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ON $*$-HOMOGENEOUS IDEALS AND $*$-POTENT DOMAINS

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To Dan Anderson, who is a fine Mathemacian and often a good coauthor

Abstract. Let $*$ be a star operation of finite character. Call a $*$-ideal I of finite type a $*$-homogeneous ideal if I is contained in a unique maximal $*$-ideal $M = M(I)$. A maximal $*$-ideal that contains a $*$-homogeneous ideal is called potent and the same name bears a domain all of whose maximal $*$-ideals are potent. One among the various aims of this article is to indicate what makes a $*$-ideal of finite type a $*$-homogeneous ideal, where and how we can find one, what they can do and how this notion came to be. We also prove some results of current interest in ring theory using some ideas from this author's joint work in [37] on partially ordered monoids. We characterize when a commutative Riesz monoid generates a Riesz group

1. Introduction

Let $*$ be a finite character star operation defined on an integral domain D throughout. (A working introduction to the star operations, and the reason for using them, will follow.) Call a nonzero $*$-ideal of finite type a $*$-homogeneous ideal, if I is contained in a unique maximal $*$-ideal. According to proposition 1 of [11], associated with each $*$-homogeneous ideal I is a unique $*$-maximal ideal $M(I) = \{x \in D|\ (x, I)^* \neq D\}$. The notion of a $*$-homogeneous ideal has figured prominently in describing unique factorization of ideals and elements in [11] and it seems important to indicate some other properties and uses of this notion and notions related to it. Call a $*$-maximal ideal M $*$-potent if M contains a $*$-homogeneous ideal and call a domain D $*$-potent if each of the $*$-maximal ideals of D is $*$-potent. The aim of this article is to study some properties of $*$-homogeneous ideals and of $*$-potent domains. We show for instance that while in a $*$-potent domain every proper $*$-ideal of finite type is contained in a $*$-homogeneous ideal, the converse may not be true. We shall also indicate how these concepts can be put to use. Before we elaborate on that, it seems pertinent to give an idea of our main tool, the star operations. Indeed, the rest of what we plan to prove will be included in the plan of the paper after the introduction to star operations.

1.1. **Introduction to star operations.** Let D be an integral domain with quotient field K , throughout. Let $F(D)$ be the set of nonzero fractional ideals of D, and let $f(D) = \{A \in F(D)|A \text{ is finitely generated}\}$. A star operation $*$ on D is a closure operation on $F(D)$ that satisfies $D^* = D$ and $(xA)^* = xA^*$ for $A \in F(D)$

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Thanks for Author One.

and $x \in K = K\backslash\{0\}$. With $*$ we can associate a new star-operation $*_s$ given by $A \mapsto A^{*_s} = \cup\{B^*|B \subseteq A, B \in f(D)\}$ for each $A \in F(D)$. We say that $*$ has finite character if $* = *_s$. Three important star-operations are the d-operation $A \mapsto A_d = A$, the v-operation $A \mapsto A_v = (A^{-1})^{-1} = \cap\{Dx|Dx \supseteq A, x \in K\}$ where $A^{-1} = \{x \in K : xA \subseteq D\}$ and the t-operation $t = v_s$. Here d and t have finite character. A fractional ideal A is a $*$-ideal if $A = A^*$and a $*$-ideal A is of finite type if $A = B^*$ for some $B \in f(D)$. If $*$ has finite character and A^* is of finite type, then $A^* = B^*$ for some $B \in f(D)$, $B \subseteq A$. A fractional ideal $A \in F(D)$ is $*$-invertible if there exists a $B \in F(D)$ with $(AB)^* = D$; in this case we can take $B = A^{-1}$. For any $*$-invertible $A \in F(D), A^* = A_v$. If $*$ has finite character and A is $*$-invertible, then A^* is a finite type $*$-ideal and $A^* = A_t$. Given two fractional ideals $A, B \in F(D)$, $(AB)^*$ denotes their $*$-product. Note that $(AB)^* = (A^*B)^* = (A^*B^*)^*$. Given two star operations $*_1$ and $*_2$ on D, we write $*_1 \leq *_2$ if $A^{*_1} \subseteq A^{*_2}$ for all $A \in F(D)$. So $*_1 \leq *_2 \Leftrightarrow (A^{*_1})^{*_2} = A^{*_2} \Leftrightarrow (A^{*_2})^{*_1} = A^{*_2}$ for all $A \in F(D)$.

