ON S-GCD DOMAINS

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Abstract. Let S be a multiplicative set in an integral domain D. A nonzero ideal I of D is said to be S-v-principal if there exist an $s \in S$ and $a \in D$ such that $sI \subseteq aD \subseteq I_v$. Call D an S-GCD domain if each finitely generated nonzero ideal of D is S-v-principal. This notion was introduced in [14]. One aim of this article is to characterize S-GCD domains, giving several equivalent conditions and showing that if D is an S-GCD domain then D_S is a GCD domain but not conversely. Also we prove that if D is an S-GCD S-Noetherian domain such that every prime w-ideal disjoint from S is a t-ideal, then D is S-factorial and we give an example of an S-GCD S-Noetherian domain which is not S-factorial. We also consider polynomial and power series extensions of S-GCD domains. We call D a sublocally s-GCD domain if D is a $\{s^n \mid n \in N\}$-GCD domain for every non-unit $s \in D\backslash\{0\}$ and show, among other things, that a non-quasilocal sublocally s-GCD domain is a generalized GCD domain (i.e., for all $a, b \in D\backslash\{0\}, aD \cap bD$ is invertible).

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1. Introduction

Let D be an integral domain with quotient field K. Let $\mathcal{F}(D)$ be the set of nonzero fractional ideals of D. For an $I \in \mathcal{F}(D)$, set $I^{-1} = \{x \in K \mid xI \subseteq A\}$. The mapping on $\mathcal{F}(D)$ defined by $I \mapsto I_v = (I^{-1})^{-1}$ is called the *v-operation* on D. A nonzero fractional ideal I is said to be a *v-ideal* or *divisorial* if $I = I_v$, and I is said to be of *v-finite type* if $I = J_v$ for some finitely generated ideal J of D. Using the v-operation we can define the *t-operation* as: For $I \in \mathcal{F}(D)$ $I_t = \cup\{F_v \mid 0 \neq F$ is a finitely generated subideal of $I\}$. It can be shown that $I \mapsto I_t$ is a mapping on $\mathcal{F}(D)$ and a star operation, like the v-operation. For properties of the v- and t-operations the reader is referred to [13, Section 34]. An ideal $I \in \mathcal{F}(D)$ is said to be a *t-ideal* if $I = I_t$ and a *t-ideal of finite type* if $I = J_t$ for a finitely generated $J \in \mathcal{F}(D)$. Also $I \in \mathcal{F}(D)$ is said to be *t-invertible* if $(II^{-1})_t = D$. A t-invertible t-ideal is known to be of finite type. A domain D is said to be a *Prüfer v-multiplication domain* (*PVMD*) if every finitely generated $I \in \mathcal{F}(D)$ is t-invertible.

An integral domain D is called a *GCD domain* if for each pair $a, b \in D^* = D\backslash\{0\}$, $GCD(a, b)$ exists. GCD domains are an important class of integral domains from classical ideal theory. In a GCD domain, every finite type v-ideal of D is principal, thus a GCD domain is a PVMD. This property can be generalized in several different ways ([2], [14]). However, we will be mostly interested in the S-GCD property ([14]). Let S be a multiplicative subset of D and I a nonzero ideal of D. We say that I is *S-principal* (resp., *S-v-principal*) if there are $s \in S$ and $a \in I$ (resp., $a \in I_v$) such that $sI \subseteq aD$. Following [14], D is an *S-GCD domain* if each finitely generated nonzero ideal of D is S-v-principal. Note that if S consists of units of D, then D is an S-GCD domain if and only if it is a GCD domain.

In the first part of this paper, we continue the study of the S-GCD property. We give an example of an S-GCD domain which is not a GCD domain. We also give equivalent conditions for an integral domain to be an S-GCD domain. We show that the following are equivalent: (1) D is an S-GCD domain, (2) for $a,\ b \in D^*$, (a,b) (resp., $aD \cap bD,\ aD : bD$) is S-v-principal (resp., S-principal), and (3) any finite intersection of principal ideals of D is S-principal. Recall from [4] that a saturated multiplicatively closed subset S of an integral domain D is said to be a *splitting set* if for each $d \in D^*$ we can write $d = sa$ for some $s \in S$ and $a \in D$ with $s'D \cap aD = s'aD$ for all $s' \in S$. A splitting set S of D is said to be an *lcm splitting set* if for each $s \in S$ and $d \in D$, $sD \cap dD$ is principal. We prove that if S is a splitting set, then D is an S-GCD domain if and only if D_S is a GCD domain, and we give an example of a domain D and a multiplicative set S which is not splitting such that D_S is a GCD domain but D is not an S-GCD domain. Based on this result we link the S-GCD domains with GCD domains, PVMDs, and UFDs.

