\}$*).*
(iii) $D[X]_{\mathcal{V}_D}$ *(respectively,* $D[X]_{\mathcal{T}_D}$*) is a field (or, equivalently,* $D[X]_{\mathcal{V}_D} = K(X)$ *(respectively,* $D[X]_{\mathcal{T}_D} = K(X)$*)).*
(iv) *Each nonzero element $z \in K$ satisfies a polynomial $f \in D[X]$ such that $\boldsymbol{c}_D(f)$ is v–invertible (respectively, t–invertible).*

Remark 4.7. Note that *quasi Prüfer domains* (i.e., integral domains having the integral closure Prüfer [18, Ayache-Cahen-Echi (1996)]) can also be characterized by using properties of the field of rational functions. In the situation of Lemma 4.2, set $\mathcal{S}_D := \{g \in D[X] \mid \boldsymbol{c}_D(g) \text{ is invertible}\}$. Then, by Lemma 4.4, the multiplicative set $\mathcal{S}_D$ of $D[X]$ is saturated if and only if D is integrally closed. Moreover, D is quasi Prüfer if and only if $D[X]_{\mathcal{S}_D}$ is a field (or, equivalently, $D[X]_{\mathcal{S}_D} = K(X)$) if and only if each nonzero element $z \in K$ satisfies a polynomial $f \in D[X]$ such that $\boldsymbol{c}_D(f)$ is invertible [97, Mott-Nashier-Zafrullah (1990), Theorem 1.7].

Looking more carefully at the content of polynomials, it is obvious that the set

$$\mathcal{N}_D := \{g \in D[X] \mid \boldsymbol{c}_D(g)^v = D\}$$

is a subset of $\mathcal{T}_D$ and it is well known that $\mathcal{N}_D$ is a saturated multiplicative set of $D[X]$ [82, Kang (1989), Proposition 2.1]. We call the *Nagata ring of D with respect to the v–operation* the ring:

$$\mathrm{Na}(D, v) := D[X]_{\mathcal{N}_D}\,.$$

We can also consider

$$\mathrm{Kr}(D, v) := \{f/g \mid f, g \in D[X],\ g \neq 0,\ \boldsymbol{c}_D(f)^v \subseteq \boldsymbol{c}_D(g)^v\}\,.$$

When v is an e.a.b. operation on D (i.e., when D is a v–domain) $\mathrm{Kr}(D, v)$ is a ring called the *Kronecker function ring of D with respect to the v–operation* [51, Gilmer (1972), Theorem 32.7]. Clearly, in general, $\mathrm{Na}(D, v) \subseteq \mathrm{Kr}(D, v)$. It is proven in [38, Fontana-Jara-Santos (2003), Theorem 3.1 and Remark 3.1] that $\mathrm{Na}(D, v) = \mathrm{Kr}(D, v)$ if and only if D is a PvMD.

Remark 4.8. (a) Concerning Nagata and Kronecker function rings, note that a unified general treatment and semistar analogs of several results were obtained in the recent years, see for instance [40, Fontana-Loper (2001)], [41, Fontana-Loper (2006)] and [42, Fontana-Loper (2007)].

(b) A general version of Lemma 4.2, in case of semistar operations, was recently proved in [13, D.F. Anderson-Fontana-Zafrullah (2008), Corollary 1.2].

4.c v–domains and uppers to zero. Recall that if X is an indeterminate over an integral domain D and if Q is a nonzero prime ideal of $D[X]$ such that $Q \cap D = (0)$ then Q is called *an upper to zero*. The "upper" terminology in polynomial rings is due to S. McAdam and was introduced in early 1970's. In a recent paper, Houston and Zafrullah introduce the *UMv–domains* as the integral domains such that the uppers to zero are maximal v–ideals and they prove the following result [76, Houston-Zafrullah (2005), Theorem 3.3].

Theorem 4.9. *Let D be an integral domain with quotient field K and let X be an indeterminate over K. The following are equivalent.*

(i) *D is a v-domain.*
(ii) *D is an integrally closed UMv–domain.*
(iii) *D is integrally closed and every upper to zero in $D[X]$ is v–invertible.*
(iii$_\ell$) *D is integrally closed and every upper to zero of the type $Q_\ell := \ell K[X] \cap D[X]$ with $\ell \in D[X]$ a linear polynomial is v–invertible.*

It would be unfair to end the section with this characterization of v-domains without giving a hint about where the idea came from.