Indeed for any finite character star-operation $*$ on D we have $d \leq * \leq t$. For a quick introduction to star-operations, the reader is referred to [27, Sections 32, 34] or [46], for a quick review. For a more detailed treatment see Jaffard [34]. A keenly interested reader may also look up [31]. These days star operations are being used to define analogues of various concepts. The trick is to take a concept, e.g., a PID and look for what the concept would be if we require that for every nonzero ideal I, I^* is principal and voila! You have several concepts parallel to that of a PID. Of these t-PID turns out to be a UFD. Similarly a v-PID is a completely integrally closed GCD domain of a certain kind. A t-Dedekind domain, on the other hand is a Krull domain and a v-Dedekind domain is a domain with the property that for each nonzero ideal A we have A_v invertible. So when we prove a result about a general star operation $*$ the result gets proved for all the different operations, d, t, v etc. Apart from the above, any terminology that is not mentioned above will be introduced at the point of entry of the concept.

Suppose that $*$ is a finite character star-operation on D. Then a proper $*$-ideal is contained in a maximal $*$-ideal and a maximal $*$-ideal is prime. We denote the set of maximal $*$-ideals of D by $*$-$Max(D)$. We have $D = \cap D_P$ where P ranges over $*$-$Max(D)$.From this point on we shall use $*$ to denote a finite type star operation. Call D of finite $*$-character if for each nonzero non unit x of D, x belongs to at most a finite number of maximal $*$-ideals. Apart from the introduction there are three sections in this paper. In section 2 we talk about $*$-homogeneous ideals, and $*$-potent domains. We characterize $*$-potent domains in this section, show that if D is of finite $*$-character then D must be potent, examine an error in a paper of the author, [24], in characterizing domains of finite $*$-character and characterize domains of finite $*$-character and give a new proof. In section 3, we show how creating a suitable definition of a $*$-homogeneous ideal will create theory of unique factorization of ideals. Calling an element $r \in D$ $*$-f-rigid ($*$-factorial rigid) if rD is a $*$-homogeneous ideal such that every proper $*$-homogeneous ideal containing r is principal we call a $*$-potent maximal $*$-ideal M (resp., domain D) $*$-f-potent if M (resp., every maximal $*$-ideal of D) contains a $*$-f-rigid element and show that over a $*$-f-potent domain a primitive polynomial f is super primitive i.e. if A_f, the content of f, is such that the generators of f have no non unit common factor then

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$(A_f)_v = D$ and indicate how to construct atomless non-pre-Schreier domain. In this section we offer a seamless patch to remove an error in the proof of result in a paper by Kang [35] and show that D is a t-superpotent if and only if $D[X]_S$ is t-f-potent, where X is an indeterminate and $S = \{f \in [X] | (A_f)_v = D\}$. We also show, by way of constructing more examples, in this section that if L is an extension of K the quotient field of D and X an indeterminate over D then D t-f-potent if and only if $D + XL[X]$ is. Finally in section 4 we define a pre-Riesz monoid as a p.o. monoid M if for any $x_1, x_2, ..., x_n \in M\backslash\{0\}$ $glb(x_1, x_2, ..., x_n) = 0$ or there is $r \in M$ with $0 < r \leq x_1, x_2, ..., x_n$ and indicate that the monoid of $*$-ideals of finite type is a pre-Riesz monoid and, of course we indicate how to use this information.

2. $*$-potent domains and $*$-homogeneous ideals

Work on this paper started in earnest with the somewhat simple observation that if D is $*$-potent then every nonzero non unit $x \in D$ is contained in some $*$-homogeneous ideal. The proof goes as follows: Because x is a nonzero non unit, x must be contained in some maximal $*$-ideal M. Now as D is $*$-potent $M = M(I)$ for some $*$-homogeneous ideal I. Consider $J = (I, x)^*$ and note that $(I, x)^* \neq D$ because $x \in M$ and $(I, x)^*$ is contained in a unique maximal $*$-ideal and this makes J a $*$-homogeneous ideal.

This leads to the question: If D is a domain with a finite character star operation $*$ defined on it such that every nonzero non unit x of D is contained in some $*$-homogeneous ideal I of D, must D be $*$-potent?

This question came up in a different guise as: when is a certain type of domain $*_s$-potent for a general star operation $*$ in [42] and sort of settled in a tentative fashion in Proposition 5.12 of [42] saying, in the general terms being used here, that: Suppose that D is a domain with a finite character $*$-operation defined on it. Then D is $*$-potent provided (1) every nonzero non unit x of D is contained in some $*$-homogeneous ideal I of D and (2) for M, $M_\alpha \in *\text{-}max(D)$, $M \subseteq \cup M_\alpha$ implies $M = M_\alpha$ for some α.

