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DOMAINS WHOSE IDEALS MEET A UNIVERSAL RESTRICTION

MUHAMMAD ZAFRULLAH

Dedicated to my friends

Abstract. Let D be an integral domain, $S(D) = I(D)$ (resp., $I_t(D)$) the set of proper nonzero ideals (resp., proper t-ideals) of D, $Max(D)$ (resp., t-$Max(D)$ the set of maximal (t-) ideals of D, and let P be a predicate on $S(D)$ with nonempty truth set $\Pi_{S(D)} \subseteq S(D)$, where P can be: "—is invertible" or "—is divisorial" etc.. We say $S(D)$ meets P (written as $S(D) \lhd P$) if $\forall s \in S(D) \exists \pi \in \Pi_{S(D)}(P)$ $(s \subseteq \pi)$. Clearly $S(D) \lhd P \Leftrightarrow Max(D)$ (resp., t-$Max(D)$) $\subseteq \Pi_{S(D)}(P)$. We show that if $S(D) \lhd P$, we have no control over $\dim D$. We also show that $I(D) \lhd P$ does not imply $I(R) \lhd P$, while $I_t(D) \lhd P$ implies $I_t(R) \lhd P$, for most choices of P, when $R = D[X]$ and have examples to show that generally $S(D) \lhd P$ does not extend to rings of fractions. We study restrictions that may control the dimension of D when $S(D) \lhd P$. We also say $S(D) \lhd P$ with a twist (written as $S(D) \lhd^t P$) if $\forall s \in S(D)$ $\exists \pi \in \Pi_D(P)(s^n \subseteq \pi$ for some $n \in N)$ and study $S(D) \lhd^t P$, along the same lines as $S(D) \lhd P$ and provide examples.

1. Introduction

The general idea of this paper is the following. Consider a property of ideals in a (commutative) ring R such as "is finitely generated". We raise and answer questions such as: A commutative ring R is Noetherian if and only if every ideal of R is finitely generated, what will be a ring every ideal of which is contained in some finitely generated ideal? It turns out that this will happen precisely when every maximal ideal of R is finitely generated. (The resulting ring may not in general be Noetherian.) Let's call the above process, "tweaking of a property". We note that while the main thrust of our paper is on tweaking of various properties of ideals of various kinds in commutative integral domains, the language adopted is such that it can be used to include questions such as: What will be the result of tweaking the property, "every left ideal is principal" to "every left ideal is contained in a principal left ideal"?

Let D be an integral domain with quotient field $K \neq D$ and let $F(D)$ be the set of nonzero fractional ideals of D. For $I \in F(D)$, the set $I^{-1} = \{x \in K | xI \subseteq D\}$ is again a fractional ideal and thus the relation v: $I \mapsto I_v$ is a function on $F(D)$. This function is called the v-operation on D. Similarly the relation t: $I \mapsto I_t = \cup\{F_v|$

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$0 \neq F$ is a finitely generated subideal of $I\}$ is a function on $F(D)$ and is called the t-operation on D. These and the operation d: $I \mapsto I$ are examples of the so called star operations. The reader may consult sections 32 and 34 of [23] or the first chapter of [19] for these operations. However, for the purposes of this introduction, we note that $I \in F(D)$ is a v-ideal (resp., t-ideal) if $I = I_v$ (resp. $I = I_t$) and if I is finitely generated, $I_v = I_t$. The rather peculiar definition of the t-operation allows one to use Zorn's Lemma to prove that each integral domain that is not a field has at least one integral t-ideal maximal among integral t-ideals, that this maximal t-ideal is prime and that every proper, integral t-ideal is contained in at least one maximal t-ideal. The set of all maximal t-ideals of a domain D is denoted by $t\text{-}Max(D)$. It can be shown that $D = \cap_{M \in t\text{-}Max(D)} D_M$. While we are at it let's also denote by $I(D)$ the set of all nonzero proper integral ideals of D and by $I_t(D)$ the set of all proper t-ideals of D.

Now let $S(D)$ represent $I(D)$ (or $I_t(D)$). Let P be a predicate that defines a non-empty truth set $\Pi_{S(D)}(P) \subseteq S(D)$, where P can be: "—is invertible" or "—is divisorial", "—is finitely generated" etc.. We say $S(D)$, for a given value or both values meets P (written as $S(D) \lhd P$) if $\forall s \in S(D) \exists \pi \in \Pi_{S(D)}(P)$ $(s \subseteq \pi)$.

(Alternatively, let P be a valid property of ideals and let $\Pi_{S(D)}(P)$ be the set of ideals of D satisfying P, we say $S(D)$ meets P (denoted as $S(D) \lhd P$) if each ideal s in $S(D)$ is contained in some ideal π that satisfies P.)

From an abstract point of view we are actually dealing with a non-empty poset $(A, \leq)$ such that every member of A precedes at least one maximal element of A. Suppose further that we designate a non-empty subset Π of A by some rule. Then every maximal member of A is in Π if and only if every member of A precedes some member of Π. Thus $S(D) \lhd P \Leftrightarrow Max(D)$ (resp., $t\text{-}Max(D)$) $\subseteq \Pi_{S(D)}(P)$. That is easy enough, but the trouble starts when we ask questions like: Suppose for example $I(D) \lhd P$ and suppose R is an extension of D must $I(R) \lhd P$? (Same question for $S(D) = I_t(D)$.) On the other hand we get the following benefit from carrying out this study: Take a property P say " finitely generated", that characterizes commutative Noetherian rings. Then $I(D) \lhd P$ gives us a, ring each of whose maximal ideal is finitely generated. It turns out that this ring is non-Noetherian unless it is of dimension one. We shall however restrict our attention to integral domains and note that D is a Krull domain if and only every t-ideal of D is t-invertible. If P stands for "is t-invertible" then, as we shall see, $I_t(D) \lhd P$ is a domain characterized by the property that every maximal t-ideal of D is t-invertible. Now you can set P as: ".. is invertible" and check for yourself that $I(D) \lhd P$ delivers a domain whose maximal ideals are all invertible but such a domain is not Dedekind unless it is of dimension one. In fact for each natural number n we can find an n dimensional domain with each maximal ideal invertible. This fascinating uncontrollability of Krull dimension is shared by most of $I(D) \lhd P$ and $I_t(D) \lhd P$ etc..

We show in section 2 that if X is an indeterminate over L a field extension of K, and $R = D + XL[X]$, and if P returns T on a maximal ideal M of D if and only if P returns T on $M + XL[X]$, $S(D) \lhd P$ if and only if $S(R) \lhd P$ for a P that holds (returns the truth value T) for principal ideals as well. Since $\dim R = \dim D + 1$, [15, Corollary 1.4], this shows that if $S(D) \lhd P$ and P returns the truth value T for each principal ideal, then one can expect no restriction on the Krull dimension of D. Next we show, in section 2, that if $R = D[X]$ and $I(D) \lhd P$, then $I(R) \ntriangleleft P$ in cases that we have considered, yet if $I_t(D) \lhd P$, then $I_t(R) \lhd P$ almost always. We give examples to show that generally $S(D) \lhd P$ does not extend to rings of fractions. We study restrictions, such as requiring the domain to be completely integrally closed or to be Noetherian etc., that control the dimension of D when $S(D) \lhd P$, in some cases. In section 3 we study $S(D) \lhd P$ with a twist (written as $S(D) \lhd^t P$) if $\forall s \in S(D)\ \exists \pi \in \Pi_{S(D)}(P)(s^n \subseteq \pi$ for some $n \in N)$ and study $S(D) \lhd^t P$ along the same lines as $S(D) \lhd P$, providing necessary examples. (Here N denotes the set of natural numbers.) Of course our terminology is usually standard, as in [31] and [23], and we provide adequate introduction to any term that is new or not quite in common use.)

2. Effects of a Universal Restriction on $S(D)$

Let us start with an introduction to general star operations so that we can reap full benefits from our toils. A star operation $*$ on D is a function on $F(D)$ that satisfies the following properties for every $I, J \in F(D)$ and $0 \neq x \in K$:
(i) $(x)^* = (x)$ and $(xI)^* = xI^*$,
(ii) $I \subseteq I^*$, and $I^* \subseteq J^*$ whenever $I \subseteq J$, and
(iii) $(I^*)^* = I^*$.

Now, an ideal $I \in F(D)$ is a $*$-ideal if $I^* = I$, so a principal ideal is a $*$-ideal for every star operation $*$. Moreover $I \in F(D)$ is called a $*$-ideal of finite type if $I = J^*$ for some $J \in f(D)$. It can be shown that (a) for every star operation $*$ and $I, J \in F(D)$, $(IJ)^* = (IJ^*)^* = (I^*J^*)^*$, (the $*$-multiplication), (b) $(I+J)^* = (I+J^*)^* = (I^*+J^*)^*$ (the $*$-sum) and (c) $(I^* \cap J^*)^* = I^* \cap J^*$($*$-intersection).