Gilmer and Hoffmann in 1975 gave a characterization of Prüfer domains using uppers to zero. This result is based on the following characterization of essential valuation overrings of an integrally closed domain D: let P be a prime ideal of D, then D_P is a valuation domain if and only if, for each upper to zero Q of $D[X]$, $Q \not\subseteq P[X]$, [51, Gilmer (1972), Theorem 19.15].

A globalization of the previous statement leads to the following result that can be easily deduced from [54, Gilmer-Hoffmann (1975), Theorem 2].

Proposition 4.10. *In the situation of Theorem 4.9, the following are equivalent:*

(i) *D is a Prüfer domain.*
(ii) *D is integrally closed and if Q is an upper to zero of $D[X]$, then $Q \not\subseteq M[X]$, for all $M \in \mathrm{Max}(D)$ (i.e., $\boldsymbol{c}_D(Q) = D$).*

In [121, Zafrullah (1984), Proposition 4] the author proves a "t–version" of the previous result.

Proposition 4.11. *In the situation of Theorem 4.9, the following are equivalent:*

(i) *D is a PvMD.*
(ii) *D is integrally closed and if Q upper to zero of $D[X]$, then $Q \not\subseteq M[X]$, for all maximal t–ideal M of D (i.e., $\boldsymbol{c}_D(Q)^t = D$).*

The proof of the previous proposition relies on very basic properties of polynomial rings.

Note that in [121, Zafrullah (1984), Lemma 7] it is also shown that, if D is a PvMD, then every upper to zero in $D[X]$ is a maximal t–ideal. As we observed in Section 1, unlike maximal v–ideals, the maximal t–ideals are ubiquitous.

Around the same time, in [73, Houston-Malik-Mott (1984), Proposition 2.6], the authors came up with a much better result, using the $*$-operations much more efficiently. Briefly, this result said that the converse holds, i.e., *D is a PvMD if and only if D is an integrally closed integral domain and every upper to zero in $D[X]$ is a maximal t–ideal.*

It turns out that integral domains D such that their uppers to zero in $D[X]$ are maximal t–ideals (called *UMt-domains* in [75, Houston-Zafrullah (1989), Section 3]; see also [35, Fontana-Gabelli-Houston (1998)] and, for a survey on the subject, [72, Houston (2006)]) and domains such that, for each upper to zero Q of $D[X]$, $\boldsymbol{c}_D(Q)^t = D$ had an independent life. In [75, Houston-Zafrullah (1989), Theorem 1.4], studying t–invertibility, the authors prove the following result.

Proposition 4.12. *In the situation of Theorem 4.9, let Q be an upper to zero in $D[X]$. The following statements are equivalent.*

(i) *Q is a maximal t–ideal of $D[X]$.*
(ii) *Q is a t–invertible ideal of $D[X]$.*
(iii) *$\boldsymbol{c}_D(Q)^t = D$.*

Based on this result, one can see that the following statement is a precursor to Theorem 4.9.

Proposition 4.13. *Let D be an integral domain with quotient field K and let X be an indeterminate over K. The following are equivalent.*

(i) *D is a PvMD.*
(ii) *D is an integrally closed UMt–domain.*
(iii) *D is integrally closed and every upper to zero in $D[X]$ is t–invertible.*
(iii$_\ell$) *D is integrally closed and every upper to zero of the type $Q_\ell := \ell K[X] \cap D[X]$, with $\ell \in D[X]$ a linear polynomial, is t–invertible.*