The proof could be something like: By (1) for every nonzero non unit x there is a $*$-homogeneous ideal I_x containing x and so $x \in M(I_x)$. So $M \subseteq \cup M(I_x)$ and by (2) M must be equal to $M(I_x)$ for some x.

Thus we have the following statement.

Theorem 2.1. . *Let $*$ be a finite character star operation defined on D. Then D is $*$-potent if D satisfies the following: (1)every nonzero non unit x of D is contained in some $*$-homogeneous ideal I of D and (2) For M, $M_\alpha \in *\text{-}max(D)$, $M \subseteq \cup M_\alpha$ implies $M = M_\alpha$ for some α.*

Condition (2) in the statement of Theorem 2.1 has had to face a lot of doubt from me, in that, is it really necessary or perhaps can it be relaxed a little?

The following example shows that condition (2) or some form of it is here to stay.

It is well known that the ring $\mathcal{E}$ of entire functions is a Bezout domain [27, Exercise 18, p 147]. It is easy to check that a principal prime in a Bezout domain is maximal. Now we know that a zero of an entire function determines a principal prime in $\mathcal{E}$ and that the set of zeros of a nontrivial entire function is discrete, including multiplicities, the multiplicity of a zero of an entire function is a positive integer [29, Theorem 6]. Thus each nonzero non unit of $\mathcal{E}$ is expressible as a countable product of finite powers of distinct principal primes. For the identity star operation d, certainly defined on $\mathcal{E}$, only an ideal I generated by a power of a principal prime can be d-homogeneous. For if I is d-homogeneous, then $I = (x_1, ..., x_n)^d = x\mathcal{E}$ a principal ideal and hence a countable product of distinct primes. Now I cannot be in a unique non principal prime for then I would have to be a countably infinite product of principal primes and so in infinitely many principal prime ideals, which are maximal. So I can only belong to a unique principal prime and has to be a finite prime power. To see that $\mathcal{E}$ falls foul of Theorem 2.1, let's put $S = \{p|p$ a prime element in $\mathcal{E}\}$. Then for each non principal prime P of $\mathcal{E}$ we have $P \subseteq \cup_{p\in S} p\mathcal{E}$ because each element of P is divisible by some member(s) of S. (I have corresponded with Prof. Evan Houston about the above material and I gratefully acknowledge that.)

Once we know more about $*$-homogeneous ideals we would know that rings do not behave in the same manner as groups do. To get an idea of how groups behave and what is the connection the reader may look up [42]. Briefly, the notion of a $*$-homogeneous ideal arose from the notion of a basic element of a lattice ordered group G (defined as $b > 0$ in G such that $(0, b]$ is a chain). A basis of G, if it exists, is a maximal set of mutually disjoint strictly positive basic elements of G. According to [19] a l.o. group G has a basis if and only if every strictly positive element of G exceeds a basic element. So if we were to take D being potent as having a basis (every proper $*$-ideal of finite type being contained in a $*$-homogeneous ideal) then every proper $*$-ideal of finite type being contained in a $*$-homogeneous ideal does not imply that D is potent.

We next tackle the question of where $*$-homogeneous ideals can be found. Call D of finite $*$-character if every nonzero non unit of D is contained in at most a finite number of maximal $*$-ideals. Again, a domain of finite $*$-character could be a domain of finite character (every nonzero non unit belongs to at most a finite number of maximal ideals) such as an h-local domain or a semilocal domain or a PID or a domain of finite t-character such as a Krull domain.

Proposition 1. *A domain D of finite $*$-character is $*$-potent.*

Proof. Let M be a maximal $*$-ideal of D and let x be a nonzero element of M. If x belongs to no other maximal $*$-ideal then xD is $*$-homogeneous and M is potent. So let us assume that M, $M_1, M_2, ..., M_n$ is the set of all maximal $*$-ideals containing x. Now consider the ideal $A = (x, x_1, ..., x_n)$ where $x_i \in M\backslash M_i$ for $i = 1, ..., n$. Obviously $A \subseteq M$ but $A \nsubseteq M_i$ because of x_i. Note that A cannot be contained in any maximal $*$-ideal other than M, for if N were any maximal $*$-ideal containing A then N would belong to $\{M, M_1, M_2, ..., M_n\}$ because of x. And N cannot be any of the M_i. Thus A^* is a $*$-homogeneous ideal contained in M and M is potent. Since M was arbitrary we have the conclusion. □

The above proof is essentially taken from the proof for part (2) of Theorem 1.1 of [5].