To each star operation $*$ we can associate a star operation $*_s$ defined by $I^{*_s} = \bigcup\{\, J^* \mid J \subseteq I \text{ and } J \in f(D)\,\}$. A star operation $*$ is said to be of finite type, or of finite character, if $I^* = I^{*_s}$ for all $I \in F(D)$. Indeed for each star operation $*$, $*_s$ is of finite character. Thus if $*$ is of finite character $I \in F(D)$ is a $*$-ideal if and only if for each finitely generated subideal J of I we have $J^* \subseteq I$. Also it is easy to see that $I_t = \bigcup\{\, J_v \mid J \subseteq I \text{ and } J \in f(D)\,\} = I_{v_s}$ and so the t-operation is an example of a star operation of finite character. Star operations of finite character, especially the t-operation, will figure prominently in our discussions. A fractional ideal I is called $*$-invertible if $(II^{-1})^* = D$. It is well known that if I is $*$-invertible for a finite character star operation $*$ then I^* and I^{-1} are of finite type and that every $*$-invertible $*$-ideal is divisorial [44]. If $*$ is a star operation of finite character then just like the t-operation, every nonzero proper integral $*$-ideal is contained in a maximal integral $*$-ideal that is prime and just like the t-ideals $D = \cap D_M$ where M varies over the maximal $*$-ideals of D. We shall be mostly concerned with the two values of $S(D)$ but will use occasionally $I_*(D)$ the set of proper, integral, $*$-ideals when we want to go general and not lose sight of the two values of $S(D)$. (Since $I_*(D) = I(D)$ $(I_t(D))$ for $* = d$ (resp., $* = t$)). Let's note that while $I_*(D) \cup \{D\}$ is a monoid under the usual $*$-multiplication of $*$-ideals with multiplicative identity D, it is a poset under inclusion. From the poset angle $(I_*(D) \cup \{D\}, +^*, \times^* \leq)$, with $A \leq B \Leftrightarrow A \supseteq B$, is a p.o. monoid and a lattice where $A +^* B = (A, B)^* = \inf(A, B) = A \wedge B$ and $\sup(A, B) = A \cap B$. The idea of using a universal restriction via a predicate germinated in [17] where we studied the set $I_*^f(D)$ of proper $*$-ideals of finite type with a preassigned non-empty subset Γ of $I_*^f(D)$, requiring that every pair of members with $A +^* B \in I_*^f(D)$, A, B be contained in some member of Γ. (This is equivalent to saying that every proper ideal in $I_*^f(D)$ is contained in a member of Γ, hence the current approach.) As these studies appeal mostly to partial order, they stand to have applications in other areas, as well.

We start with a simple example to set the scene. Let's consider, for a star operation $*$ of finite character, $I_*(D)$ and define $\Pi_{I_*(D)}(P)$ with $P =$ "—is principal" and and suppose that $I_*(D) \lhd P$. Then every maximal $*$-ideal of D is principal, as we have already observed. But the story doesn't end here. The event of $I_*(D) \lhd P$ imparts some properties to D, such as: the only atoms (irreducible elements) in D are primes and hence generators of maximal $*$-ideals. For this let d be an atom and let $d|ab$ for some $a, b \in D$. If $d \nmid a$ and $d \nmid b$, then, $D = ((d, a)^*(d, b)^*)^* = (d^2, da, db, ab)^* \subseteq dD$ a contradiction, because $d|ab$. Thus an irreducible element is a prime in D, if $I_*(D) \lhd P$ for any star operation $*$ of finite character. Now for $* = d$ the identity operation $I(D) \lhd P$ gives a domain D in which every proper nonzero ideal is contained in a principal ideal, something stronger than what Cohn [12] called a pre-Bezout domain. In fact $I(D) \lhd P$ gives a domain something that is even stronger than what was called a special pre-Bezout, or spre-Bezout domain in [17]. Similarly if $I_t(D) \lhd P$, then D is something stronger than a PSP-domain (every primitive polynomial over D is super-primitive), also discussed in [17]. Recall that a polynomial f is super primitive if $(A_f)_v = D$, where A_f is the content, the ideal generated by the coefficients of f. Now it is easy to see that if such a domain is atomic it is at least a UFD (when $I_t(D) \lhd P$) and at most a PID (When $I(D) \lhd P$). (If the last sentence is not clear wait till the paper unfolds itself.) Now, can we find domains that satisfy these properties and yet are not atomic? Yes indeed!

Example 2.1. Let Z, Q denote the ring of integers and its quotient field respectively and let X be an indeterminate over Q, then the ring $D = Z + XQ[X]$ is such that $I(D) \lhd P$, where $P =$ "—is principal".

Illustration: According to [14, Theorem 4.21] the nonzero prime ideals of D are of the form $pZ + XQ[X], XQ[X]$ and maximal height one principal primes of the form $f(X)D$ where $f(X)$ is irreducible in $Q[X]$ and $f(0) = 1$. Now $XQ[X]$ is not maximal and the rest of them are. So all the maximal ideals are principal and so $I(D) \lhd P$ with P given above. That D is not atomic can be concluded from the fact that X cannot be expressed as a product of atoms.

Now according to [14], $\dim D = 2$ and we said that if $I(D) \lhd P$, then there maybe no restriction on $\dim D$. The answer to this question is provided in a more general form below.

Let's first collect some simple results, observations and notation. We say that P returns T on an ideal of $I(D)$ if the truth value of P for that ideal is T. For the sake of easy reference, let's start with an observation that we have already made.

Lemma 2.2. *Let $(A, \leq)$ be a non-empty poset such that every element of A precedes some maximal element of A and suppose that we can designate a non-empty subset Π of A by some rule. Also let $Max(A)$ denote the set of all maximal elements of A. Then every member of A precedes some member of Π if and only if $Max(A) \subseteq \Pi$. Thus $I(D) \lhd P$ if and only if P returns T for each member of $Max(D)$ and $I_t(D) \lhd P$ if and only if P returns T for each member of t-$Max(D)$.*

This, somewhat simple observation may, in some instances, have some interesting consequences.

Lemma 2.3. *(1) If a maximal ideal M of D is a t-invertible t-ideal, then M is invertible. (2) If $P_1 =$ "—is t-invertible" and $P_2 =$ "— is invertible", then $I(D) \lhd P_1 \Leftrightarrow I(D) \lhd P_2$ and (3) $I(D) \lhd P \Rightarrow I_t(D) \lhd P$ for any predicate P whose truth set consists of t-ideals.*

Proof. (1) Suppose M is a t-invertible t-ideal then $(MM^{-1})_t = D$. If $MM^{-1} \neq D$ then MM^{-1} must be contained in a maximal ideal N. But since $M \subseteq MM^{-1}, N = M$. So $MM^{-1} \subseteq M$. But as M is also a t-ideal, $D = (MM^{-1})_t \subseteq M$, a contradiction.

(2) By Lemma 2.2, $I(D) \lhd P_i \Leftrightarrow P_i$ returns T for each maximal ideal M and for each $i = 1, 2$. So $I(D) \lhd P_1 \Rightarrow$ every maximal ideal is a t-invertible t-ideal and by (1) every maximal ideal is invertible. So $I(D) \lhd P_1 \Rightarrow I(D) \lhd P_2$. The converse is obvious because every invertible ideal is a t-invertible t-ideal.

(3) Suppose that $I(D) \lhd P$ then, in particular, for every maximal t-ideal M, P returns T. □

Proposition 1. *(1) Let, on $I(D)$, $P =$ "— is a principal ideal (resp., t-invertible t-ideal, t-ideal of finite type, t-ideal, finitely generated ideal,divisorial ideal). Then $I(D) \lhd P$ if and only if every maximal ideal of D is a principal ideal (resp., invertible ideal, t-ideal of finite type, t-ideal, finitely generated ideal, divisorial ideal) of D. (2) Let, on $I(D)$, $P =$ "— is a principal ideal (resp., invertible ideal, t-invertible t-ideal, t-ideal of finite type, finitely generated ideal, divisorial ideal). Then $I_t(D) \lhd P \Leftrightarrow$ every maximal t-ideal is a principal ideal (resp., invertible ideal, t-invertible t-ideal, t-ideal of finite type, finitely generated ideal, divisorial ideal).*

Proof. In the presence of Lemma 2.2 and Lemma 2.3, it appears totally unnecessary to repeat the arguments required for the proofs of (1) and (2). □

Note that in case of (1) every maximal ideal being a t-ideal of finite type ensures that every maximal t-ideal of D is actually a maximal ideal. Indeed if we suppose that $\wp$ is a maximal t-ideal that is not maximal, then $\wp$ is contained in a maximal ideal, say M, but M is already a t-ideal.

We have restricted our attention to the star operations that are easily defined for usual extensions. One of the usual extensions is the $D + XL[X]$ construction, where L is an extension of K and X an indeterminate over L. It is a special case of the $D + M$ construction of [11]. To be able to fully appreciate how it works, one needs to learn a little about the construction $D + XL[X]$. Let D, L, X be be as above. Then $R = D + XL[X] = \{f \in L[X] | f(0) \in D\}$ is an integral domain. Indeed R has two kinds of nonzero prime ideals P , ones that intersect D trivially and ones that don't. If $P \cap D \neq (0)$ then $P = P \cap D + XL[X]$ [15, Lemma 1.1] and obviously P is maximal if and only if $P \cap D$ is. It can be shown, as was indicated prior to the proof of Corollary 16 in [4], that if $P = P \cap D + XL[X]$, then P is a maximal t-ideal of R if and only if $P \cap D$ is a maximal t-ideal of D and indeed as $P_v = (P \cap D)_v + XL[X]$, P is divisorial if and only if $(P \cap D)$ is. Moreover, prime ideals of R that are not comparable with $XL[X]$, i.e. ones that intersect D trivially, are of the form $(1 + Xg(X))R$ where $1 + Xg(X)$ is an irreducible element of $L[X]$, [15, Lemmas 1.2, 1.5]. (This can also be seen as follows: If P is a prime that intersects D trivially, then P extends to a prime $\wp$ of $K + XL[X]$ that is incomparable with $XL[X]$. Now $K + XL[X]$ is one dimensional and every element of $K + XL[X]$ is of the form $lX^r(1 + Xg(X))$ where. $l \in L$, $r \geq 0$ and $1 + Xg(X)$

is obviously a product of primes from $L[X]$. Next $lX^r(1+Xg(X)) \in \wp$ forces $(1+Xg(X)) \in \wp$, because $X \notin \wp$. But then $\wp$ is principal generated by a prime of the form $1+h(X)$ and this also is a prime in R, thus $1+Xh(X) \in \wp \cap R = P$. Now as P contains a principal prime $(1+Xh(X))R$ that extends to a maximal height one prime in $K+XL[X]$, $P=(1+Xh(X))R$.) Also as $XL[X]$ is of height one $XL[X]$ is a t-ideal and $\dim R = \dim D + 1$, by [15, Corollary 1.4]. Let us say that a predicate P respects principals if P returns T on principal ideals.