We give an S-version of a well known result about GCD domain where S is generated by prime elements of D : D is a UFD if and only if D is an S-GCD domain and D satisfies the ACCP property. Recall from [20] that a domain D is said to be a *Mori domain* if it satisfies the ascending chain condition on integral divisorial ideals. We show that if D is a Mori domain and S a multiplicative set, then D is an S-GCD domain if and only if D_S is a UFD. Note that if D is an S-GCD domain, then D is not necessarily a PVMD. For example, if we take D an integral domain which is not a PVMD (e.g., any non-integrally closed domain) and $S = D^*$ a multiplicative subset of D, then D is an S-GCD domain which is not a PVMD. Let D be an integral domain, S a splitting multiplicative subset of D and T the m-complement of S. We show that if D is an S-GCD domain as well as a T-GCD domain, then D is a GCD domain. We also prove that if S is an lcm splitting set of an integral domain D, then D is an S-GCD domain if and only if D is a GCD domain and consequently, D is an S-GCD domain if and only if $D[X]$ is an S-GCD domain. On the other hand, let D be an integral domain and S a multiplicative subset of D. The mapping on $\mathcal{F}(D)$ defined by $I \mapsto I_w = \{x \in K \mid xJ \subseteq I$ for some finitely generated ideal J of D such that $J_v = D\}$ is called the *w-operation* on D. According to [15] a nonzero ideal I of D is *S-w-principal* if there exist an $s \in S$ and $a \in D$ such that $sI \subseteq aD \subseteq I_w$. We also define D to be an *S-factorial domain* if each nonzero ideal of D is S-w-principal. Recall from [6] that an ideal I of D is called *S-finite* if $sI \subseteq J \subseteq I$ for some finitely generated ideal J of D and some $s \in S$. Also, D is called *S-Noetherian* if each ideal of D is S-finite. We show that if D is an S-GCD S-Noetherian domain such that every prime w-ideal disjoint from S is a t-ideal, then D is S-factorial, but we give an example of an S-GCD S-Noetherian domain which is not S-factorial. Note that the S-GCD property does not carry over to the power series ring. In fact, there is an example of a GCD domain D such that $D[[X]]$ is not a GCD domain [19, Theorem 8]. (This is the case when S consists of units of D). We give with an additional condition a necessary and sufficient condition for the power series ring $D[[X]]$ to be an S-GCD domain. First, recall from [14], the power series ring $D[[X]]$ is said to satisfy the *property* $(*)$, if for all integral v-invertible v-ideals I and J of $D[[X]]$ such that $(IJ)_0 \neq (0)$, we have $((IJ)_0)_v = ((IJ)_v)_0$ where $I_0 = \{f(0) \mid f \in I\}$. We show that if D is a Krull domain such that $D[[X]]$ satisfies $(*)$ and S a multiplicative subset of D, then D is an S-GCD domain if and only if $D[[X]]$ is an S-GCD domain (Theorem 2.6). In particular, in a Krull domain D such that $D[[X]]$ satisfies $(*)$, D is a UFD if and only if $D[[X]]$ is a UFD.

In the third section of this paper, we define the notion of a sublocally s-GCD domain. Let D be an integral domain and s a nonzero element of D. We say that D is an *s-GCD domain* if for each pair $a, b \in D^*$, there is a positive integer n and an element $c \in aD \cap bD$ such that $s^n(aD \cap bD) \subseteq cD$. So if $\mathcal{S} = \{s^n \mid n \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of positive integers, then it is obvious that D is an s-GCD domain if and only if D is an $\mathcal{S}$-GCD domain. An integral domain

D is said to be a *sublocally s-GCD domain* if for every nonzero non-unit $s \in D$, D is an s-GCD domain. We show that D is a sublocally s-GCD domain if and only if D is an S-GCD domain for every nontrivial multiplicative set S of D (i.e., S contains at least one non-unit). Recall from [1] that an integral domain D is said to be a *generalized GCD domain* (*G-GCD domain*) if for each pair $a, b \in D^*$ we have $aD \cap bD$ invertible. We prove that a sublocally s-GCD domain that is not quasi-local is a G-GCD domain. In particular a semi-quasi-local sublocally s-GCD domain that is not quasi-local is a GCD domain.

2. On S-GCD domains

We begin this section by recalling the following definitions in order to give an S-version of a known classical result about GCD domains. First, let us recall that for a multiplicative set S in D, Anderson and Dumitrescu [6] call an ideal I of D *S-finitely generated* (resp., *S-principal*) if $sI \subseteq J \subseteq I$ for some finitely generated (resp., principal) ideal $J \subseteq I$ and some $s \in S$ and D is called an *S-Noetherian domain* (resp., S-PID) if each ideal of D is S-finite (resp., S-principal).

Definition 2.1. [14] Let D be an integral domain and S a multiplicative subset of D. We say that a nonzero ideal I of D is *S-v-principal* if there exist an $s \in S$ and $a \in D$ such that $sI \subseteq aD \subseteq I_v$. We also define D to be an *S-GCD domain* if each finitely generated nonzero ideal of D is S-v-principal.

Remark 2.1.
(1) If S consists of units of D, then D is an S-GCD domain if and only if D is a GCD domain. At the other extreme if $S = D^*$, then every domain is an S-GCD domain.
(2) In an S-GCD domain, every finite type divisorial ideal is S-principal.

Let D be an integral domain and S a multiplicative subset of D. The mapping on $\mathcal{F}(D)$ defined by $I \mapsto I_w = \{x \in K \mid xJ \subseteq I$ for some finitely generated ideal J of D such that $J_v = D\}$ is called the *w-operation* on D. Recall from [15] that, a nonzero ideal I of D is *S-w-principal* if there exist an $s \in S$ and $a \in D$ such that $sI \subseteq aD \subseteq I_w$. We also define D to be an *S-factorial domain* if each nonzero ideal of D is S-w-principal.

Example 2.1. Let S be a multiplicative subset of an integral domain D.
(1) If D is a GCD domain, then D is an S-GCD domain.
(2) Since for all fractional ideals I of D, $I_w \subseteq I_v$, every S-w-principal ideal of D is S-v-principal. So every S-factorial domain is an S-GCD domain.
(3) Let T be a multiplicative subset of D containing S. As every S-v-principal ideal of D is T-v-principal, then every S-GCD domain is a T-GCD domain.

The converse of (1) in the previous example is not true in general. Indeed for any domain D and $S = D^*$, D is an S-GCD domain.