Now how do we get a domain of finite $*$-character? The answer is somewhat longish and interesting. Bazzoni conjectured in [13] and [14] that a Prufer domain D is of finite character if every locally principal ideal of D is invertible. [30] were the first to verify the conjecture using partially ordered groups. Almost simultaneously [32] proved the conjecture for r-Prufer monoids, using Clifford semigroups of ideals and soon after I chimed in with a very short paper [47]. The ring-theoretic techniques used in this paper not only verified the Bazzoni conjecture but also helped prove Bazzoni-like statements for other, suitable, domains that were not necessarily PVMDs. (Recall that D is a PVMD if every t-ideal A of finite type of D is t-invertible i.e. $(AA^{-1})_t = D$.) In the course of verification of the conjecture I mentioned a result due to Griffin from [28] that says:

Theorem 2.2. *A PVMD D is of finite t-character if and only if each t-invertible t-ideal of D is contained in at most a finite number of mutually t-comaximal t-invertible t-ideals of D.*

As indicated in the introduction of [47] the set of t-invertible t-ideals of a PVMD is a lattice ordered group under t-multiplication and the order defined by reverse containment of the ideals involved and that the above result for PVMDs came from the use of Conrad's F-condition. Stated for lattice ordered groups Conrad's F-condition says: Every strictly positive element exceeds at most a finite number of mutually disjoint elements. This and Theorem 2.2, eventually led the authors of [24], to the following statement.

Theorem 2.3. *(cf. Theorem 1 of* [24]*) Let D be an integral domain, $*$ a finite character star operation on D and let Γ be a set of proper, nonzero, $*$-ideals of finite type of D such that every proper nonzero $*$-finite $*$-ideal of D is contained in some member of Γ . Let I be a nonzero finitely generated ideal of D with $I^* \neq D$. Then I is contained in an infinite number of maximal $*$-ideals if and only if there exists an infinite family of mutually $*$-comaximal ideals in Γ containing I.*

This theorem was a coup, it sort of catapulted the consideration of finiteness of character from Prufer-like domains to consideration of finiteness of $*$-character in general domains. But alas, there was an error in the proof. There was no reason for the error as I had used the technique, Conrad's F-condition, involved in the proof of Theorem 2.3 at other places such as [22], [41] and, later, [25] but there it was. I realized the error while working on a paper on p.o. groups, that I eventually published with Y.C. Yang as [42]. I wrote to my coauthor of [24], proposing a corrigendum. But for one reason or another the corrigendum never got off the ground. Fortunately Chang and Hamdi have recently published [16] including Theorem 1 of [24] as Lemma 2.3 with proof exactly the way I would have liked after the corrigendum was used.

Perhaps as a kind gesture those authors have not pointed out the error in the proof of [24, Theorem 1], but a careless use of Zorn's Lemma must be pointed out so that others do not fall in a similar pit. Now going over the whole thing anew might be painful, so I reproduce below the proposed brief corrigendum and point out any other s made that I could not see at that time.

> "There is some confusion in lines 8-15 of the proof of Theorem 1. In the following we offer a fix to clear the confusion and give a rationale for the fix.
>
> The fix: Read the proof from the sentence that starts from line 8 as follows: Let S be the family of sets of mutually $*$-comaximal homogeneous members of Γ containing I. Then S is non empty by ($\sharp\sharp$). Obviously S is partially ordered under inclusion. Let $A_{n_1} \subset A_{n_2} \subset ... \subset A_{n_r} \subset ...$ be an ascending chain of sets in S. Consider $T = \cup A_{n_r}$. We claim that the members of T are mutually $*$-comaximal. For take $x, y \in T$, then $x, y \in A_{n_i}$, for some

i, and hence are $*$-comaximal. Having established this we note that by ($\sharp$), T must be finite and hence must be equal to one of the A_{n_j}. Thus by Zorn's Lemma, S must have a maximal element $U = \{V_1, V_2, ..., V_n\}$. Disregard the next two sentences and read on from: Next let M_i be the maximal $*$-ideal....