Theorem 2.4. *Let P be a predicate that respects principals, L an extension field of K, X an indeterminate over L and let $R = D + XL[X]$. Then (i) given that P returns T on a maximal ideal M of D if and only if P returns T on $M + XL[X]$, $I(D) \lhd P \Leftrightarrow I(R) \lhd P$ (ii) given that P returns T on a maximal t-ideal M of D if and only if P returns T on $M + XL[X]$, $I_t(D) \lhd P \Leftrightarrow I_t(R) \lhd P$.*

Proof. (Perhaps, before a "formal" proof of (i), an example might help. Take a predicate P, say: ... is finitely generated. Then P respects principals because a principal ideal is finitely generated. Now given a maximal ideal M of D the ideal $M + XL[X] = MR$ [15, Lemma 1.1 and Theorem 1.3] is a maximal ideal of R and obviously M is finitely generated if and only if MR is. Then $I(D) \lhd P$ implies $I(R) \lhd P$ because every maximal ideal of D being finitely generated implies every maximal ideal of R of the form $M + XL[X]$ being finitely generated and as all the maximal ideals that intersect D trivially are principal, we conclude that every maximal ideal of R is finitely generated i.e. $I(R) \lhd P$. Conversely suppose $I(R) \lhd P$. That is every maximal ideal of R is finitely generated. Then in particular maximal ideals of R that intersect D non trivially are finitely generated. But the ideals of R that intersect D non trivially are of the form $M + XL[X] = MR$ [15, Lemma 1.1 and Theorem 1.3]. Now each MR being finitely generated implies that each maximal ideal M of D is finitely generated. But this means $I(R) \lhd P \Rightarrow I(D) \lhd P$.)

(i) Suppose $I(D) \lhd P$, then P returns T for every maximal ideal M of D and hence for every maximal ideal of R of the form $M + XL[X]$, by the given. Since P respects principal ideals we conclude that P returns T for every maximal ideal of R. (Since every maximal ideal of R not of the form $M + XL[X]$ is principal.) That is $I(R) \lhd P$. Conversely suppose that $I(R) \lhd P$. Then P returns T for all maximal ideals $\mathcal{M}$ of R, in particular for the ones that intersect D non-trivially. But those are precisely of the form $\mathcal{M} = \mathbf{m} + XL[X]$ where $\mathbf{m} = \mathcal{M} \cap D$ is maximal and as P returns T for $\mathbf{m} + XL[X]$ if and only if P returns T for $\mathbf{m}$, and as the $\mathbf{m}s$ are precisely the maximal ideals of D we conclude that $I(D) \lhd P$. The proof of (ii) follows the same lines as those adopted in the proof of (i). However, just for completeness we include it. Suppose $I_t(D) \lhd P$ then P returns T for every maximal t-ideal M of D and hence for every maximal t-ideal of R of the form $M + XL[X]$. Since P respects principal ideals we conclude that P returns T for every maximal t-ideal of R. That is $I_t(R) \lhd P$. Conversely suppose that $I_t(R) \lhd P$. Then P returns T for all maximal t-ideals $\mathcal{M}$ of R, in particular for the ones that intersect D non-trivially. But those are precisely of the form $\mathcal{M} = \mathbf{m} + XL[X]$ where $\mathbf{m} = \mathcal{M} \cap D$ is a maximal t-ideal and as P returns T for $\mathbf{m} + XL[X]$ if and only if P returns T for $\mathbf{m}$, and as the $\mathbf{m}s$ are precisely the maximal t-ideals of D we conclude that $I_t(D) \lhd P$. □

The above "theorem" is not much of a theorem, really. But it tells us what to check for, before making a statement such as $I(D) \lhd P \Leftrightarrow I(R) \lhd P$.

Corollary 1. *(i)with D, L, X, R as in Theorem 2.4 and with $P =$ "— is a principal ideal (resp., t-invertible t-ideal, t-ideal of finite type, t-ideal, finitely generated ideal,divisorial ideal) $I(D) \lhd P \Leftrightarrow I(R) \lhd P$ and (ii) with D, L, X, R as in Theorem 2.4 and with $P =$ "— is a principal ideal (resp., invertible ideal, t-invertible t-ideal, t-ideal of finite type, finitely generated ideal,divisorial ideal) $I_t(D) \lhd P \Leftrightarrow I_t(R) \lhd P$*

Proof. (i)Note that in each case P returns T for a principal ideal. Moreover for A an ideal of D, because $A_v + XL[X] = (A + XL[X])_v$ and $A_t + XL[X] = (A + XL[X])_t$ and because $A + XL[X] = A(D + XL[X])$, A being finitely generated, invertible (or being a v-ideal of finite type) results in $A + XL[X]$ being of that kind and vice versa, we conclude that the requirements of Theorem 2.4 are met. (Indeed as a maximal ideal being a t-invertible t-ideal is invertible, we haven't let anything unverified.) For (ii) note that all the checking is as in (i), even the t-invertible t-ideal case falls under t-ideals of finite type and t-ideals of finite type are all v-ideals. So nothing more needs be done. □

Remark 2.5. Note that if D is not a field, as we have assumed from the start, then, whatever be D, $D + XL[X]$ is not Noetherian. This is because $D + XL[X]$ affords a strictly ascending chain of ideals such as $(X) \subseteq (X/d) \subseteq (X/d^2) \subseteq ... \subseteq (X/d^n)$ for any nonzero non unit d of D. Now as the maximal ideals of a Noetherian domain D are finitely generated so are the maximal ideals of $D + XL[X]$, by Corollary 1. This gives us an example (a) of a non-Noetherian domain whose maximal ideals are all finitely generated. That is not all, we can construct chains of such domains, of any length, starting with a domain whose maximal ideals are all finitely generated. To make things simple let $L = K$. Let R_0 be a domain with the property that every maximal ideal of R_0 is finitely generated and let $R_1 = R_0 + X_0 qf(R_0)[X_0]$, where X_0 is an indeterminate over $qf(R_0)$, $R_2 = R_1 + X_1 qf(R_1)[X_1]$, where X_1 is an indeterminate over $qf(R_1)$ and obviously every maximal ideal of R_2 is finitely generated because R_1 has this property. If proceeding in this manner, we reach $R_n = R_{n-1} + X_{n-1} qf(R_{n-1})[X_{n-1}]$, where X_{n-1} is an indeterminate over $qf(R_{n-1})$ we can construct the next. As a result of this recursive procedure we have a chain of domains: $R_0 \subseteq R_1 \subseteq ... \subseteq R_n \subseteq R_{n+1} \subseteq ...$, where each of R_i gets the property of having all maximal ideals finitely generated from the previous, for $i > 0$. Next recall that (b) D is a Mori domain if D has ACC on integral divisorial ideals. Obviously Noetherian domains and less obviously Krull domains are Mori. It can be shown that D is a Mori domain if and only if for every nonzero integral ideal A of D there is a finitely generated ideal $F \subseteq A$ such that $A_v = F_v$ [35, Lemma 1]. This translates to: every t-ideal is a t-ideal of finite type [3, Corollary 1.2]. Thus if D is Mori, then every maximal t-ideal of D is of finite type. To show that the property of having every maximal t-ideal of finite type does not characterize Mori domains one can construct $R = D + XK[X]$ indicating, via Corollary 1, that every maximal t-ideal of R is of finite type but R is not Mori because R affords an ascending chain like: $(X) \subseteq (X/d) \subseteq (X/d^2) \subseteq ... \subseteq (X/d^n)$ for any nonzero non unit of D. We can actually construct, as in (a) above, chains of domains satisfying this property.

There are other uses Corollary 1 can be put to, but we shall let the reader discover those, if need arises. We now concentrate on the next extension $R = D[X]$ where X is the usual indeterminate over D.