The following theorem gives equivalent conditions for an integral domain to be an S-GCD domain. It is well-known that if we take S a subset of the group of units of D, then these conditions are all equivalent to D being a GCD domain.

Theorem 2.1. *Let D be an integral domain and S a multiplicative subset of D. Then the following assertions are equivalent.*
(1) *D is an S-GCD domain.*
(2) *Any finite intersection of nonzero principal ideals of D is S-principal.*
(3) *For $a,\ b \in D^*$, $aD \cap bD$ is S-principal.*
(4) *For $a,\ b \in D^*$, $aD + bD$ is S-v-principal.*
(5) *For $a,\ b \in D^*$, $aD : bD$ is S-principal.*

Proof: We show that (1) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (2) and (3) $\Longleftrightarrow$ (5).
(1) $\Longrightarrow$ (4) Obvious. Conversely, let $I = b_1D + \cdots + b_nD$ be a nonzero finitely generated ideal of D. By hypothesis, there exist an $s_1 \in S$ and $a_1 \in D$ such that $s_1(b_1D + b_2D) \subseteq a_1D \subseteq (b_1D + b_2D)_v$. Then $s_1I \subseteq a_1D + b_3D + \cdots + b_nD \subseteq I_v$. By induction, there exist an $s_2 \in S$ and $a_2 \in D$ such that $s_2(a_1D + b_3D + \cdots + b_nD) \subseteq a_2D \subseteq (a_1D + b_3D + \cdots + b_nD)_v$. Let $t = s_1s_2 \in S$. Then $tI \subseteq a_2D \subseteq I_v$, and hence I is S-v-principal.
(4) $\Longrightarrow$ (3) Let $a,\ b \in D^*$. Since $I = aD + bD$ is *S-v-principal*, then there exist an $s \in S$ and $d \in D^*$ such that $sI \subseteq dD \subseteq I_v$. Thus $I^{-1} \subseteq \frac{1}{d}D \subseteq \frac{1}{s}I^{-1}$. Therefore $sI^{-1} \subseteq \frac{s}{d}D \subseteq I^{-1}$. But $I^{-1} = \frac{1}{a}D \cap \frac{1}{b}D = \frac{1}{ab}(aD \cap bD)$. So $s(aD \cap bD) \subseteq \frac{sab}{d}D \subseteq aD \cap bD$, and hence $aD \cap bD$ is S-principal.
(3) $\Longrightarrow$ (4) Let $a,\ b \in D^*$, and let $I = aD + bD$. By hypothesis $aD \cap bD$ is S-principal, so there exist an $s \in S$ and $d \in D^*$ such that $s(aD \cap bD) \subseteq dD \subseteq aD \cap bD$. Since $I^{-1} = \frac{1}{a}D \cap \frac{1}{b}D = \frac{1}{ab}(aD \cap bD)$, then $sI^{-1} \subseteq \frac{d}{ab}D \subseteq I^{-1}$. This implies that $sI \subseteq \frac{sab}{d}D \subseteq I_v$. Hence I is S-v-principal.
(2) $\Longrightarrow$ (3) Obvious. Conversely, let $a_1, ..., a_n \in D^*$. We show that $I = a_1D \cap \cdots \cap a_nD$ is *S-principal*. By hypothesis there exist an $s_1 \in S$ and an $\alpha_1 \in D$ such that $s_1(a_1D \cap a_2D) \subseteq \alpha_1D \subseteq a_1D \cap a_2D$. Then $s_1I \subseteq (s_1(a_1D \cap a_2D)) \cap a_3D \cap \cdots \cap a_nD \subseteq \alpha_1D \cap a_3D \cap \cdots \cap a_nD \subseteq I$. By induction, there exist an $s_2 \in S$ and an $\alpha_2 \in D$ such that $s_2(\alpha_1D \cap a_3D \cap \cdots \cap a_nD) \subseteq \alpha_2D \subseteq \alpha_1D \cap a_3D \cap \cdots \cap a_nD$. Let $t = s_1s_2 \in S$. Then $tI \subseteq \alpha_2D \subseteq I$, and hence I is S-principal.
(3) $\Longleftrightarrow$ (5) It is sufficient to remark that for each $a,\ b \in D$, $aD \cap bD = (aD : bD)(bD)$.

Corollary 2.1. *For an integral domain D, the following statements are equivalent.*

1. *D is a GCD domain.*
2. *Any finite intersection of nonzero principal ideals of D is principal.*
3. *For $a,\ b \in D^*$, $(aD + bD)_v$ is principal.*
4. *For $a,\ b \in D^*$, $aD : bD$ is principal.*

Theorem 2.2. *Let D be an integral domain and S a multiplicative subset of D. If D is an S-GCD domain, then D_S is a GCD domain.*

Proof: Let I and J be nonzero principal ideals of D_S, say $I = aD_S$ and $J = bD_S$ for $a, b \in D^*$. Then $I \cap J = (aD_S) \cap (bD_S) = (aD \cap bD)_S$ is principal since by Theorem 2.1, $aD \cap bD$ is *S-principal*.

Our next result gives, with an additional condition, a necessary and sufficient condition for an integral domain D to be an S-GCD domain.

Theorem 2.3. *Let D be an integral domain and S a multiplicative subset of D such that for each $d \in D^*$, there is an $s \in S$ such that $s(dD_S \cap D) \subseteq dD$. Then the following assertions are equivalent.*

1. *D is an S-GCD domain.*
2. *D_S is a GCD domain.*

Proof: (1) $\Longrightarrow$ (2) This always holds by Theorem 2.2. (2) $\Longrightarrow$ (1) Let aD and bD be nonzero principal ideals of D. Since $(aD \cap bD)_S = aD_S \cap bD_S$ is a principal ideal of D_S, then there exists a $d \in aD \cap bD$ such that $(aD \cap bD)_S = dD_S$. Thus $(aD \cap bD)_S \cap D = (dD_S) \cap D$. But by assumption $s(dD_S \cap D) \subseteq dD$ for some $s \in S$. Then $s(aD \cap bD) \subseteq s(dD_S \cap D) \subseteq dD \subseteq aD \cap bD$.