Rationale for the Fix: Using sets of mutually $*$-comaximal elements would entail some unwanted maximal elements as the following example shows: Let $x = 2^2 5^2$ in Z the ring of integers. Then $\mathcal{S} = \{\{(2^2 5^2)\}, \{(25^2)\}, \{(2^2 5)\}\{(2^2)\}, \{(5^2)\}, \{(2^2), (5^2)\}, \{(2)\}, \{(5)\}, \{(2), (5^2)\}, \{(2^2), (5)\}, \{(2), (5)\}\}$. In this case, while $\mathcal{S}$ includes legitimate maximal elements: $\{(2^2), (5^2)\}$, $\{(2), (5^2)\}$, $\{(2^2), (5)\}, \{(2), (5)\}$ it also includes $\{(2^2 5^2)\}, \{(25^2)\}, \{(2^2 5)\}$ which fit the definition of maximal elements. The reason why the fix should work is that given any set $T = \{A_1, A_2, ..., A_m\}$ of mutually $*$-comaximal $*$-finite ideals, by ($\sharp\sharp$) there is a set of mutually $*$-comaximal homogeneous $*$-finite ideals $\{H_1, H_2, ..., H_n\}$ in Γ, where $n \geq m$ such that each H_j contains some A_i. Also as a homogeneous ideal cannot be contained in two disjoint ideals we do not face the above indicated problem and Zorn's Lemma gives the required maximal elements."

(To be sure that the above "proposal" was not created after seeing the Chang Hamdi paper check the image of the E-mail sent to Prof. Dumitrescu and a pdf version of the corrigendum here [48], at the end of that doument.)

The other error was essentially confusing the size of a set with the set, on my part. I must admit that my coauthor told me to say, after finding that there was at least one homogeneous ideal containing a given $*$-ideal A of finite type, that one can find a largest set of mutually $*$-comaximal homogeneous ideals containing A. But I just don't care about doing that unless the conclusion is very simple.

It's only fitting that I end this saga with a more satisfying statement and/or proof of [24, Theorem 1]. Lurking behind the façade of the set Γ and the other conditions were the following definitions and statements. Call a $*$-ideal I of finite type ($*$-) homogeneous, as we have already done, if I is contained in a unique maximal $*$-ideal $M = M(I)$.

Lemma 2.4. *A $*$-ideal I of finite type is $*$-homogeneous if and only if for each pair X, Y of proper $*$-ideals of finite type containing I we have that $(X+Y)^*$ is proper.*

Proof. Let I be $*$-homogeneous, then any proper finite type $*$-ideals X, Y containing I are $*$-homogeneous contained in $M(I)$ and so $(X+Y)^* \subseteq M(I)$. Conversely if the condition holds and I is contained in two distinct maximal $*$-ideals N_1, N_2. For $n \in N_1 \backslash N_2$ we have $(n, N_2)^* = D$, so there is a finite set $J \subseteq N_2$ such that $(n, J)^* = D$, because $*$ is of finite type. But then $X = (I, n)^* \subseteq N_1$ and $Y = (I, J)^* \subseteq N_2$ both containing I but $(X+Y)^* = D$ a contradiction. $\square$

Remark 2.5. Note that if A and B are proper $*$-ideals such that $(A+B)^* = D$ and if C is any proper $*$-ideal containing B then $(A+C)^* = D$, since $(A+C)^* = (A+B+C)^*$.

Theorem 2.6. *Let $*$ be a finite type star operation defined on an integral domain D. Then D is of finite $*$-character if and only if every $*$-ideal of finite type of D is contained in at most a finite number of mutually $*$-comaximal $*$-ideals of finite type.*

Proof. (I) We first show that every $*$-ideal of finite type of D is contained in at least one $*$-homogeneous ideal of D. For suppose that there is a $*$-ideal A of finite type of D that is not contained in any $*$-homogeneous ideals of D. Then obviously A is not $*$-homogeneous. So there are at least two proper $*$-ideals A_1, B_1 of finite type such that $(A_1 + B_1)^* = D$ and $A \subseteq A_1, B_1$. Obviously, neither of A_1, B_1 is homogeneous. As B_1 is not $*$-homogeneous there are at least two $*$-comaximal proper $*$-ideals B_{11}, B_{12} of finite type containing B_1. Now by Remark 2.5 A_1, B_{11}, B_{12} are mutually $*$-comaximal proper $*$-ideals containing A and by assumption none of these is $*$-homogeneous. Let B_{123} and B_{22} be two $*$-comaximal proper $*$-ideals containing B_{12}. Then by Remark 2.5 and by assumption, A_1, B_{11}, B_{22}, B_{123} are proper mutually $*$-comaximal $*$-ideals containing A and none of these ideals is homogeneous, and so on. Thus at stage n we have a collection: $A_1, B_{11}, B_{22}, ...,$ $B_{nn}, B_{12...n,n+1}$ that are proper mutually $*$-comaximal $*$-ideals containing A and none of these ideals is homogeneous. The process is never ending and has the potential of delivering an infinite number of mutually $*$-comaximal proper $*$-ideals of finite type containing A, contrary to the finiteness condition. Whence the conclusion.