Note that the equivalence (i)$\Leftrightarrow$(ii) is in [75, Houston-Zafrullah (1989), Proposition 3.2]. (ii)$\Leftrightarrow$(iii) is a consequence of previous Proposition 4.12. Obviously, (iii)$\Rightarrow$(iii$_\ell$). (iii$_\ell$)$\Rightarrow$(i) is a consequence of the characterization already cited that an integral domain D is a PvMD if and only if each nonzero two generated ideal is t–invertible [92, Malik-Mott-Zafrullah (1988), Lemma 1.7]. As a matter of fact, consider a nonzero two generated ideal $I := (a, b)$ in D, set $\ell := a + bX$ and $Q_\ell := \ell K[X] \cap D[X]$. Since D is integrally closed, then $Q_\ell = \ell \boldsymbol{c}_D(\ell)^{-1} D[X]$ by [111, Querré (1980), Lemme 1, page 282]. If Q_ℓ is t–invertible (in $(D[X])$, then it is easy to conclude that $\boldsymbol{c}_D(\ell) = I$ is t–invertible (in D).

Remark 4.14. Note that Prüfer domains may not be characterized by straight modifications of conditions (ii) and (iii) of Proposition 4.13. As a matter of fact, if there exists in $D[X]$ an upper to zero which is also a maximal ideal, then the domain D is a G(oldman)-domain (i.e., its quotient field is finitely generated over D), and conversely [84, Kaplansky (1970), Theorems 18 and 24]. Moreover, every upper to zero in $D[X]$ is invertible if and only if D is a GGCD domain [10, D.D. Anderson-Dumitrescu-Zafrullah (2007), Theorem 15].

On the other hand, a variation of condition (iii$_\ell$) of Proposition 4.13 does characterize Prüfer domains: *D is a Prüfer domain if and only if D is integrally closed and every upper to zero of the type $Q_\ell := \ell K[X] \cap D[X]$ with $\ell \in D[X]$ a linear polynomial is such that $\boldsymbol{c}_D(Q_\ell) = D$* [73, Houston-Malik-Mott (1984), Theorem 1.1].

5 v–domains and GCD–theories

In [22, Borevich-Shafarevich (1966), page 170], a *factorial monoid* $\mathfrak{D}$ is a commutative semigroup with a unit element $\mathfrak{1}$ (and without zero element) such that every element $\mathfrak{a} \in \mathfrak{D}$ can be uniquely represented as a finite product of atomic (= irreducible) elements $\mathfrak{q}_i$ of $\mathfrak{D}$, i.e., $\mathfrak{a} = \mathfrak{q}_1\mathfrak{q}_2 ... \mathfrak{q}_r$, with $r \geq 0$ and this factorization is unique up to the order of factors; for $r = 0$ this product is set equal to $\mathfrak{1}$. As a consequence, it is easy to see that this kind of uniqueness of factorization implies that $\mathfrak{1}$ is the only invertible element in $\mathfrak{D}$, i.e., $\mathfrak{U}(\mathfrak{D}) = \{\mathfrak{1}\}$. Moreover, it is not hard to see that, in a factorial monoid, any two elements have GCD and every atom is a prime element [57, Halter-Koch (1998), Theorem 10.7].

Let D be an integral domain and set $D^\bullet := D \setminus \{0\}$. In [22, Borevich-Shafarevich (1966), page 171] an integral domain D is said to have a *divisor theory* if there is a factorial monoid $\mathfrak{D}$ and a semigroup homomorphism, denoted by $(-)$: $D^\bullet \to \mathfrak{D}$, given by $a \mapsto (a)$, such that:

(D1) $(a) \mid (b)$ in $\mathfrak{D}$ if and only if $a \mid b$ in D for $a, b \in D^\bullet$.
(D2) If $\mathfrak{g} \mid (a)$ and $\mathfrak{g} \mid (b)$ then $\mathfrak{g} \mid (a \pm b)$ for $a, b \in D^\bullet$ with $a \pm b \neq 0$ and $\mathfrak{g} \in \mathfrak{D}$.
(D3) Let $\mathfrak{g} \in \mathfrak{D}$ and set

$$\overline{\mathfrak{g}} := \{x \in D^\bullet \text{ such that } \mathfrak{g} \mid (x)\} \cup \{0\}.$$

Then $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$ if and only if $\mathfrak{a} = \mathfrak{b}$ for all $\mathfrak{a}, \mathfrak{b} \in \mathfrak{D}$.