Corollary 2.2. *Let D be an integral domain and S a multiplicative set of D. Suppose that each t-ideal of D has v-finite type. Then if D_S is a GCD domain, D is an S-GCD domain.*

Proof: For $d \in D^*$, $dD_S \cap D$ is a *t-ideal*. Let $dD_S \cap D = (a_1, ..., a_n)_v$. Now for each $1 \leq i \leq n$, $a_i \in dD_S \cap D$, so there exists an $s_i \in S$ such that $s_i a_i \in dD$. Put $s = s_1 \cdots s_n$. Then $s(a_1, ..., a_n) \subseteq dD$ and hence $s(dD_S \cap D) = s(a_1, ..., a_n)_v \subseteq dD$.

We next give an example of a domain D and a multiplicative set S generated by a principal prime such that D_S is a GCD domain but D is not an S-GCD domain.

Example 2.2. Let $R = \mathbb{Z}_{(p)} + Y\mathbb{Q}[[Y]]$ where $\mathbb{Z}$ is the ring of integers, p a prime number, $\mathbb{Q}$ the field of rational numbers, and Y an indeterminate over $\mathbb{Q}$. It is easy to see that R is a discrete rank two valuation domain. Let $S = \{p^n \mid 0 \leq n \in \mathbb{Z}\}$ and note that S is a multiplicative set of R such that $R_S = \mathbb{Q}[[Y]]$. Now let $D = R + XR_S[X] = \mathbb{Z}_{(p)} + Y\mathbb{Q}[[Y]] + X\mathbb{Q}[[Y]][X]$. As $D_S = R_S[X] = \mathbb{Q}[[Y]][X]$, a polynomial ring over a valuation domain, we conclude that D_S is a GCD domain. Now consider the ideal $(Y) \cap (X)$ in D. Now X and Y are non-units, $X \nmid Y$ and $Y \nmid X$ and every power of p divides both X and Y. So if s is a power of p, then $\frac{XY}{s} \in (Y) \cap (X)$, as $\frac{XY}{s} \in (X)$ because $\frac{XY}{s} = X\frac{Y}{s}$ and $\frac{XY}{s} \in (Y)$ because $\frac{XY}{s} = \frac{X}{s}Y$. Now let $a \in (Y) \cap (X)$. Then $a = Yf = Xg$ where $f, g \in D$. Taking f as a function of X over R_S, we note $f = Xh(X)$ where $h(X) \in R_S[X]$. So $a = YXh(X)$. For some $s \in S$, $sh(X) \in D$, so $a = (YX/s)k(X)$ where $k(X) = sh(X) \in D$. As for any $t \in S$, $a/t = (YX/st)k(X)$ we conclude that for any $t \in S$ and any $a \in (Y) \cap (X)$ we have $a/t \in (Y) \cap (X)$ so $p((X) \cap (Y)) = (X) \cap (Y)$. Now if D were an S-GCD domain, then $(Y) \cap (X)$ would be S-principal, that is, for some $a \in (Y) \cap (X)$ and $s \in S$ we would have $s((Y) \cap (X)) \subseteq aD \subseteq (Y) \cap (X)$. But then $(Y) \cap (X) \subseteq (a/s)D$. Note that we have assumed that a belongs to $(Y) \cap (X)$ and shown that if $a \in (Y) \cap (X)$, then $a/s \in (Y) \cap (X)$. This implies that $(Y) \cap (X) = (a/s)D$. So $p(a/s)D = (a/s)D$ and hence $pD = D$, a contradiction.

Recall from [4] that a saturated multiplicatively closed subset S of an integral domain D is said to be a *splitting set* if for each $d \in D^*$ we can write $d = sa$ for some $s \in S$ and $a \in D$ with $s'D \cap aD = s'aD$ for all $s' \in S$.

Corollary 2.3. *Let D be an integral domain and S a splitting set in D. Then D is an S-GCD domain if and only if D_S is a GCD domain.*

Proof: Since S is a splitting set in D, then by [4, Theorem 2.2], there exists a multiplicatively closed subset T of D such that for each $d \in D^*$ we can write $d = st$ for some $s \in S$ and $t \in T$ with $dD_S \cap D = tD$. Let $d \in D^*$. Then $d = st$ for some $s \in S$ and $t \in T$ with $dD_S \cap D = tD$. So $s(dD_S \cap D) = stD = dD$. Hence by Theorem 2.3, D is an S-GCD domain if and only if D_S is a GCD domain.

Recall from [20] that a domain D is said to be a *Mori domain* if it satisfies the ascending chain condition on integral divisorial ideals. Note that if D is a Mori domain and S a multiplicative set of D, then D_S is a Mori domain [20].

Proposition 2.1. *Let D be a Mori domain and let S be a multiplicative set. Then D is an S-GCD domain if and only if D_S is a UFD.*

Proof: If D is S-GCD, then D_S is a GCD Mori domain and so a UFD. Conversely, suppose D_S is a UFD. By [20, Théorème 1] every t-ideal of D is a finite type v-ideal. Hence by Theorem 2.3, D is an S-GCD domain.

The following result is an immediate consequence of the previous proposition. Note that in [21], the author gave an example of Mori domain D such that $D[X]$ is not a Mori domain.