Call two $*$-homogeneous ideals A, B similar if $(A, B)^* \neq D$, that is if A and B belong to the same maximal $*$-ideal. The relation $R =$ "A is similar to B" is obviously an equivalence relation on the set S of $*$-homogeneous ideals containing A. Form a set T of $*$-homogeneous ideals by selecting one and exactly one $*$-homogeneous ideal from each equivalence class of R. Then T is a set of mutually $*$-comaximal $*$-homogeneous ideals containing A and so must be finite because of the finiteness condition. Let $|T| = n$ and claim that n is the largest number of mutually $*$-comaximal $*$-ideals of finite type containing A. For if not then there is say a set U of mutually $*$-comaximal $*$-ideals of finite type that contain A and $|U| = r > n$. Then there is at least one member B of U that is $*$-comaximal with each member of T. (Since no two $*$-comaximal $*$-ideals share the same maximal $*$-ideal.) But then, by (I), there is a $*$-homogeneous ideal J containing B. By Remark 2.5, J is $*$-comaximal with each member of T, yet by the construction of T a $*$-homogeneous ideal containing A must be similar to a member of T, a contradiction. Finally if $P_1, P_2, ..., P_n$ are maximal $*$-ideals such that each contains a member of T then these are the only maximal $*$-ideals containing A. For if not then there is a maximal $*$-ideal $M \neq P_i$ containing A and there is $x \in M \backslash P_i$, $i = 1, 2, ..., n$. But then $(A, x)^*$ is a finite type $*$-ideal containing A and $*$-comaximal with each member of T, yet by (I) $(A, x)^*$ must be contained in a $*$-homogeneous ideal that is $*$-comaximal with each member of T, a contradiction. For the converse note that if a nonzero non unit $x \in D$ is contained in infinitely many mutually $*$-comaximal ideals then D cannot be of finite $*$-character, because a maximal $*$-ideal cannot contain two or more $*$-comaximal ideals. □

So, if we must construct a $*$-homogeneous ideal we know where to go. Otherwise there are plenty of $*$-potent domains, with one kind studied in [33] under the name $*$-super potent domains. Let's note here that there is a slight difference between the definitions. Definition 1.1 of [33] calls a finitely generated ideal I $*$-rigid if I is contained in a unique maximal $*$-ideal. But it turns out that if I is $*$-rigid, then I^* is $*$-homogeneous and if J is $*$-homogeneous then J contains a finitely generated ideal K such that K is exactly in the same maximal $*$-ideal containing J, making K $*$-rigid, see also [49].)

3. What $*$-homogeneous ideals can do

This much about $*$-homogeneous ideals and potent domains leads to the questions: What else can $*$-homogeneous ideals do? $*$-homogeneous ideals arise and figure prominently in the study of finite $*$-character of integral domains. The domains of $*$-finite character where the $*$-homogeneous ideals show their full force are the $*$-Semi Homogeneous ($*$-SH) Domains.

It turns out, and it is easy to see, that if I and J are two $*$-homogeneous ideals that are similar, i.e. that belong to the same unique maximal $*$-ideal (i.e. $M(I) = M(J)$ in the notation and terminology of [11]) then $(IJ)^*$ is $*$-homogeneous belonging to the same maximal $*$-ideal. With the help of this and some auxiliary results it can then be shown that if an ideal K is a $*$-product of finitely many $*$-homogeneous ideals then K can be uniquely expressed as a $*$-product of mutually $*$-comaximal $*$-homogeneous ideals. Based on this a domain D is called a $*$-semi homogeneous ($*$-SH) domain if every proper principal ideal of D is expressible as a $*$-product of finitely many $*$-homogeneous ideals. It was shown in [11, Theorem 4] that D is a $*$-SHD if and only if D is a $*$-h-local domain (D is a locally finite intersection of localizations at its maximal $*$-ideals and no two maximal $*$-ideals of D contain a common nonzero prime ideal.) Now if we redefine a $*$-homogeneous ideal so that the $*$-product of two similar, newly defined, $*$-homogeneous ideals is a $*$-homogeneous ideal meeting the requirements of the new definition, we have a new theory.

To explain the process of getting a new theory of factorization merely by producing a suitable definition of a $*$-homogeneous ideal we give below one such theory.