Given a divisor theory, the elements of the factorial monoid $\mathfrak{D}$ are called *divisors of the integral domain D* and the divisors of the type (a), for $a \in D$ are called *principal divisors of D*.

Note that, in [115, Skula (1970), page 119], the author shows that the axiom **(D2)**, which guarantees that $\overline{\mathfrak{g}}$ is an ideal of D, for each divisor $\mathfrak{g} \in \mathfrak{D}$, is unnecessary. Furthermore, note that divisor theories were also considered in [96, Močkoř (1993), Chapter 10], written in the spirit of Jaffard's volume [78].

Borevich and Shafarevich introduced domains with a divisor theory in order to generalize Dedekind domains and unique factorization domains, along the lines of Kronecker's classical theory of "algebraic divisors" (cf. [86, Kronecker (1882)] and also [119, Weyl (1940)] and [33, Edwards [1990)]). As a matter of fact, they proved that

(a) if an integral domain D has a divisor theory $(-)\colon D^\bullet \to \mathfrak{D}$ then it has only one (i.e., if $(\!(-)\!)\colon D^\bullet \to \mathfrak{D}'$ is another divisor theory then there is an isomorphism $\mathfrak{D} \cong \mathfrak{D}'$ under which the principal divisors in $\mathfrak{D}$ and $\mathfrak{D}'$, which correspond to a given nonzero element $a \in D$, are identified) [22, Borevich-Shafarevich (1966), Theorem 1, page 172];
(b) an integral domain D is a unique factorization domain if and only if D has a divisor theory $(-)\colon D^\bullet \to \mathfrak{D}$ in which every divisor of $\mathfrak{D}$ is principal [22, Borevich-Shafarevich (1966), Theorem 2, page 174];
(c) an integral domain D is a Dedekind domain if and only if D has a divisor theory $(-)\colon D^\bullet \to \mathfrak{D}$ such that, for every prime element $\mathfrak{p}$ of $\mathfrak{D}$, $D/\overline{\mathfrak{p}}$ is a field [22, Borevich-Shafarevich (1966), Chapter 3, Section 6.2].

Note that Borevich and Shafarevich do not enter into the details of the determination of those integral domains for which a theory of divisors can be constructed [22, Borevich-Shafarevich (1966), page 178], but it is known that they coincide with the Krull domains (see [118, van der Waerden (1931), §105], [17, Aubert (1983), Theorem 5], [90, Lucius (1998), §5], and [85, Krause (1989)] for the monoid case). In particular, note that, for a Krull domain, the group of non-zero fractional divisorial ideals provides a divisor theory.

Taking the above definition as a starting point and recalling that **(D2)** is unnecessary, in [90, Lucius (1998)], the author introduces a more general class of domains, called the domain with GCD–theory.

An integral domain D is said to have a *GCD–theory* if there is a GCD–monoid $\mathfrak{G}$ and a semigroup homomorphism, denoted by $(-)\colon D^\bullet \to \mathfrak{G}$, given by $a \mapsto (a)$, such that:

(G1) $(a) \mid (b)$ in $\mathfrak{G}$ if and only if $a \mid b$ in D for $a, b \in D^\bullet$.
(G2) Let $\mathfrak{g} \in \mathfrak{G}$ and set $\overline{\mathfrak{g}} := \{x \in D^\bullet \text{ such that } \mathfrak{g} \mid (x)\} \cup \{0\}$. Then $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$ if and only if $\mathfrak{a} = \mathfrak{b}$ for all $\mathfrak{a}, \mathfrak{b} \in \mathfrak{G}$.

Let $\mathfrak{Q} := \boldsymbol{q}(\mathfrak{G})$ be the group of quotients of the GCD–monoid $\mathfrak{G}$. It is not hard to prove that the natural extension a GCD–theory $(-)\colon D^\bullet \to \mathfrak{G}$ to a group homomorphism $(-)' : K^\bullet \to \mathfrak{Q}$ has the following properties:

(qG1) $(\alpha)' \mid (\beta)'$ with respect to $\mathfrak{G}$ if and only if $\alpha \mid \beta$ with respect to D for $\alpha, \beta \in K^\bullet$.
(qG2) Let $\mathfrak{h} \in \mathfrak{Q}$ and set $\overline{\mathfrak{h}} := \{\gamma \in K^\bullet \text{ such that } \mathfrak{h} \mid (\gamma)'\} \cup \{0\}$ (the division in $\mathfrak{Q}$ is with respect to $\mathfrak{G}$). Then $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$ if and only if $\mathfrak{a} = \mathfrak{b}$ for all $\mathfrak{a}, \mathfrak{b} \in \mathfrak{Q}$.