Proposition 2. *(1) Let $I(D) \lhd P$ where $P =$ "— is a proper nonzero principal ideal (resp., t-invertible t-ideal, t-ideal, t-ideal of finite type, divisorial ideal), let X be an indeterminate over D and let $R = D[X]$. Then it never is the case that $I(R) \lhd P$ for $P =$ "— is a proper nonzero principal ideal (resp., t-invertible t-ideal, t-ideal, t-ideal of finite type, divisorial ideal) and (2) Let $I_t(D) \lhd P$ where $P =$ "— is a t-invertible t-ideal (resp., t-ideal, t-ideal of finite type, divisorial ideal), let X be an indeterminate over D and let $R = D[X]$. Then $I_t(R) \lhd P$ where $P =$ "— is a t-invertible t-ideal (resp., t-ideal, t-ideal of finite type, divisorial ideal) and conversely.*

Proof. (1) Let $I(D) \lhd P$ where $P =$ "— is a proper nonzero principal ideal (resp., t-invertible t-ideal, t-ideal, t-ideal of finite type, divisorial ideal). Then every maximal ideal $\wp$ of D is a t-ideal. Now consider the prime ideal $\wp[X]$ in $R = D[X]$ and note that $\wp[X]$ can never be a maximal ideal because $D[X]/\wp[X] \cong (D/\wp)[X]$ is a polynomial ring over a field and so must have an infinite number of maximal ideals. This forces $\wp[X]$ to be properly contained in an infinite number of maximal ideals M_α of $D[X]$. Let M be one of them. Then $M = (f, \wp[X])$. Now, if it were the case that $I(R) \lhd P$ for $P =$ "— is a proper t-ideal", then every maximal ideal of R would be a t-ideal. This would make M a t-ideal with $M \cap D = \wp \neq (0)$. But then, according to Proposition 1.1 of [28], $M = (M \cap D)[X] = \wp[X]$, a contradiction to the fact that $\wp[X] \subsetneq M$. For (2) note that if $I_t(D) \lhd P$ where P is as specified, then every maximal t-ideal $\wp$ of D is a t-invertible t-ideal (resp., t-ideal, t-ideal of finite type, divisorial ideal). Now let M be a maximal t-ideal of R. If $M \cap D = (0)$, then M is a t-invertible t-ideal and hence a t-ideal (and divisorial, being a finite type t-ideal), by Theorem 1.4 of [28]. Next if M is such that $M \cap D \neq (0)$, then $M = (M \cap D)[X]$ where $M \cap D$ is a maximal t-ideal of D and hence a t-ideal, and obviously is divisorial if and only if M is divisorial [25, Proposition 4.3]. Conversely suppose that $I_t(R) \lhd P$ for P as specified. Then every maximal t-ideal M of R is a t-invertible t-ideal (resp., t-ideal, t-ideal of finite type, divisorial ideal). Now let $\wp$ be a maximal t-ideal of D. Then $\wp[X]$ is a maximal t-ideal of R by Proposition 1.1 of [28] and hence divisorial. But this leads to $\wp[X] = (\wp[X])_v = \wp_v[X]$ $((\wp[X])_t = \wp_t[X])$ and hence to $\wp = \wp_v$. (We have chosen to focus of divisorial ideals (t-ideals), as all the other cases are divisorial (or t-ideals) and a maximal t-ideal of R that intersects D trivially is divisorial of finite type and hence a t-ideal.) Moreover if a maximal t-ideal M of R intersects D non-trivially then $M = (M \cap D)[X]$ as above and of course M is a t-ideal (t-ideal of finite type, divisorial) if and only if $M \cap D$ is) [25, Proposition 4.3]. □

I cannot find a way to prove or disprove the following: Let $R = D[X]$, and let $P =$ "— is a finitely generated ideal" then $I(D) \lhd P \not\Rightarrow I(R) \lhd P$.

Now we are ready to show that if $R = D_{\mathcal{S}}$, for a multiplicative set $\mathcal{S}$ of D where $S(D) \lhd P$ for $P =$ "— is a proper nonzero principal ideal (resp., t-invertible t-ideal, t-ideal, t-ideal of finite type, divisorial ideal), then it may not generally be the case that $S(R) \lhd P$. Let's first recall from Lemma 2.3 that if a maximal ideal is a t-invertible t-ideal then it is actually invertible. Before we start constructing examples, let's take a look at the tool that we use in the following example. Let

K be a proper subfield of a field L, let X be an indeterminate over L and let $T = K + XL[X]$. The ring T is an example of an atomic domain that is not a UFD (see [13, page 353]) and an example of a $D + M$ construction. That T is one dimensional follows from [15, Corollary 1.4], that every maximal ideal of T different from $XL[X]$ is principal of height one follows from Lemmas 1.2 and 1.5 of [15] and that $XL[X]$ is divisorial can be easily checked, because $XL[X] = (X, lX)_v$ where $l \in L \backslash K$.

Example 2.6. Let L be a field extension of K with $[L : K] = \infty$, let X be an indeterminate over L and consider $R = D + XL[X]$. Set $S = D\backslash(0)$. If every maximal ideal of D is principal (invertible, finitely generated) then so is every maximal ideal of R. But that is not the case for every maximal ideal of R_S. For $R_S = K + XL[X]$ has a maximal ideal that is a t-ideal but neither principal nor finitely generated, because $[L : K] = \infty$. (It is easy to see that every invertible ideal is principal in T, [10, Example 1.10].)

The following example has been taken, almost verbatim, from [29, Example 3.3]. To decipher this example, recall that D is a PVMD (Prufer v-multiplication domain) if every nonzero finitely generated ideal of D is t-invertible. A good source for this concept is [32].

Example 2.7. . There does exist at least one example of a domain D such that each maximal ideal of D is a t-ideal but for some maximal ideal M we have MD_M not a t-ideal. One such example is that of an essential domain that is not a PVMD. (Recall that an integral domain D is essential if D has a set G of primes such that D_p is a valuation domain for each $P \in G$ and $D = \cap_{P \in G} D_P$.). Now the example in question was constructed by Heinzer and Ohm in [26] and further analyzed in [32] and [22]. As it stands, the example has all except one maximal ideals of height one and hence t-ideals and the other maximal ideal M is a height 2 prime t-ideal. Indeed this is the maximal ideal M such that D_M is a 2-dimensional regular local ring and so with a maximal ideal that is not a t-ideal. Showing that while $I(D) \lhd P$ for $P =$ "— is a t-ideal of D, $I(D_M) \ntriangleleft P$.

For the next example recall from [42] that an integral domain D is a pre-Schreier domain if for all $a, b_1, b_2 \in D\backslash\{0\}$, $a|b_1b_2$ implies that $a = a_1a_2$, with $a_i \in D$ such that $a_i|b_i$. Also call a D-module M locally cyclic if for any elements $x_1, x_2, ..., x_n \in M$ there is a $d \in M$ such that $x_i = r_i d$.

Example 2.8. For $\mathbb{R}$ the field of real numbers, let $\mathbb{R} + M$, be a non-discrete rank one valuation domain, as constructed in say Example 4.5 of [42]. As decided in the above-mentioned example, $T = \mathbb{Q} + M$ (where $\mathbb{Q}$ is the field of rational numbers) is a pre-Screier domain with M divisorial and by [42, Theorem 4.4] locally cyclic. But then M cannot be a v-ideal of finite type. For if $M = (x_1, x_2, ..., x_n)_v$, then there would be a $d \in M$ such that $M = (x_1, x_2, ..., x_n)_v \subseteq (d) \subseteq M$, contradicting the construction in Example 4.5 of [42]. Now let p be a prime element in $\mathbb{Z}$, the ring of integers, and consider the local ring $R = \mathbb{Z}_{(p)} + M$. Indeed the maximal ideal of R is principal and hence can pass as a t-ideal of finite type, a t-invertible t-ideal. But if S is the multiplicative set of R generated by p, neither of these properties are shared by the maximal (t-) ideal M of $R_S = \mathbb{Q} + M$.

Now the fact that $I(D) \lhd P$ can go through the $D + XL[X]$ construction with the various descriptions of P can be used to construct, for example, a domain of any (finite) dimension with t-maximal ideals principal. If that reminds an attentive reader of comments (3) and (4) of Remarks 8 of [33], then so be it. The point however is that the events of $I(D) \lhd P$ and $I_t(D) \lhd P$, with suitable descriptions of P, do not have the usual Ascending Chain Conditions on ideals (principal or t-) ideals. One may wonder if there are any simple restrictions that will get the beast under control. Yet to prepare to see that, here is another simple set of results that can come in handy when we are dealing with completely integrally closed integral domains. Of course before we bring in those results some introduction is in order. Recall that an integral domain D with quotient field K is completely integrally closed if whenever $rx^n \in D$ for $x \in K$, $0 \neq r \in D$, and every integer $n \geq 1$, we have $x \in D$. It can be shown that an intersection of completely integrally closed domains is completely integrally closed. The go to reference for Krull domains is Fossum's book [21] where you can find that D is a Krull domain if D is a locally finite intersection of localizations at height one primes such that D_P is a discrete valuation domain at each height one prime. Thus a Krull domain is completely integrally closed. Glaz and Vasconcelos [24] called an integral domain D an H-domain if for an ideal A with $A^{-1} = D$, (or equivalently $A_v = D$) then A contains a finitely generated subideal F such that $A^{-1} = F^{-1}$. They showed that a completely integrally closed H-domain is a Krull domain. In [27, Proposition 2.4] it was shown that D is an H-domain if and only if every maximal t-ideal of D is divisorial. We have in the following a basic result and some of its derivatives.

Proposition 3. *(a) Let D be a completely integrally closed domain. Then (1) D is a Krull domain if and only if $I_t(D) \lhd P$ for $P =$ ″— is a proper divisorial ideal, (2) D is a locally factorial Krull domain if and only if $I_t(D) \lhd P$ for $P =$ ″— is a proper invertible integral ideal of D, (3) D is a Krull domain if and only if $I_t(D) \lhd P$ for $P =$ ″— is a proper t-invertible t-ideal of D, (b) (4) Let D be such that D_M is a Krull domain for each maximal ideal M of D. Then D is a Krull domain if and only if $I_t(D) \lhd P$ for $P =$ ″— is a proper divisorial ideal of D* [18] *(5) Let D be an intersection of rank one valuation domains. Then D is a Krull domain if and only if $I_t(D) \lhd P$ for $P =$ ″— a proper divisorial ideal of D, (6) Let D be an almost Dedekind domain. Then D is a Dedekind domain if and only if $I(D) \lhd P$ for $P =$ ″— is a proper divisorial ideals of D.*

Proof. The idea of proof, in each case, is that every maximal t-ideal (maximal ideal) being contained in a proper divisorial ideal must be equal to it and combining this with the fact that D is completely integrally closed we get the Krull domain conclusion. For the locally factorial domain conclusion in (2) we note that every maximal t-ideal of D is invertible and so divisorial. This gives the Krull conclusion and a Krull domain is locally factorial if and only if every height one prime of D is invertible [1, Theorem 1]. For the Dedekind domain conclusion in (6), we note that every maximal ideal is of height one and divisorial, being invertible, so every maximal ideal is a t-ideal and so the domain is Krull and one dimensional. The converse in each case is obvious, in that if D is a Krull domain then D is completely integrally closed and every maximal t-ideal of D is, a t-invertible t-ideal and hence, divisorial. (If D is locally factorial, as in (2), every maximal t-ideal of D is invertible and hence divisorial.) And if D is Dedekind, then D is completely integrally closed and every maximal ideal is invertible and hence divisorial. □

It is well known that D is a Krull domain if and only if every t-ideal of D is a t-product of prime t-ideals of D [36]. As we have seen, the prime t-ideals in a Krull domain happen to be all t-invertible t-ideals, and hence maximal t-ideals and divisorial [28, Proposition 1.3]. Also, according to [41, Theorem 1.10], D is a locally factorial Krull domain if, and only if, every t-ideal of D is invertible. Finally, D being completely integrally closed may not control the dimension of D when every maximal ideal is a t-ideal. Because the ring of entire functions is an infinite dimensional Bezout domain and completely integrally closed [23, page 146]. (Also, in a Bezout domain every maximal ideal is a t-ideal.)