Let's recall first that if $A = (a_1, ..., a_m)$ is a finitely generated ideal then $A_{(r)}$ denotes $(a_1^r, ..., a_m^r)$. Let's also recall that if A is $*$-invertible then $(A^r)^* = (a_1^r, ..., a_m^r)^*$ [7, Lemma 1.14].

Definition 3.1. Call a $*$-homogeneous ideal I $*$-almost factorial general homogeneous ($*$-afg homogeneous) if (afg1) I is $*$ -invertible, and (afg2) for each finite type $*$-homogeneous ideal $J \subseteq M(I)$ we have for some $r \in N$, $(I^r + J)_m^*$ is principal for some $m \in N$, m depending upon the choice of generators of $(I^r + J)$.

(You can also redefine it as: Definition A. Call a $*$-homogeneous ideal I $*$-almost factorial general homogeneous ($*$-afg homogeneous) if (afg1) I is $*$ -invertible and (afg2) for each finitely generated $*$-homogeneous ideal $J \subseteq M(I)$ such that $J^* \supseteq I^r$, for some $r \in N$, we have $(J)_m^*$ principal for some $m \in N$. (Here you may add that m may vary with each choice of generators of J. And redo the following accordingly.)

Lemma 3.2. *Let I be $*$-invertible and J any f.g. ideal then* $((IJ)_r)^* = (I^r J_r)^*$

Proof. Let $I = (a_1, ..., a_m)$, $J = (b_1, ..., b_n)$. Then $IJ = (\{a_i b_j | i = 1, ..., m;\ j = 1, ...n\})$ and $(IJ)_r^* = \{a_i^r b_j^r | i = 1, ..., m;\ j = 1, ...n\}^* = ((a_1^r, ..., a_m^r)(b_1^r, ..., b_n^r))^* = (I^r J_r)^*$, because I is $*$-invertible. $\square$

Using the above definition, we can be sure of the following.

Proposition 2. *The following hold for a $*$-afg ideal I. (1) $(I^r)^*$ is principal for some positive integer r, (2) for any finitely generated ideal $J \subseteq M(I)$, we have $(I^m)^* \subseteq (J_m)^*$ or $(J_m)^* \subseteq (I^m)^*$ for some positive integer m, (3) if J is a $*$-invertible $*$-ideal that contains I, then J itself is a $*$-afg ideal and (4) if J is a $*$-afg ideal similar to I (i.e., $J \subseteq M(I)$), then $(IJ)^*$ is $*$-afg similar to both I and J.*

Proof. (1) If I is $*$-afg, $(I^r)^* = (f)$, for some $r \in N$ and $f \in D$ by definition and we can choose r to be minimum. $((I + I)_r = (f))$

(2) By definition, if I is $*$-afg, we also have $((I^m + J_m)^* = (d)$, for each finitely generated ideal J. Dividing both sides by d we get $(I^m/d + J_m/d)^* = D$. Now as I^m and J_m are contained in $M(I)$, and no other maximal $*$-ideal, so $(I^m/d)^*$ and $(J_m/d)^*$ have no choice but to be in $M(I)$, if non-trivial. So, $(I^m/d)^* = D$ or $(J_m/d)^* = D$. Thus if $(I^m)^* = dD$, then $(I^m)^* = (d) \supseteq (J_m)^*$ and if $(J_m)^* = dD$ then $I^m \subseteq J_m = dD$. Thus by (afg2) $(I^m)^*$ is principal and contains $(J_m)^*$ or $(J_m)^*$ is principal and contains $(I^m)^*$, for some $m \in N$.

(3) Note that as $J \supseteq I$ we have $J^r \supseteq I^r$ for all positive integers r. Next for every finitely generated ideal F such that $F^* \supseteq J^{\ s}$ for some s we have $F^* = (J^s + G)^* = (J^s + I^s + G)^* = (I^s + (J^s + G))^*$ and so $(F_m)^* = (d)$ for some positive integer m and for each $*$-homogeneous ideal G.) (4) If I, J are two similar $*$-afg homogeneous ideals then $(IJ)^*$ is similar to both I and J. $(IJ)^*$ is $*$-invertible and $*$-homogeneous and of course similar to both I and J. We have to show that for each $*$-homogeneous ideal G, for some $r \in N$, $(I^rJ^r + G)^*_m = (d)$ for some m. Let $F = (I^rJ^r + G)^*$. By (2) we know that $I^m \supseteq J^m$ or $J^m \supseteq I^m$, say $I^m \supseteq J^m$. Now consider $(F_m)^* = (I^{mr}J^{mr} + G_m)^* \supseteq (J^{2mr} + G_m)$ or $(F_m)^* = (J^{2mr} + H)^*$ and by definition $(F_{mt})^*$ is principal for some t. □

Now define a $*$-afg semi homogeneous domain ($*$-afg-SHD) as: D is a $*$-afg-SHD if every nonzero non unit of D is expressible as a $*$-product of finitely many $*$-afg homogeneous ideals. Indeed D is a $*$-afg-SHD is a $*$-SHD whose $*$-homogeneous ideals are $*$-afg homogeneous. (S. Xing, a student of Wang Fanggui, is working with me on this topic. Xing, incidentally, is also at Chengdu University, China. Now Dan Anderson has also joined in and there's a possibility that the definition will be completely twisted out of shape.)