Corollary 2.4. *Let D be a Mori domain such that $D[X]$ is Mori and let S be a multiplicative set of D. Then D is an S-GCD domain if and only if $D[X]$ is an S-GCD domain.*

Remark 2.2. Let D be an integral domain and S a multiplicative subset of D. If $D_S = qf(D)$, then D is an S-PID (S-principal ideal domain). In particular, D is an S-GCD domain.

Our next result gives an S-version of a well known result about GCD domains where S is generated by prime elements of D, that is, an integral domain D is UFD if and only if D is a GCD domain satisfying ACCP.

Proposition 2.2. *Suppose that D satisfies ACCP. If S is a set generated by primes, then D is a UFD if and only if D_S is a UFD.*

Proof: Follows from Corollary 1.7 and Proposition 3.2 of [5].

An integral domain D is said to be a *weak finite conductor domain* if for each pair $a, b \in D^*$, $aD \cap bD$ is a v-ideal of finite type. Also recall that an element $x \in D^*$ is called *primal* if for $y, z \in D^*$, $x|yz$ implies that $x = rs$ where $r|y$ and $s|z$. An integrally closed integral domain D with all nonzero elements primal was called a *Schreier domain* by Cohn [11]. A domain D whose nonzero elements are all primal is called *pre-Schreier.* A primal element r is called *completely primal* if every factor of r is primal. A prime element is completely primal. We shall also have occasion to use: If S is a multiplicative set of D generated by completely primal elements of D and if D_S is pre-Schreier, then so is D [8, Theorem 4.2]. This theorem was originally proved by Cohn [11, Theorem 2.6] for Schreier domains.

Theorem 2.4. *Let D be a weak finite conductor domain and let S be generated by a set of prime elements of D. Then D is a GCD domain if and only if D is an S-GCD domain.*

Proof: Since prime elements are completely primal, S is generated by completely primal elements. Now if D is an S-GCD domain, then D_S is a GCD domain. This implies that D_S is a pre-Schreier domain. So by [8, Theorem 4.4], D is pre-Schreier. Hence by [22, Theorem 3.6], D is a GCD domain. The converse is obvious.

Note that if D is an S-GCD domain, then D is not necessary a PVMD. Indeed, Let D be an integral domain which is not a PVMD (e.g., any non-integrally closed domain) and let $S = D^*$ a multiplicative subset of D. It is easy to show that D is an S-principal ideal domain. So D is an S-GCD domain which is not a PVMD. The next proposition links the S-GCD property with PVMDs. First, let us recall that a splitting set S of D is said to be an *lcm splitting* set if for each $s \in S$ and $d \in D$, $sD \cap dD$ is principal [4]. We start with a special case of PVMDs.

Proposition 2.3. *Let D be an integral domain, S a splitting set of D, and T the m-complement of S. If D is an S-GCD domain as well as a T-GCD domain, then D is a GCD domain.*

Proof: Since D is a T-GCD domain, then by Theorem 2.2, D_T is a GCD domain. So by [4, Proposition 2.4], S is an lcm splitting set. On the other hand, as D is an S-GCD domain, D_S is a GCD domain. So by [4, Theorem 4.3], D is a GCD domain.

Remark 2.3. Note that if S is an lcm splitting set of an integral domain D, then D is an S-GCD domain if and only if D is a GCD domain. Indeed, if D is an S-GCD domain, then D_S is a GCD domain. So by [4, Theorem 4.3], D is a GCD domain. The other implication is obvious.

Corollary 2.5. *Let D be an integral domain and S be an lcm splitting set of D. Then D is an S-GCD domain if and only if $D[X]$ is an S-GCD domain.*

Proof: By [7, Theorem 2.2], S is an lcm splitting set in $D[X]$. So by the previous remark, D is an S-GCD domain if and only if D is a GCD domain which is equivalent to $D[X]$ is a GCD domain if and only if $D[X]$ is an S-GCD domain.

Let D be an integral domain. A splitting set S of D is called a *t-lcm splitting* set if $sD \cap dD$ is t-invertible for all $s \in S$ and $0 \neq d \in D$. This concept was mentioned by Chang, Dumitrescu and Zafrullah in [10], where it was also mentioned that if S is a t-lcm splitting set, then D is a PVMD if and only if D_S is a PVMD.

Proposition 2.4. *Let D be an integral domain and S be a t-lcm splitting set of D. If D is an S-GCD domain, then D is a PVMD.*

Proof: If D is S-GCD, then D_S is a GCD domain. So D_S is a PVMD and hence D is a PVMD.

Proposition 2.5. *Let D be an integral domain and S a multiplicative subset of D. If D is an S-GCD domain, then for each $x \in \overline{D}$ (the integral closure of D), there exists an $s \in S$ such that $sx \in D$.*

Proof: Since D is an S-GCD domain, then by Theorem 2.3, D_S is a GCD domain. Thus D_S is integrally closed. This implies that $D_S = \overline{D_S} = \overline{D}_S$, and hence for each $x \in \overline{D}$, there exists an $s \in S$ such that $sx \in D$.