In [90, Lucius (1998), Theorem 2.5], the author proves the following key result, that clarifies the role of the ideal $\overline{\mathfrak{a}}$. (Call, as before, *divisors of* D the elements of the GCD–monoid $\mathfrak{G}$ and *principal divisors of* D the divisors of the type (a), for $a \in D^\bullet$.)

Proposition 5.1. *Let* D *be an integral domain with GCD–theory* $(-)\colon D^\bullet \to \mathfrak{G}$*, let* $\mathfrak{a}$ *be any divisor of* $\mathfrak{G}$ *and* $\{(a_i)\}_{i\in I}$ *a family of principal divisors with* $\mathfrak{a} = GCD(\{(a_i)\}_{i\in I})$*. Then* $\overline{\mathfrak{a}} = (\{a_i\}_{i\in I})^v = \overline{\mathfrak{a}}^v$*.*

As a part consequence of Proposition 5.1, we have a characterization of a v–domain as a domain with GCD-theory [90, Lucius (1998), Theorem and Definition 2.9].

Theorem 5.2. *Given an integral domain* D*,* D *is a ring with GCD–theory if and only if* D *is a* v*–domain.*

The "only if part" is a consequence of Proposition 5.1 (for details see [90, Lucius (1998), Corollary 2.8]).

The proof of the "if part" is constructive and provides explicitly the GCD–theory. The GCD–monoid is constructed via Kronecker function rings. Recall that, when v is an e.a.b. operation (i.e., when D is a v–domain (Theorem 2.4)), the Kronecker function ring with respect to v, $\mathrm{Kr}(D,v)$, is well defined and it is a Bézout domain [51, Gilmer (1972), Lemma 32.6 and Theorem 32.7]. Let $\mathfrak{K}$ be the monoid $\mathrm{Kr}(D,v)^\bullet$, let $\mathfrak{U} := \mathfrak{U}(\mathrm{Kr}(D,v))$ be the group of invertible elements in $\mathrm{Kr}(D,v)$ and set $\mathfrak{G} := \mathfrak{K}/\mathfrak{U}$. The canonical map:

$$[-] : D^\bullet \to \mathfrak{G} = \frac{\mathrm{Kr}(D,v)^\bullet}{\mathfrak{U}},\ \ a \mapsto [a] \ (= \text{the equivalence class of } a \text{ in } \mathfrak{G})$$

defines a GCD–theory for D, called the *Kroneckerian GCD–theory* for the v–domain D. In particular, the GCD of elements in D is realized by the equivalence class of a polynomial; more precisely, under this GCD–theory, given $a_0, a_1, ..., a_n \in D^\bullet$, $\mathrm{GCD}(a_0, a_1, ..., a_n) := \mathrm{GCD}([a_0], [a_1], ..., [a_n]) = [a_0 + a_1X + ... + a_nX^n]$.

It is classically known [22, Borevich-Shafarevich (1966), Chapter 3, Section 5] that the integral closure of a domain with divisor theory in a finite extension of fields is again a domain with divisor theory. For integral domains with GCD–theory a stronger result holds.

Theorem 5.3. *Let* D *be an integrally closed domain with field of fractions* K *and let* $K \subseteq L$ *be an algebraic field extension and let* T *be the integral closure of* D *in* L*. Then* T *is a* v*–domain (i.e., domain with GCD–theory) if and only if* D *is a* v*–domain (i.e., a domain with GCD–theory).*

The proof of the previous result is given in [90, Lucius (1998), Theorem 3.1] and it is based on the following facts:

In the situation of Theorem 5.3,

(a) For each ideal I of D, $I^{v_D} = (IT)^{v_T} \cap K$ [88, Krull (1936), Satz 9, page 675];
(b) If D is a v–domain, then the integral closure of $\mathrm{Kr}(D, v_D)$ in the algebraic field extension $K(X) \subseteq L(X)$ coincides with $\mathrm{Kr}(T, v_T)$ [90, Lucius (1998), Theorem 3.3].