Another condition that helps control the dimension is requiring some kind of an ascending chain condition. Call D a t-ACC domain if D satisfies ACC on its t-invertible t-ideals.

Lemma 2.9. *Let D be a t-ACC domain and let I be a proper t-invertible t-ideal of D. Then $\cap(I^n)_t = (0)$. Consequently, in a domain satisfying t-ACC, if A is a proper divisorial ideal of D and I a t-invertible t-ideal then $(AI)_v = A$ implies $I = D$.*

Proof. Because a t-invertible t-ideal is a v-ideal of finite type with I^{-1} of finite type there is no harm in using v for t. Now let $\cap(I^n)_v \neq 0$ and let x be a nonzero element in $\cap(I^n)_v$. Then there is a chain of t-invertible t-ideals $xI^{-1} \subseteq (xI^{-2})_v \subseteq ... \subseteq x(I^{-r})_v$ ... which must stop after a finite number of steps, because of the t-ACC restriction. Say $x(I^{-n})_v = x(I^{-n-1})_v$. Cancelling x from both sides we get $(I^{-n})_v = (I^{-n-1})_v$. Multiplying both sides by I^{n+1} and applying the v-operation we get $I = D$, a contradiction that arises from assuming that there is a nonzero element in $\cap(I^n)_v$. For the consequently part note that $(AI)_v = A$ implies that $A \subseteq (I^n)_v$ for all positive integers n. □

Proposition 4. *Let D be a t-ACC domain. Then (1) D is a PID if and only if $I(D) \lhd P$ for $P =$ "— is a proper nonzero principal ideal" and (2) D is a Dedekind domain if and only if $I(D) \lhd P$ for $P =$ "— is a proper invertible ideal" and (3) D is a Krull domain if and only if $I_t(D) \lhd P$ for $P =$ "— is a proper t-invertible t-ideal".*

Proof. We shall prove (3) and explain why it should work for the other two cases. For (3) note that $I_t(D) \lhd P$ for $\Leftrightarrow \forall A \in I_t(D)$ $(A \neq D \Rightarrow \exists \pi \in \Pi(A \subseteq \pi))$ where Π is the set determined by $P =$ "— is a proper t-invertible t- (resp., nonzero principal, invertible) ideal". Then, by the condition, every maximal t-ideal (maximal ideal) of D is t-invertible (resp., principal, invertible). By Lemma 2.9 we have for each maximal t-ideal M (maximal ideal M) $\cap(M^n)_v = (0)$ (resp., $\cap M^n = (0)$, since powers of principal (invertible) ideals are v-ideals). Thus each maximal t-ideal (maximal ideal) is of height one. Thus D is of t-dimension one (resp., of dimension one). Now, in each case, MD_M is of height one and principal, forcing D_M to be a rank one valuation domain for each maximal t-ideal (maximal ideal) M. This makes D completely integrally closed, for $D = \cap D_M$ where M ranges over maximal t-ideals (maximal ideals). Now apply Proposition 3, using the fact that each maximal t-ideal (maximal ideal) is divisorial, being a t-invertible t-ideal (principal (invertible) ideal). The converse is obvious in each case. □

Proposition 5. *Let D be a t-ACC domain. Then (1) D is a UFD if and only if $I_t(D) \lhd P$ for $P =$ "— is a proper nonzero principal ideal" and (2) D is a locally* *factorial Krull domain if and only if $I_t(D) \lhd P$ for $P =$ "— is a proper invertible ideal".*

Proof. We shall prove (1) and explain why it should work for the other case. For (1) note that $I_t(D) \lhd P$ for $P =$ "— proper nonzero principal (invertible) ideal" $\Leftrightarrow \forall A \in I_t(D)$ $(A \neq D \Rightarrow \exists \pi \in \Pi(A \subseteq \pi))$ where Π is the set determined by P returning T. Then, by the condition, every maximal t-ideal of D is principal (invertible). By Lemma 2.9 we have for each maximal t-ideal M, $\cap M^n = (0)$, since powers of principal (invertible) ideals are v-ideals. Thus each maximal t-ideal is of height one. Thus D is of t-dimension one. Now, in each case, MD_M is of height one and principal, forcing D_M to be a rank one valuation domain for each maximal t-ideal. This makes $D = \cap D_M$, where M ranges over maximal t-ideals, a completely integrally closed domain. Now apply Proposition 3, using the fact that each maximal t-ideal is divisorial, being principal or invertible. This gets us the Krull conclusion. Now recall that in a Krull domain D, $A_t = (P_1^{n_1}...P_r^{n_r})_t$. Then, in case of (2), D is locally factorial by [41, Theorem 1.10] and, in case of (1), D is factorial because every principal ideal is a product of prime powers. The converse, in each case, is obvious in that a UFD (locally factorial Krull domain) is Krull every maximal t-ideal of whose is principal (resp., invertible). □

As already mentioned, an integral domain D that satisfies ACC on integral divisorial ideals is called a Mori domain. Obviously a Noetherian domain is a Mori domain. It is easy to check that for every nonzero integral ideal A of a Mori domain D there are elements $a_1, .., a_r \in A$ such that $A_v = (a_1, .., a_r)_v$. So the inverse of a nonzero ideal of a Mori domain is a v-ideal of finite type. Hence a v-invertible ideal in a Mori domain is t-invertible. It is well known that a domain D is a Krull domain if, and only if, every nonzero ideal of D is t-invertible (see e.g. [34, Theorem 2.5]) and thus a Krull domain is Mori too. Noting that a Mori domain is a t-ACC domain and that Noetherian is Mori too, we have the following direct corollaries.

Corollary 2. *Let D be a Mori domain. Then (1) D is a PID if and only if $I(D) \lhd P$ for $P =$ "— is a proper nonzero principal ideal", (2) D is a Dedekind domain if and only if $I(D) \lhd P$ for $P =$ "— is a proper invertible ideal", (3) D is a Krull domain if and only if $I_t(D) \lhd P$ for $P =$ "— is a proper t-invertible t-ideal", (4) D is a UFD if and only if $I_t(D) \lhd P$ for $P =$ "— is a proper nonzero principal ideal" and (5) D is a locally factorial Krull domain if and only if $I_t(D) \lhd P$ for $P =$ "— is proper invertible ideal".*

Corollary 3. *Let D be a Noetherian domain. Then (1) D is a PID if and only if $I(D) \lhd P$ for $P =$ "— is a proper nonzero principal ideal" and (2) D is a Dedekind domain if and only if $I(D) \lhd P$ for $P =$ "— is a proper invertible ideal".*

Corollary 4. *Let D be a Mori domain. Then (1) D is a UFD if and only if $I_t(D) \lhd P$ for $P =$ "— is a proper nonzero principal ideal", (2) D is a locally factorial Krull domain if and only if $I_t(D) \lhd P$ for $P =$ "— is a proper invertible integral ideal", (3) D is a Krull domain if and only if $I_t(D) \lhd P$ for $P =$ "— is a proper t-invertible t-ideal".*

Finally, consider the following scheme of results.

Proposition 6. *Suppose that D satisfies ACCP (ACC on principal ideals). Then (1) D is a PID if and only if $I(D) \lhd P$ for $P =$ "— is a proper nonzero principal* *ideal" and (2) D is a UFD if and only if $I_t(D) \lhd P$ for $P =$ "— is a proper nonzero principal ideal".*

Proof. $I_t(D) \lhd P$ for $P =$ "— is a proper nonzero principal ideal"$\Leftrightarrow \forall A \in I_*(D)$ $(A \neq D \Rightarrow \exists \pi \in \Pi(A \subseteq \pi))$ where Π is the set of proper nonzero principal ideals of D fixed by P and $* = d$ or t. Then, by the condition, for any maximal (maximal t-ideal) M of D we have $M \subseteq \pi$ for some $\pi \in \Pi$ and so $M = \pi D$. Claim that, because of the ACCP, M is of height one. (For if not, then there is $Q \subseteq \cap \pi^n D$. So for every nonzero $x \in Q$, x is divisible by every power of π, giving rise to an infinite ascending chain $xD \subsetneq \frac{x}{\pi}D \subsetneq \frac{x}{\pi^2} \subsetneq ... \subsetneq \frac{x}{\pi^n}D \subsetneq ...$ which is impossible in the presence of ACCP on D.) Now MD_M is principal and of height one, making D_M a rank one discrete valuation domain and making $D = \cap D_M$ completely integrally closed with every maximal (t-) ideal principal. This makes D a Krull domain with every height one prime a principal ideal and so a UFD. Finally, a UFD with every height one prime maximal is a PID. The converse, in each case is straightforward. $\square$

The above Proposition may revive an old question touched on in [34]: If D has ACCP and M a maximal t-ideal, must M be of height one? We couldn't answer it then and we had to resort to using the "strong ACCP": D has ACCP and D_M has ACCP for every maximal t-ideal M. Now I have taken the route of using the t-ACC and this gives rise to: If D has t-ACC, must D_M have ACCP for each maximal t-ideal M?