Theorem 2.5. *Let D be an integral domain and S a multiplicative subset of D such that D is an S-GCD S-Noetherian domain. Then the following hold.*

(1) *D_S is a factorial domain.*
(2) *If every prime w-ideal of D disjoint from S is a t-ideal, then D is an S-factorial domain.*

Proof: (1) It follows from the fact D_S is GCD and Noetherian.
(2) We show that every prime w-ideal of D disjoint from S is S-principal and this will prove the result via [15, Theorem 3.2]. Let P be a prime w-ideal of D with $P \cap S = \emptyset$. Then P is a t-ideal by assumption. Since D is S-Noetherian, P is S-finite. So there is a finitely generated subideal $J \subseteq P$ and an $s \in S$ such that $sP \subseteq J \subseteq P$. Because D is an S-GCD domain, there is $t \in S$, $d \in J_v$ ($\subseteq P$) such that $tJ_v \subseteq (d)$. Now as already $sP \subseteq J \subseteq P$ we get $stP \subseteq tJ \subseteq P$. Applying the t-operation we get $(stP)_t \subseteq (tJ)_t \subseteq P_t$. Since J is finitely generated $(tJ)_t = (tJ)_v \subseteq (d)$ and hence $stP \subseteq (tJ)_v \subseteq (d) \subseteq P$. As $stP \subseteq (d) \subseteq P$, P is actually S-principal. As P is a w-ideal that is also a t-ideal, the expression $stP \subseteq (d) \subseteq P$ stays the same whether we apply the t-operation or the w-operation. Whence P is S-w-principal, the requirement of [15, Theorem 3.2] is met and D is an S-factorial domain.

Corollary 2.6. *D is S-factorial in each of the following cases.*

(1) *D is a GCD domain that is S-Noetherian.*
(2) *D is an S-GCD S-Noetherian domain and w-dim(D)$= 1$.*

Proof:

(1) Since D is a GCD domain, D is an S-GCD domain. Also every prime w-ideal is a t-ideal, because a GCD domain is a PVMD and in a PVMD, $w = t$. So all the requirements of the previous theorem are met.
(2) If w-$dim(D) = 1$, then every prime w-ideal is a t-ideal and so the conditions of the previous theorem are met.

Proposition 2.6. *Let D be S-factorial. Then every prime w-ideal P that is disjoint from S is a t-ideal.*

Proof: Let P be a prime w-ideal of D disjoint from S. Since D is S-factorial, P must be S-principal. That is, for some $s \in S$ and $d \in P$ we should have $sP \subseteq (d)$. But then PD_S is a principal prime in the UFD D_S ([15]) and hence of height one. This makes $P = PD_S \cap D$ of height one, and hence a t-ideal.

We next give an example of an S-GCD S-Noetherian domain which is not S-factorial.

Example 2.3. Let $\mathbb{Q}$ denote the field of rational numbers and let X, Y and Z be indeterminates over $\mathbb{Q}$. Set $R = \mathbb{Q}(\sqrt{2})[[X, Y, Z]] = \mathbb{Q}(\sqrt{2}) + M$ and $D = \mathbb{Q} + M$. It is easy to see that $D = \mathbb{Q} + M$ is a local Noetherian domain with integral closure R. Since the maximal ideal M is common to both D and R, $M = MR$ and so are the following prime ideals contained in M. $P_1 = XR$, $P_2 = (X, Y)R$. We have $P_1 \subsetneq P_2 \subsetneq M$. We claim that P_2 is not a *t-ideal* of D while M is a *t-ideal* of D. This follows from the following observations. Since $ht_R(P_2) = 2$ we have $R = R : P_2 = P_2 : P_2 = D : P_2$. Similarly $R = R : M = M : M = D : M$. Now as, with respect to D, $M^{-1} \supsetneq D$ we must have $M_v \subsetneq D$. But since $D = \mathbb{Q} + M$ is local $M_v = M$. Next as, with respect to D, $P_2^{-1} = M^{-1,}$ we have $(P_2)_v = M_v = M$. But as $P_2 \subsetneq M$ we conclude that P_2 is not a t-ideal. Now let $S = \{Z^n \mid n \geq 0\}$. Then S is a multiplicative set in D and $D_S = (\mathbb{Q} + M)_S = \mathbb{Q}(\sqrt{2})[[X, Y, Z]]_S$ a quotient ring of a UFD and so is a UFD. Since D is Noetherian (and hence Mori) and D_S a UFD, by Proposition 2.1 D is an S-GCD domain. But D is not S-factorial by Proposition 2.6, because P_2 is a w-prime ideal of D that is disjoint from S but not a t-ideal.

Remark 2.4. The S-GCD property does not carry over to the power series ring. In fact, there is an example of a GCD domain D such that $D[[X]]$ is not a GCD domain [19, Theorem 8]. (This is the case when S consists of units of D).

Let D be an integral domain with quotient field K. Recall from [14] that the power series ring $D[[X]]$ is said to satisfy *property* $(*)$ if for all integral v-invertible v-ideals I and J of $D[[X]]$ such that $(IJ)_0 \neq (0)$, we have $((IJ)_0)_v = ((IJ)_v)_0$ where $I_0 = \{f(0) \mid f \in I\}$. For example, $\mathbb{Z}[i\sqrt{5}][[X]]$ satisfies property $(*)$ [14, Example 3.1]. We close this section with the following two results.

Theorem 2.6. *Let D be a Krull domain such that $D[[X]]$ satisfies $(*)$ and S a multiplicative subset of D. Then D is an S-GCD domain if and only if $D[[X]]$ is an S-GCD domain.*

Proof: By [14, Theorem 4.4], $S\text{-}Cl_t(D) \simeq S\text{-}Cl_t(D[[X]])$. So by [14, Theorem 4.2] D is an S-GCD domain if and only if $S\text{-}Cl_t(D) = 0$ if and only if $S\text{-}Cl_t(D[[X]]) = 0$ which is equivalent to $D[[X]]$ being an S-GCD domain.