Remark 5.4. (a) The notions of GCD–theory and divisor theory, being more in the setting of monoid theory, have been given a monoid treatment [57, Halter-Koch (1998), Exercises 18.10, 19.6 and Chapter 20].

(b) Note that a part of previous Theorem 5.3 appears also as a corollary to [59, Halter-Koch (2003),Theorem 3.6]. More precisely, let $\mathsf{cl}^v(D)$ $(:= \bigcup\{F^v : F^v) \mid F \in \boldsymbol{f}(D)\})$ be the v–(integral) closure of D. We have already observed (Theorem 2.4 and Remark 2.6) that an integral domain D is a v–domain if and only if $D = \mathsf{cl}^v(D)$. Therefore Theorem 5.3 is an easy consequence of the fact that, in the situation of Theorem 5.3, it can be shown that $\mathsf{cl}^v(T)$ is the integral closure of $\mathsf{cl}^v(D)$ in L [59, Halter-Koch (2003), Theorem 3.6].

(c) In [90, Lucius (1998), §4], the author develops a "stronger GCD–theory" in order to characterize PvMD's. A *GCD-theory of finite type* is a GCD–theory, (...), with the property that each divisor $\mathfrak{a}$ in the GCD–monoid $\mathfrak{G}$ is such that $\mathfrak{a} = \mathrm{GCD}((a_1), (a_2), ..., (a_n))$ for a finite number of nonzero elements $a_1, a_2, ..., a_n \in D$. For a PvMD, the group of non-zero fractional t–finite t–ideals provides a GCD–theory of finite type. (Note that the notion of a GCD–theory of finite type was introduced in [17, Aubert (1983)] under the name of "quasi divisor theory". A thorough presentation of this concept, including several characterizations of P$*$MD's, is in [57, Halter-Koch (1998), Chapter 20].)

The analogue of Theorem 5.2 can be stated as follows: *Given an integral domain D, D is a ring with GCD–theory of finite type if and only if D is a PvMD.* Also in this case, the GCD–theory of finite type and the GCD–monoid are constructed explicitly, via the Kronecker function ring $\mathrm{Kr}(D, v)$ (which coincides in this situation with the Nagata ring $\mathrm{Na}(D, v)$), for the details see [90, Lucius (1998), Theorem 4.4]. Moreover, in [90, Lucius (1998), Theorem 4.6] is given another proof of Prüfer theorem [109, Prüfer (1932), §11], analogous to Theorem 5.3: *Let D be an integrally closed domain with field of fractions K and let $K \subseteq L$ be an algebraic field extension and let T be the integral closure of D in L. Then T is a PvMD (i.e., domain with GCD–theory of finite type) if and only if D is PvMD (i.e., domain with GCD–theory of finite type).* Recall that a similar result holds for the special case of Prüfer domains [51, Gilmer (1972), Theorem 22.3].

6 Ideal-theoretic characterizations of v–domains

Important progress in the knowledge of the ideal theory for v–domains was made in 1989, after a series of talks given by the second named author while visiting several US universities. The results of various discussions of that period are contained in the "A to Z" paper [4, Anderson-Anderson-Costa-Dobbs-Mott-Zafrullah (1989)], which contains in particular some new characterizations of v-domains and of completely integrally closed domains. These characterizations were then expanded into a very long list of equivalent statements, providing further characterizations of (several classes of) v–domains [12, Anderson-Mott-Zafrullah (1989)].

Proposition 6.1. *Let D be an integral domain. Then, D is a v–domain if and only if D is integrally closed and $(F_1 \cap F_2 \cap ... \cap F_n)^v = F_1^v \cap F_2^v \cap ... \cap F_n^v$ for all $F_1, F_2, ..., F_n \in \boldsymbol{f}(D)$ (i.e., the v–operation distributes over finite intersections of finitely generated fractional ideals).*

The "if part" is contained in the "A to Z" paper (Theorem 7 of that paper, where the converse was left open). The converse of this result was proved a few years later in [94, Matsuda-Okabe (1993), Theorem 2].