3. A Universal Restriction with Conditions

Call a directed p.o. group G an almost l.o. group if for each finite subset $X = \{x_1, ..., x_r\} \subseteq G^+$ there is a positive integer $n = n(X)$ such that $inf(x_1^n, ... x_r^n) \in G^+$. Almost l.o. groups were introduced in [16] and further studied in [39]. One can talk about a commutative p.o. monoid M with least element 1 and a pre-assigned set Π such that for all $x_1, ..., x_r \in M$ with $\mathcal{L}(x_1, ..., x_r) \neq 1$, there being a $\pi \in \Pi$ such that $x_1^n, ..., x_r^n \geq \pi$. As ring theory provides a plethora of examples of this concept, we turn to ring theory.

Let D be a domain with a finite type star operation $*$ defined on it, let $I_*(D)$ be the set of proper $*$-ideals of D and let $\Pi_{I_*(D)}(P)$ be a non-empty subset of $I_*(D)$ defined by a predicate P such that for each $A \in I_*(D)$ there is $n = n(A)$ with $A^n \subseteq \pi$ for some $\pi \in \Pi_{I_*(D)}(P)$. Let us say $I_*(D)$ meets P with a twist when this happens and denote it by $I_*(D) \lhd^t P$. We start with a motivating example of this notion.

Example 3.1. Let R be a Dedekind domain with torsion class group, let K be the quotient field of R and let X be an indeterminate over K. Then the ring $D = R + XK[X]$ is such that $I(D) \lhd^t P$ where $P =$ "— is a principal ideal".

Illustration: Recall, as we have already done, from [14, Theorem 4.21] that maximal ideals of D are of the form $M + XK[X]$, where M is a maximal ideal of R, or of the form $(1 + Xf(X))D$ where $1 + Xf(X)$ is irreducible in $K[X]$. Now since for each maximal ideal M of R we have $M^n \subseteq dR$ for some positive integer n and some nonzero $d \in R$ we have $(M + XK[X])^n = M^n + XK\{X\} \subseteq dR + XK[X]$. Next since for each maximal ideal $\mathcal{M}$ of D, either $\mathcal{M}$ is principal, and hence is contained in a principal ideal in $\Pi_{I_*(D)}(P)$ or $\mathcal{M}$ is such that $\mathcal{M}^n$ is contained in a principal ideal for some positive integer n, the same must hold for every ideal I of D.

The above example leads to the following statement.

Proposition 7. *(1)$I(D) \lhd^t P$ where $P =$ "—- is a proper nonzero finitely generated ideal" if and only if for every maximal ideal M of D we have $M^n \subseteq \pi \in \Pi_{I(D)}(P)$ and (2) let L be an extension field of $K = qf(D)$ and let X be an indeteminate over L. Then $I(D) \lhd^t P$ where $P =$ "—- is a proper nonzero finitely generated ideal" if and only if $I(R) \lhd^t P$ where $R = D + XL[X]$.*

Proof. (1) Suppose that for every ideal A of D we have some $n = n(A)$ and a $\pi_A \in \Pi_{I(D)}$ such that $A^n \subseteq \pi_A$ then the same holds if A is a maximal ideal of D. Conversely suppose that for every maximal ideal M of D we have some $n = n(M)$ and some $\pi_M \in \Pi_{I(D)}$ such that $M^{n(M)} \subseteq \pi_M$ and let A be a proper nonzero ideal of D. Then $A \subseteq M$ for some maximal ideal M of D and $A^{n(M)} \subseteq M^{n(M)} \subseteq \pi_M$. For (2) let $I(D) \lhd^t P$, where P is as given, then, by (1), for every maximal ideal $\wp$ of D there is $n = n(\wp)$ such that $\wp^n \subseteq \pi_\wp$ for some $\pi_\wp \in \Pi_{I(D)}$. Since every maximal ideal of $D + XL[X]$ is either principal, and hence finitely generated, or of the form $\wp + XL[X]$ where $\wp$ is a maximal ideal of D [15, Lemmas 1.2, 1.5], for every ideal A of R there is $n = 1$ or $n(\wp)$ such that $A^n \subseteq \pi_A \in \Pi_{I(R)}$, so $I(R) \lhd^t P$. Conversely suppose that $I(R) \lhd^t P$. Then, in particular, for maximal ideals $\mathcal{M}$ of the form $\wp + XL[X]$ there are positive integers $n(\mathcal{M})$ such that $(\wp + XL[X])^{n(\mathcal{M})}$ $= \wp^{n(\mathcal{M})} + XL[X] \subseteq \pi_\mathcal{M} \in \Pi_{I(R)}$. But then $\pi_\mathcal{M} \cap D \neq (0)$ forcing $\pi_\mathcal{M} = \pi + XL[X] = \pi(D + XL[X])$ [15, Lemma1.1], where π is finitely generated because $\pi_\mathcal{M}$ is. This gives $\wp^{n(\mathcal{M})} + XL[X] \subseteq \pi + XL[X]$ and modding out $XL[X]$ we get $\wp^{n(\mathcal{M})} \subseteq \pi \in \Pi_{I(D)} = \{\pi \neq (0) | \pi + XL[X] \in \Pi_{I(R)}\}$. □

Proposition 8. *(1) $I(D) \lhd^t P$ where $P =$ "— is a proper t-ideal of finite type" if and only if for every maximal ideal M of D, there is $n = n(M)$ such that $M^n \subseteq \pi \in \Pi_{I(D)}(P)$, (2) $I_t(D) \lhd^t P$ where $P =$ "— is a proper t-ideal of finite type" if and only if for every maximal t-ideal M of D we have $M^n \subseteq \pi \in \Pi_{I(D)}(P)$, (3) let L be an extension field of K and let X be an indeteminate over L. Then $I(D) \lhd^t P$ where $P =$ "— is a proper t-ideal of finite type" if and only if $I(R) \lhd^t P$ where $R = D + XL[X]$ and (4) let L be an extension field of K and let X be an indeteminate over L. Then $I_t(D) \lhd^t P$ where $P =$ "— is a proper t-ideal of finite type" if and only if $I_t(R) \lhd^t P$ where $R = D + XL[X]$.*

(1) The proof works as the proof of (1) of Proposition 7. (2) The proof works in the same manner as that of (1) of Proposition 7, except that here the maximal t-ideals are in the focus. (3) Let $I(D) \lhd^t P$ where $P =$ "— is a proper t-ideal of finite type". To show that $I(R) \lhd^t P$ all we need show is that for every maximal ideal $\mathcal{M}$ of R, there is a positive integer $n = n(\mathcal{M})$ such that $\mathcal{M}^n \subseteq \pi_\mathcal{M} \in \Pi_{I(R)}(P)$. Now, as we have shown in the proof of (2) of Proposition 7, a maximal ideal $\mathcal{M}$ of R is either principal and hence contained in some member of $\Pi_{I(R)}$ or of the form $\mathcal{M} = M + XL[X]$, where M is a maximal ideal of D. But then, for $n = n(M)$ we have $M^n \subseteq \pi D$, where π is a t-ideal of finite type in $\Pi_{I(D)}$, forcing $\mathcal{M}^n = M^n + XL[X] \subseteq \pi R$. Because π is a t-ideal of finite type of D, so is $\pi R = \pi + XL[X]$, see e.g. proof of Lemma 3.5 of [46]. Conversely, suppose that $I(R) \lhd^t P$ where $P =$ "— is a proper t-ideal of finite type". Here, in particular, for a maximal ideal $\mathcal{M}$ of the form $\mathcal{M} = M + XL[X]$ we have a positive integer $n = n(\mathcal{M})$ such

that $\mathcal{M}^{n(\mathcal{M})} = M^{n(\mathcal{M})} + XL[X] \subseteq \pi_{\mathcal{M}}$ where $\pi_{\mathcal{M}}$ is a t-ideal of finite type of R. Obviously as $M^{n(\mathcal{M})} = (M^{n(\mathcal{M})} + XL[X]) \cap D \subseteq \pi_{\mathcal{M}} \cap D$, and as $M^{n(\mathcal{M})} \cap D \neq (0)$, we conclude that $\pi_{\mathcal{M}} \cap D \neq (0)$. Thus $\pi_{\mathcal{M}} = \pi_{\mathcal{M}} \cap D + XL[X]$ by [15, Lemma 1.1]. And as observed in the proof of Lemma 3.5 of [46] $\pi_{\mathcal{M}} = \pi_{\mathcal{M}} \cap D + XL[X]$ is a t-ideal of R if and only if $\pi_{\mathcal{M}} \cap D$ is a t-ideal of D. That $\pi_{\mathcal{M}}$ is of finite type if and only if $\pi_{\mathcal{M}} \cap D$ is, follows from the fact that $\pi_{\mathcal{M}} = (a_1, ..., a_n)_v + XL[X]$. Finally, for (4), let $I_t(D) \lhd^t P$ where $P =$ "— is a proper t-ideal of finite type" and as maximal t-ideals of R that intersect D trivially are prime ideals of R that intersect D trivially, are not comparable with $XL[X]$, and hence are principal we need to concentrate on maximal t-ideals $\mathcal{M}$ of R that intersect D non-trivially. But those are precisely $\mathcal{M} = (\mathcal{M} \cap D) + XL[X]$ and as $\mathcal{M} = \mathcal{M}_t = (\mathcal{M} \cap D)_t + XL[X]$ we have $(\mathcal{M} \cap D)_t = (\mathcal{M} \cap D)$. Thus $\mathcal{M} = M + XL[X]$ where M is a maximal t-ideal of D. But, by the hypothesis, there is a positive integer $n = n(M)$ such that $M^{n(M)} \subseteq \pi_M$ for some $\pi_M \in \Pi_{I_t(D)}(P)$. This forces $\mathcal{M}^{n(M)} = M^{n(M)} + XL[X] \subseteq \pi_M + XL[X]$ which is a t-ideal of finite type and hence in $\Pi_{I(R)}$. For the converse we take the same line as in the proof of (3) and note that for each maximal t-ideal M of D, $\mathcal{M} = M + XL[X]$ is a maximal t-ideal of R and as $\mathcal{M}^n = (M + XL[X])^n \subseteq \pi_{\mathcal{M}}$ for some $\pi_{\mathcal{M}} \in \Pi_{I_t(R)}(P)$, $M^n \subseteq \pi_{\mathcal{M}} \cap D \neq (0)$. Now, as in (3), $\pi_{\mathcal{M}} \cap D$ can be shown to be a t-ideal of finite type and hence in $\Pi_{I_t(D)}(P)$.