Corollary 2.7. *Let D be a Krull domain such that $D[[X]]$ satisfies $(*)$. Then D is a UFD if and only if $D[[X]]$ is a UFD.*

3. Sublocally s-GCD domains

Let D be an integral domain and s a nonzero element of D. We say that D is an *s-GCD domain* if for each pair $a, b \in D^*$, there is a positive integer n and an element $c \in aD \cap bD$ such that $s^n(aD \cap bD) \subseteq cD$. So if $\mathcal{S} = \{s^n \mid n \in \mathbb{N}\}$, then D is an s-GCD domain if and only if D is an $\mathcal{S}$-GCD domain. If s is a unit of D, then D is an s-GCD domain if and only if D is a GCD domain. If D is an s-GCD domain, then D_s is a GCD domain.

Definition 3.1. An integral domain D is a *sublocally s-GCD domain* if for every nonzero non-unit $s \in D$, D is an s-GCD domain.

A multiplicative subset S of D is called *nontrivial* if S contains at least one non-unit.

Proposition 3.1. *An integral domain D is sublocally s-GCD if and only if D is an S-GCD domain for every nontrivial multiplicative set S of D.*

Proof: If D is s-GCD for every nonzero non-unit s, then D is S-GCD for every nontrivial multiplicative set containing s. Thus a sublocally s-GCD domain is an S-GCD domain for each multiplicative set S. The converse is clear.

Recall from [1] that an integral domain D is said to be a *generalized GCD domain (G-GCD domain)* if for each pair $a, b \in D^*$ we have $aD \cap bD$ invertible, or equivalently, every finite intersection of (integral) invertible ideals of D is invertible [1].

Theorem 3.1. *A sublocally s-GCD domain that is not quasi-local is a G-GCD domain.*

To prove this we need the following lemmas.

Lemma 3.1. *A sublocally s-GCD domain that is not quasi-local is locally GCD*

Proof: Let M be a maximal ideal of D. Since D is not quasi-local there is a non-unit $s \in D\backslash M$. Then D is an s-GCD domain and hence D_s a GCD domain. Hence $D_M = (D_s)_{M_s}$ is a GCD domain.

For the next lemma we need to collect some necessary notions. A domain is called *t-local* if it maximal ideal is a t-ideal. Clearly a t-local domain is quasi-local and its maximal ideal is a t-ideal. It is well known that for a set $\{x_1, \ldots, x_n\} \subseteq D^*$, $(x_1, ..., x_n)_v = D$ if and only if $D = \cap_{i=1}^n D_{x_i}$, see [3, Lemma 2.1].

Lemma 3.2. *A sublocally s-GCD domain that is not t-local is a PVMD.*

Proof: If D is not t-local, then for each maximal t-ideal m of D there is a non-unit $x \in D \backslash m$. But then $(x, m)_t = D$. So there are nonzero $x_1, x_2, ..., x_n \in m$ such that $(x, x_1, x_2, ..., x_n)_v = D$. But then $D = D_x \cap (\cap_{i=1}^n D_{x_i})$. As D is a sublocally s-GCD domain, D_x and the D_{x_i} are all GCD domains and hence PVMDs. So by [12, Theorem 4.1(2)] their intersection D is a PVMD.

Proof of Theorem 3.1: By Lemma 3.1, D is locally GCD domain. Also by Lemma 3.2, D is a PVMD. But by [22, Corollary 3.4], a locally GCD domain that is a PVMD is a G-GCD domain.

Corollary 3.1. *A semi-quasi-local sublocally s-GCD domain that is not quasi-local is a GCD domain.*

Proof: By Theorem 3.1, D is a G-GCD domain. So for each pair $a, b \in D^*$, $aD \cap bD$ is invertible and invertible ideals are principal in a semi-quasi-local domain. Hence D is a GCD domain.

Corollary 3.2. *Let D be a sublocally s-GCD domain that is not t-local and contains a non-unit completely primal element. Then D is a GCD domain.*

Proof: Let x be a completely primal non-unit in D. As D is sublocally s-GCD domain, D_x is a GCD domain and hence by [11, Theorem 2.6] D is Schreier. But a Schreier PVMD is a GCD domain [22].

Note that a one-dimensional quasi-local domain is a sublocally s-GCD domain, but such a domain need not be a G-GCD domain, or equivalently, a GCD domain. Recall from [3] that a domain D is said to be *locally factorial* if for each non-unit $x \in D^*$ we have D_x is a UFD. Also by Proposition 2.1, if D is a Mori domain and S a multiplicative set in D, then D is an S-GCD domain if and only if D_S is a UFD. Using this result we can prove the following proposition.

Proposition 3.2. *The following assertions are equivalent for a Mori domain that is not quasi-local.*

(1) *D is a sublocally s-GCD domain.*
(2) *D is a locally factorial domain.*

Corollary 3.3. *A locally factorial domain that is not quasi-local is a sublocally s-GCD domain.*

Proof: Since D is not quasi local, there are nonzero non units $x_1, x_2, ..., x_n$ such that $(x_1, ..., x_n)$ $= D$. By [3, Lemma 2.1] $D = \cap D_{x_i}$. Because, D is locally factorial each of D_{x_i} is a UFD and so Krull. But then being a finite intersection of Krull domains, D is Krull and hence Mori. Now apply Proposition 3.2.

Definition 3.2. Let D be an integral domain and s a nonzero element of D. We say that D is an *s-factorial domain* if D is an $\{s^n \mid n \in \mathbb{N}\}$-factorial domain. We also define D to be a *sublocally s-factorial domain* if D is s-factorial for each nonzero non-unit $s \in D$.