Note that, even for a Noetherian 1-dimensional domain, the v–operation may not distribute over finite intersections of (finitely generated) fractional ideals. For instance, here is an example due to W. Heinzer cited in [8, D.D. Anderson-Clarke (2006), Example 1.2], let k be a field, X an indeterminate over k and set $D := k[\![X^3, X^4, X^5]\!]$, $F := (X^3, X^4)$ and $G := (X^3, X^5)$. Clearly, D is a non-integrally closed 1-dimensional local Noetherian domain with maximal ideal $M := (X^3, X^4, X^5) = F + G$. It is easy to see that $F^v = G^v = M$, and so $F \cap G = (X^3) = (F \cap G)^v \subsetneq F^v \cap G^v = M$.

Recently, D.D. Anderson and Clarke have investigated the star operations that distribute over finite intersections. In particular, in [7, D.D. Anderson-Clarke (2005), Theorem 2.8], they proved a star operation version of the "only if part" of Proposition 6.1 and, moreover, in [7, D.D. Anderson-Clarke (2005), Proposition 2.7] and [8, D.D. Anderson-Clarke (2006), Lemma 3.1 and Theorem 3.2] they established several other general equivalences that, particularized in the v–operation case, are summarized in the following:

Proposition 6.2. *Let D be an integral domain.*

(a) *$(F_1 \cap F_2 \cap ... \cap F_n)^v = F_1^v \cap F_2^v \cap ... \cap F_n^v$ for all $F_1, F_2, ..., F_n \in \boldsymbol{f}(D)$ if and only if $(F :_D G)^v = (F^v :_D G^v)$ for all $F, G \in \boldsymbol{f}(D)$.*

(b) *The following are equivalent.*

  (i) *D is a v–domain.*
  (ii) *D is integrally closed and $(F :_D G)^v = (F^v :_D G^v)$ for all $F, G \in \boldsymbol{f}(D)$*
  (iii) *D is integrally closed and $((a, b) \cap (c, d))^v = (a, b)^v \cap (c, d)^v$ for all nonzero $a, b, c, d \in D$.*

(iv) *D is integrally closed and $((a,b)\cap(c))^v = (a,b)^v \cap (c)$ for all nonzero $a,b,c \in D$.*
(v) *D is integrally closed and $((a,b) :_D (c))^v = ((a,b)^v :_D (c))$ for all nonzero $a,b,c \in D$.*

Note that PvMD's can be characterized by "t–versions" of the statements of Proposition 6.2 (b) [8, D.D. Anderson-Clarke (2006), Theorem 3.3]. Moreover, in [8, D.D. Anderson-Clarke (2006)], the authors also asked several questions related to distribution of the v–operation over intersections. One of these questions [7, D.D. Anderson-Clarke (2005), Question 3.2(2)] can be stated as: *Is it true that, if D is a v-domain, then $(A\cap B)^v = A^v \cap B^v$ for all $A,B \in \boldsymbol{F}(D)$?*

In [95, Mimouni (2007), Example 3.4], the author has recently answered by the negative, constructing a Prüfer domain with two ideals $A,B \in \boldsymbol{F}(D)$ such that $(A\cap B)^v \neq A^v \cap B^v$.

In a very recent paper [5, Anderson-Anderson-Fontana-Zafrullah (2008)], the authors classify the integral domains that come under the umbrella of v–domains, called there $\ast$–*Prüfer domains* for a given star operation $\ast$ (i.e., integral domains such that every nonzero finitely generated fractional ideal is $\ast$–invertible). Since v–Prüfer domains coincide with v–domains, this paper provides also direct and general proofs of several relevant quotient-based characterizations of v–domains given in [12, Anderson-Mott-Zafrullah (1989), Theorem 4.1]. We collect in the following theorem several of these ideal-theoretic characterizations in case of v–domains. For the general statements in the star setting and for the proof the reader can consult [5, Anderson-Anderson-Fontana-Zafrullah (2008), Theorems 2.2 and 2.8].