Apart from the examples constructed in the above proposition there are examples of domains $I_*(D) \lhd^t P$ for $P =$ "— is a $*$-ideal of finite type". Some of these examples are simple and straightforward and some are not so simple. Presented in the following is a sampling of them. If D is Noetherian and $P =$ "— is a finitely generated ideal, then $I(D) \lhd^t P$. Recall, again, that D is a Mori domain if it satisfies ACC on its integral divisorial ideals. Obviously Noetherian domains are Mori and less obviously Krull domains are Mori. Recall also that D is Mori if and only if for every nonzero integral ideal A of D there is a finitely generated ideal $F \subseteq A$ such that $A_v = F_v$, if and only if every t-ideal of D is a t-ideal of finite type [43]. Thus if D is a Mori domain then $I_t(D) \lhd^t P$ where $P =$ "— is a t-ideal of finite type". Note that since for a finitely generated nonzero ideal A of any domain $A_t = A_v$, every t-ideal of a Mori domain is divisorial. In what follows we shall also need the fact that if I is a $*$-ideal for some star operation $*$, then $\sqrt{I}$ is a $*_s$-ideal (see Theorem 1 of [45]). Thus if I is divisorial, or a t-ideal then $\sqrt{I}$ is a t-ideal.

Proposition 9. *Let D be a Mori domain. Then $I(D) \lhd^t P$ with $P =$ "— is a t-ideal" if and only if every maximal ideal of D is divisorial.*

Proof. If every maximal ideal M of D is a t-ideal then, D being Mori, M is a t-ideal of finite type and hence in $\Pi_{I(D)}(P)$, returning T for P. Whence $I(D) \lhd^t P$. Conversely suppose that D is Mori and $I(D) \lhd^t P$ where P is as given and let M be a maximal ideal of D. Then by the condition $M^n \subseteq A$ where A is a t-ideal. This gives $M = \sqrt{M^n} \subseteq \sqrt{A}$. Since M is maximal, we have $M = \sqrt{A}$ which is a t-ideal. Since M is arbitrary we have the result. $\square$

The event of $I(D) \lhd^t P$ for $P =$ "—- is a t-ideal of finite type" does not put any constraint on the height of maximal ideals of a Mori domain. Indeed there do exists examples of Noetherian domains with maximal t-ideals of height greater than one, see e.g. [20, Example 3.5].

Corollary 5. *Let D be a Noetherian integral domain. Then $I(D) \lhd^t P$ with $P =$ "—- is a t-ideal of finite type" if and only if every maximal ideal of D is divisorial.*

Indeed as in a polynomial ring over $D \neq K$, every maximal ideal being a radical of a t-ideal of any kind is not possible because that would make every maximal ideal of the polynomial ring a t-ideal as we have seen in section 2. On the other hand, we have the following statement.

Proposition 10. *Let $R = D[X]$. If $P =$ "—- is a t-ideal (resp., t-invertible t-ideal, divisorial ideal)" Then $I_t(D) \lhd^t P \Rightarrow I_t(R) \lhd^t P$ and if D is integrally closed, $I_t(R) \lhd^t P \Rightarrow I_t(D) \lhd^t P$.*

Proof. (a). Let M be a maximal t-ideal of $D[X]$ and suppose that $M \cap D \neq (0)$. Then $M = \wp[X]$ where $\wp = M \cap D$ is a maximal t-ideal of D [28]. Since $I_t(D) \lhd^t P$ we conclude that for some $n = n(\wp)$, $\wp^n$ is contained in a t-invertible t-ideal (resp. t-ideal, divisorial ideal) A. But then, $M^n = \wp^n[X] \subseteq A[X]$. Next let M be a maximal t-ideal of $D[X]$ such that $M \cap D = (0)$. Then M is a t-invertible t-ideal and hence divisorial by Theorem 1.4 of [28] and $M^n \subseteq M$ for all n. Next suppose that $I_t(R) \lhd^t P$ for the specified P. Then, in particular, for every maximal t-ideal $\wp$ of D we have the maximal t-ideal $M = \wp[X]$ and, by the condition, there is $n = n(M)$ such that M^n is contained in a t-ideal (resp., t-invertible t-ideal, divisorial ideal) A of $D[X]$. Since $M^n \cap D \neq (0)$, $A \cap D \neq (0)$ and since D is integrally closed $A = (A \cap D)[X]$ and $A \cap D$ is a t-ideal (resp., t-invertible t-ideal, divisorial ideal), if A is [5, Corollary 3.1]. □

Indeed as the behavior of $D + XL[X]$ is the same under $S(D) \lhd^t P$ as it was under $S(D) \lhd P$, one can construct examples to show that if R is a ring of fractions of D, $S(D) \lhd^t P$ may not imply $S(R) \lhd^t P$ in general. This leaves us to check what happens if we restrict a domain to be completely integrally closed and satisfy $S(D) \lhd^t P$ for a suitable P. To appreciate the following proposition we need to have an idea of the divisor class group of a Krull domain being torsion. For this too the reference to go to is [21]. For our purposes the divisor class group being torsion means that for each proper divisorial ideal I there is some positive integer n such that $(I^n)_v$ is principal. The other concept to know is the local class group $G(D) = Cl(D)/Pic(D)$ of a Krull domain D, introduced and studied by Bouvier in [9]. Now $G(D)$ being torsion is equivalent to $(I^n)_v$ being invertible, for some integer n, for each proper divisorial ideal I.

Proposition 11. *(a) Let D be a completely integrally closed domain. Then (1) D is a Krull domain if and only if $I_t(D) \lhd^t P$ for $P =$ "— is a proper divisorial ideal", (2) D is a Krull domain if and only if $I_t(D) \lhd^t P$ for $P =$ "— is a proper t-invertible t-ideal", (3) D is a Krull domain, with torsion divisor class group, if and only if $I_t(D) \lhd^t P$ for $P =$ "— is a proper principal ideal", (b) Let D be an intersection of rank one valuation domains. Then (4) D is a Krull domain, if and only if $I_t(D) \lhd^t P$ for $P =$ "— a proper v-ideal of finite type" and (5) D is a Krull domain, with torsion local class group, if and only if $I_t(D) \lhd^t P$ for $P =$ "— a proper invertible ideal", (c) Let D be completely integrally closed. Then (6) D is a Dedekind domain if and only if $I(D) \lhd^t P$ for $P =$ "— is a proper divisorial ideal" (resp. invertible ideal) and (7) D is a Dedekind domain with torsion class group if and only if $I(D) \lhd^t P$ for $P =$ "— is a proper principal ideal".*

Proof. (1). Let D be a completely integrally closed domain and let $I_t(D) \lhd^t P$ for $P =$ "— is a proper divisorial ideal". (I.e. suppose that for every t-ideal I there is n such that I^n is contained in a principal ideal.) Now let M be a maximal t-ideal of D. We claim that M is divisorial, for if not then $M_v = D$. But, by the condition, M^n is contained in a proper divisorial ideal π. Thus $(M^n)_v \subseteq \pi$ because π is a divisorial ideal. On the other hand $(M^n)_v = ((M_v)^n)_v = D$, contradicting the assumption that π is a proper divisorial ideal. Whence $M_v \neq D$, forcing $M = M_v$. Now as M is arbitrary, we conclude that D is an H domain [27]. Finally, according to [24], D is Krull. Conversely if M is a maximal t-ideal of a Krull domain then M is divisorial and so is $(M^n)_v$ which returns T for P for any n. (2). Because a proper t-invertible t-ideal is divisorial too and because every prime t-ideal of a Krull domain is t-invertible and so must be every maximal t-ideal M, with $(M^n)_v$ a t-invertible t-ideal, we conclude that the proof of (1) applies. (3). For sufficiency, note that a proper principal ideal is divisorial. So D is at least a Krull domain, by part (1). Now let M be a maximal t-ideal of D. Then, by the condition, M^n is contained in a proper nonzero principal ideal π and clearly $M^n \subseteq \pi \subseteq M$. Thus M is the radical of a principal ideal and Theorem 3.2 of [2] applies to give the conclusion that the divisor class group of D is torsion. Conversely if D is a Krull domain whose divisor class group is torsion, then via Theorem 3.2 of [2] (or via [21, Proposition 6.8]) one finds that for each maximal t-ideal M we have $(M^n)_v = \pi$ a principal ideal verifying that M^n is contained in a proper principal ideal for each maximal t-ideal M of D. Note in part (b) that D being completely integrally closed is provided by the given. Then (4) can be proved just like (1) and that leaves (5). Now in (5) we prove just like (3) that D is a Krull domain and then use the condition to show that M is the radical of an invertible ideal. This would give, via Theorem 3.3 of [2] the conclusion that $G(D)$ is torsion. For necessity in this case we appeal to Theorem 3.3 of [2] to conclude that $I_t(D) \lhd^t P$. For (6) and (7) note that every maximal t-ideal is maximal, and divisorial, because every maximal ideal is divisorial. So, in each case, D is a one dimensional Krull domain and hence a Dedekind domain. Now in case of (7) we can conclude, as in the proof of (3), that every maximal ideal is the radical of a principal ideal. The converse in each case is obvious, if not dealt with. □