Remark 3.1. Note that every sublocally s-factorial domain is a locally factorial domain. If D is not a quasi-local domain, then the sublocally s-factorial and locally factorial notions coincide. For in both cases we end up with a Krull domain. Since the two notions agree for a one-dimensional quasi-local domain, they agree for any one-dimensional domain. But we need to determine if they match up in the non-t-local case. Of course, as in the proof of Corollary 3.3, the non-t-local locally factorial domains are Krull domains, being intersections of finitely many Krull domains. Now being locally factorial and Krull they are sublocally s-factorial, being sublocally s-GCD (by Corollary 3.3) and Krull ([15, Corollary 3.5]). That leaves the case of t-local domains of dimension greater than one. For that we have the following example.

Example 3.1. Let $R = \mathbb{Z}_{(p)} + Y\mathbb{Q}[[Y]]$ and $S = \{p^n \mid 0 \leq n \in \mathbb{Z}\}$ be as in Example 2.2. Let $D = R + XR_S[[X]]$. Then $D_S = R_S + XR_S[[X]] = R_S[[X]] = \mathbb{Q}[[Y]][[X]] = \mathbb{Q}[[X,Y]]$ a UFD. Also because S is generated by a prime and D_S is a UFD, D must be a Schreier domain. That D is not a GCD domain follows from the fact that $XD \cap YD$ is not principal. First note that $XD \cap YD \subseteq (XD \cap YD)_S \cap D = XYD_S \cap D = \{fXY/p^n | \ f \in D, n \in \mathbb{N}\} \subseteq XD \cap YD$. If $h \in XD \cap YD$, then $h/p \in XD \cap YD$. So if $p(XD \cap YD) = XD \cap YD = hD$, then $phD = hD$ and hence $pD = D$, a contradiction. In fact for the same reason we cannot find $h \in XD \cap YD$ such that for some $n \in \mathbb{N}$, $p^n(XD \cap YD) \subseteq hD$. Now $D_S = D_p = \mathbb{Q}[[X,Y]]$ is factorial. Moreover for any nonzero non-unit f other than a power of p, f is divisible by X, Y or both in $\mathbb{Q}[[X,Y]]$, so D_f is a quotient ring of $\mathbb{Q}[[X,Y]]$ and hence factorial. Thus the ring D is locally factorial. As we have seen above, D is not an S-GCD domain for $S = \{p^n | \ 0 \leq n \in \mathbb{Z}\}$, so D is not a sublocally s-GCD domain and hence not sublocally s-factorial.

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References

[1] D. D. Anderson, π-domains, divisorial ideals and overrings, Glasgow Math. J., **19** (1978), 199-203.
[2] D. D. Anderson and D. F. Anderson, Generalized GCD-domains, Comment. Math. Univ. St. Pauli **28** (1979), 215-221.
[3] D. D. Anderson and D. F. Anderson, Locally factorial integral domains, J. Algebra, **90** (1984), 265-283.
[4] D. D. Anderson, D. F. Anderson and M. Zafrullah, Splitting the t-class group, J. Pure Appl. Algebra **74** (1991), 17-37.
[5] D. D. Anderson, D. F. Anderson and M. Zafrullah, Factorization in integral domains II, J. Algebra **152** (1992), 78-92.
[6] D. D. Anderson and T. Dumitrescu, S-Noetherian rings, Comm. Algebra **30** (2002), 4407-4416.
[7] D. D. Anderson and M. Zafrullah, Splitting sets in integral domains, Proc. Amer. Math. Soc. **129** (2001), 2209-2217.
[8] D. D. Anderson and M. Zafrullah, The Schreier property and Gauss' lemma, Bollettino U. M. I. **10**-B (2007), 43-62.

[9] V. Barucci, L. Izelgue and S. Kabbaj, Some factorization properties of $A+XB[X]$ domains, Lecture Notes in Pure and Applied Mathematics vol. **185**, Marcel Dekker, New York (1997), pp. 69-78.
[10] G. W. Chang, T. Dumitrescu and M. Zafrullah, Splitting sets in integral domains, J. Pure Appl. Algebra **187** (2004), 71-86.
[11] P. M. Cohn, Bezout rings and their subrings, Proc. Cambridge Phil. Soc. **64** (1968), 251-264.
[12] S. El Baghdadi, M. Fontana and M. Zafrullah, Intersections of quotient rings and Prüfer v-multiplication domains, J. Algebra Appl. **15**, 1650149 (2016) [18 pages].
[13] R. Gilmer, *Multiplicative Ideal Theory*, Maecel Dekker, New York, (1972).
[14] A. Hamed and S. Hizem, On the class group and S-class group of formal power series rings, J. Pure Appl. Algebra **221** (2017), 2869-2879.
[15] H. Kim, M. O. Kim and J. O. Lim, On S-strong Mori Domains, J. Algebra **416** (2014), 314-332.
[16] S. Malik, J. Mott and M. Zafrullah, On t-invertibility, Comm. Algebra **16** (1988) 149-170.
[17] J. Mott and M. Zafrullah, On Prüfer v-multiplication domains, Manuscripta Math. **35** (1981), 1-26.
[18] M. Nagata, Some types of simple ring extensions, Houston J. Math. **1** (1975), 131-136.
[19] M. H. Park, D. D. Anderson and B. G. Kang, GCD-domains and power series rings, Comm. Algebra **30** (2002), 5955-5960.
[20] J. Querré, Sur une propriété des anneaux de Krull, Bull. Sci. Math. **2** (1971), 341-354.
[21] M. Roitman, On Mori domains and commutative rings with CC$^{\perp}$ II, J. Pure Appl. Algebra **61** (1989), 53-77.
[22] M. Zafrullah, On a property of pre-Schreier domains, Comm. Algebra **15** (1987), 1895-1920.

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