Theorem 6.3. *Given an integral domain D, the following properties are equivalent.*

(i) *D is a v–domain.*
(ii) *For all $A \in \boldsymbol{F}(D)$ and $F \in \boldsymbol{f}(D)$, $A \subseteq F^v$ implies $A^v = (BF)^v$ for some $B \in \boldsymbol{F}(D)$.*
(iii) *$(A:F)^v = (A^v : F) = (AF^{-1})^v$ for all $A \in \boldsymbol{F}(D)$ and $F \in \boldsymbol{f}(D)$.*
(iv) *$(A:F^{-1})^v = (A^v : F^{-1}) = (AF)^v$ for all $A \in \boldsymbol{F}(D)$ and $F \in \boldsymbol{f}(D)$.*
(v) *$(F:A)^v = (F^v : A) = (FA^{-1})^v$ for all $A \in \boldsymbol{F}(D)$ and $F \in \boldsymbol{f}(D)$.*
(vi) *$(F^v : A^{-1}) = (FA^v)^v$ for all $A \in \boldsymbol{F}(D)$ and $F \in \boldsymbol{f}(D)$.*
(vii) *$((A+B) : F)^v = ((A:F)+(B:F))^v$ for all $A,B \in \boldsymbol{F}(D)$ and $F \in \boldsymbol{f}(D)$.*
(viii) *$(A:(F\cap G))^v = ((A:F)+(A:G))^v$ for all $A \in \boldsymbol{F}(D)$ and $F,G \in \boldsymbol{f}^v(D)$ $(:= \{H \in \boldsymbol{f}(D) \mid H = H^v\})$.*
(ix) *$(((a) :_D (b)) + ((b) :_D (a)))^v = D$ for all nonzero $a,b \in D$.*
($\mathrm{x}_{\boldsymbol{f}}$) *$((F\cap G)(F+G))^v = (FG)^v$ for all $F,G \in \boldsymbol{f}(D)$.*
($\mathrm{x}_{\boldsymbol{F}}$) *$((A\cap B)(A+B))^v = (AB)^v$ for all $A,B \in \boldsymbol{F}(D)$.*
($\mathrm{xi}_{\boldsymbol{f}}$) *$(F(G^v \cap H^v))^v = (FG)^v \cap (FH)^v$ for all $F,G,H \in \boldsymbol{f}(D)$.*
($\mathrm{xi}_{\boldsymbol{fF}}$) *$(F(A^v \cap B^v))^v = (FA)^v \cap (FB)^v$ for all $F \in \boldsymbol{f}(D)$ and $A,B \in \boldsymbol{F}(D)$.*

(xii) *If $A, B \in \boldsymbol{F}(D)$ are v–invertible, then $A \cap B$ and $A + B$ are v–invertible.*
(xiii) *If $A, B \in \boldsymbol{F}(D)$ are v–invertible, then $A + B$ is v–invertible.*

Note that some of the previous characterizations are remarkable for various reasons. For instance, (xiii) is interesting in that while an invertible ideal (respectively, t–invertible t–ideal) is finitely generated (respectively, t–finite) a v–invertible v–ideal may not be v–finite. Condition ($\mathrm{x}_{\boldsymbol{F}}$) in the star setting gives $((A \cap B)(A + B))^* = (AB)^*$ for all $A, B \in \boldsymbol{F}(D)$ and for $* = d$ (respectively, $* = t$), it is a (known) characterization of Prüfer domains (respectively, PvMD's), but for $* = v$ is a brand-new characterization of v–domains. More generally, note that, replacing in each of the statements of the previous theorem v with the identity star operation d (respectively, with t), we (re)obtain several characterizations of Prüfer domains (respectively, PvMD's).

Franz Halter-Koch has recently shown a great deal of interest in the paper [5, Anderson-Anderson-Fontana-Zafrullah (2008)] and, at the Fez Conference in June 2008, he has presented further systematic work in the language of monoids, containing in particular the above characterizations [62, Halter-Koch (2009)].

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