For a star operation $*$ of finite type, defined on D, call D of finite $*$-character if every nonzero non unit of D belongs to at most a finite number of maximal $*$-ideals of D. We shall be mostly concerned with $* = t$ or d though some of the considerations here may apply to the general approach. In any case we may define $*$-dimension as the supremum of the lengths of chains of $*$-ideals that are prime. Call D a weakly Krull domain (WKD) if $D = \cap_{P \in X^1(D)} D_P$ and the intersection is locally finite. It turns out that D is of finite t-character and of t-dimension one [6]. We shall also need to use the *nth* symbolic power $Q^{(n)}$of a prime Q defined by $Q^{(n)} = Q^n D_Q \cap D$. We shall need also to recall that a nonzero finitely generated ideal I is said to be rigid (t-rigid) if I is contained in a unique maximal (t-) ideal. A maximal (t-) ideal is said to be (t-) potent if it contains a (t-) rigid ideal. Finally a domain D is said to be (t-) potent if each of its maximal (t-) ideals is (t-) potent.

Proposition 12. *(1)Let $I(D) \lhd^t P$ where $P =$ "— is a proper nonzero principal ideal" (resp. invertible ideal, t-invertible t-ideal) . If D has t-ACC, then D is a t-potent domain whose maximal ideals M are divisorial such that $\cap (M^n)_v = (0)$ and (2) Let $I_t(D) \lhd^t P$ where $P =$ "— is a proper nonzero principal ideal" (resp. invertible ideal, t-invertible t-ideal) . If D has t-ACC, then D is a t-potent domain whose maximal t-ideals M are divisorial such that $\cap (M^n)_v = (0)$.*

Proof. For (1) let $I(D) \lhd^t P$ where $P =$ "— is a proper nonzero principal ideal" (resp. invertible ideal, t-invertible t-ideal) and suppose that D has t-ACC. As we concluded in the proof of Proposition 11, every maximal ideal M is divisorial. Next, for every maximal ideal M we have $M^n \subseteq \pi \in \Pi_{I(D)}(P)$. This shows also that M is t-potent. Next $(M^n)_v \subseteq \pi$, because π is divisorial. So $\cap(M^{nr})_v \subseteq \cap(\pi^n)_v$. Since π is a t-invetible t-ideal and since D is t-ACC, Lemma 2.9 applies to give $\cap(\pi^n)_v = (0)$. Whence $\cap(M^n)_v = (0)$. For (2) note that $I_t(D) \lhd^t P$ implies that $M^n \subseteq \pi \in \Pi_{I_t(D)}(P)$ for each maximal t-ideal M. Since π is divisorial, M must be. The rest of the proof follows the same lines as taken in the proof of (1). □

The above result does not give much. But with some give and take it can.

Proposition 13. *(a)Let $I(D) \lhd^t P$ where $P =$ "— is a proper nonzero principal ideal" and suppose that D has t-ACC. Then the following are equivalent: (1) D is one dimensional, (2) for every maximal ideal M, M^n being contained in a principal ideal dD implies $Q^{(n)} \subseteq dD$ for every nonzero prime Q contained in M, (3) D is a one dimensional WKD and (4) Every power of every nonzero prime ideal Q of D is a primary ideal and (b) Let $I_t(D) \lhd^t P$ where $P =$ "— is a proper nonzero principal ideal" and suppose that D has t-ACC. Then the following are equivalent: (1) D has t-dimension one, (2) for every maximal t-ideal M, M^n being contained in a principal ideal dD implies $Q^{(n)} \subseteq dD$ for every nonzero prime Q contained in M, (3) D is a WKD.*

Proof. (a) That (1) ⇒ (2) is clear. For (2) ⇒ (3), we show that there D is one dimensional. Assume by way of contradiction that there is a nonzero non-maximal prime Q contained in a maximal ideal M. Let $M^n \subseteq dD$ for a non unit $d \in D$ and let $0 \neq x \in Q^{(n)}$. Then $x \in dD$. Since $d \notin Q$, $(x/d)d \in Q^{(n)}$ forces $x/d \in Q^{(n)}$. Repeating the argument over and over again we get $\frac{x}{d}D \subseteq \frac{x}{d^2}D \subseteq \frac{x}{d^3}D \subseteq ... \subseteq \frac{x}{d^n}D \subseteq \frac{x}{d^{n+1}}D \subseteq ...$ which is impossible in the presence of t-ACC. Thus D is one dimensional and hence of t-dimension one. Now a t-potent domain of t-dimension one is a WKD by [30, Theorem 5.3]. That (3) ⇒ (4), is direct because D is one dimensional. For (4) ⇒ (1), suppose that there is a nonzero non-maximal prime ideal Q and proceed as in the proof of (2) ⇒ (3) to get the desired contradiction. For the proof of (b) note that (1) ⇒ (2) is obvious and (2) ⇒ (3) goes exactly along the lines taken in the proof of (2) ⇒ (3) of (a), while (3) ⇒ (1) is obvious too. □

Lest a reader considers Proposition 13 an empty result we hasten to give examples to allay such feelings. For the following set of examples we need to know that an extension of domains $A \subseteq B$ is called a root extension if for each $b \in B$ there is a positive integer $n = n(b)$ such that $b^n \in A$. Let's call $A \subseteq B$ a fixed root extension if there is a fixed positive integer n such that $b^n \in A$, for all $b \in B$. Also an integral domain D is called an Almost Principal Ideal (API-)domain if for each subset $\{a_\alpha\}$ of $D\backslash\{0\}$ there is a positive integer n such that $(\{a_\alpha^n\})$ is principal. According to [7, Theorem 4.11] if $A \subseteq B$ is a fixed root extension and B is a subring of the integral closure of A, then A is an API domain if and only if B is.

Example 3.2. Of course (1) every Dedekind domain D with torsion class group is an example of a one dimensional WKD such that $I(D) \lhd^t P$ where $P =$ "— is a proper nonzero principal ideal" . (2) In section 4 of [7] there are studied several examples of Noetherian API domains that are not integrally closed. The simplest of these being $\mathbb{Z}[2i] = \mathbb{Z} + 2i\mathbb{Z}$. Since for each $a + bi \in \mathbb{Z}[i]$ we have

$(a+bi)^2 = a^2 - b^2 + 2abi \in \mathbb{Z}[2i]$, this gives the conclusion that $\mathbb{Z}[2i]$ is Noetherian and that $\mathbb{Z}[2i] \subseteq \mathbb{Z}[2i]$ is a fixed root extension. Because $\mathbb{Z}[i]$ is a PID, Corollary 4.13 of [7] applies to give the conclusion that $\mathbb{Z}[2i]$ is an API domain. That $\mathbb{Z}[2i]$ is one dimensional, follows from Theorem 2.1 of [7]. Now let M be a maximal ideal of $\mathbb{Z}[2i]$. Then M is finitely generated, say $M = (x_1, x_2, ..., x_r)$ then $(x_1^n, ..., x_r^n)$ is principal and, using Lemma 2.3 of [38], we conclude that $M^{nr} \subseteq (x_1^n, ..., x_r^n)$. (3) Finally, let K be a field of characteristic $p > 0$ and let L be a purely inseparable field extension of K such that $L^p \subseteq K$ and consider $T = K + XL[X]$. According to the information gathered prior to Example 2.6, the only non-principal maximal ideal of T is $XL[X] = (X, lX)_v$ where $l \in L \backslash K$. Obviously $(X^p, (lX)^p)_v = X^p$ and an application of Lemma 2.3 of [38] or direct computation gives $(XL[X])^{2p} \subseteq ((XL[X])^{2p})_v = ((X, lX)^{2p})_v \subseteq X^p$. The above can serve also as examples for part (b), but all fastfaktorielle rings of [37] dubbed as almost factorial domains in [21] can serve as examples as almost factorial domains are nothing but Krull domains with torsion divisor class groups. For non-Krull examples for (b) recall that, according to [40], an integral domain D is called an AGCD domain if for each pair $a, b \in D \backslash \{0\}$ there is a positive integer $n = n(a, b)$ such that $a^n D \cap b^n D$ is principal (equivalently for every nonzero finitely generated ideal $(a_1, ..., a_r)$ there is $n = n\ (a_1, ..., a_r)$ such that $(a_1^n, ..., a_r^n)_v$ is principal). Any Noetherian AGCD domain would serve as an example for (b). Reason: take a maximal t-ideal M, it's finitely generated. Say $M = (a_1, ..., a_r)$, for some n we must have $(a_1^n, ..., a_r^n)_v = dD$, principal. But then $M^{nr} \subseteq (a_1^n, ..., a_r^n) \subseteq (a_1^n, ..., a_r^n)_v = dD$, by Lemma 2.3 of [38].

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Department of Mathematics, Idaho State University, Pocatello, 83209 ID
E-mail address: mzafrullah@usa.net
URL: https://www.lohar.com