Next, each of the definitions of homogeneous elements can actually give rise to $*$-potent domains in the same manner as the $*$-super potent domains of [33]. In [33], for a star operation $*$ of finite character, a $*$-homogeneous ideal is called $*$-rigid. The $*$-maximal ideal containing a $*$-homogeneous ideal I may be called a $*$-potent maximal $*$-ideal, as we have already done. Next we may call the $*$-homogeneous ideal I $*$-super-homogeneous if each $*$-homogeneous ideal J containing I is $*$-invertible and we may call a $*$-potent domain D $*$-super potent if every maximal $*$- ideal I of D contains a $*$-super homogeneous ideal. But then one can study $*$-A-potent domains where A refers to a $*$-homogeneous ideal that corresponds to a particular definition. For example a $*$-homogeneous ideal J is said to be of type 1 in [11] if $\sqrt{J} = M(J)$. So we can talk about $*$-type 1 potent domains as domains each of whose maximal $*$-ideals contains a $*$-homogeneous ideal of type 1. The point is, to each suitable definition say A of a $*$-homogeneous ideal we can study the $*$-A-potent domains as we studied the $*$-super potent domains in [33]. Of course the theory corresponding to definition A would be different from that of other $*$-potent domains. For example each of the maximal $*$-ideal of the $*$-type 1 potent domain would be the radical of a $*$-homogeneous ideal etc. Now as it is usual we present some of the concepts that have some direct and obvious applications, stemming from the use of $*$-homogeneous ideals. For this we select the $*$-f-potent domains for a study.

3.1. **$*$-f-potent domains.** Let $*$ be a finite type star operation defined on an integral domain D. Call a nonzero non unit element r of D $*$-factorial rigid ($*$-f-rigid) if r belongs to a unique maximal $*$-ideal and every finite type $*$-homogeneous ideal containing r is principal. Indeed if r is a $*$-f-rigid element then rD is a $*$-f-homogeneous ideal and hence a $*$-super homogeneous ideal. So the terminology and the theory developed in [11] applies. Note here that every non unit factor s of a $*$-f-rigid element r is $*$-f-rigid because of the definition. Note also that if r, s are similar $*$-f-rigid elements (i.e. rD, sD are similar $*$-f-homogeneous ideals) then rs is a $*$-f-rigid element similar to r and s and so if r is $*$-f-rigid then r^n is $*$-f-rigid for any positive integer n.

Example 3.3. . Every prime element is a t-f-rigid element.

Call a maximal $*$-ideal M $*$-f-potent if M contains a $*$-f-rigid element and a domain D $*$-f-potent if every maximal $*$-ideal of D is $*$-f-potent.

Example 3.4. . UFDs PIDs, Semirigid GCD domains, prime potent domains are all t-f-potent.

(domains in which every maximal t-ideal contains a prime element may be called prime potent. Indeed a prime element generates a maximal t-ideal [31, 13.5]. (So a domain in which every maximal t-ideal contains a prime element is simply a domain in which every maximal t-ideal is principal.)

The definition suggests right away that if r is $*$-f-rigid and x any element of D then $(r, x)^* = sD$ for some $s \in D$ and applying the v-operation to both sides we conclude that $GCD(r, x) = (r, x)_v$ of r exists with every nonzero element x of D and that for each pair of nonzero factors u, v of r we have $u|v$ or $v|u$; that is r is a rigid element of D, in Cohn's terminology [18]. Indeed it is easy to see, if necessary with help from [11], that a finite product of $*$-f-rigid elements is uniquely expressible as a product of mutually $*$-comaximal $*$-f-rigid elements, up to order and associates and that if every nonzero non unit of D is expressible as a product of $*$-f-rigid elements then D is a semirigid GCD domain of [44]. Also, as we shall show below, a t-f-potent domain of t-dimension one (i.e. every maximal t-ideal is of height one) is a GCD domain of finite t-character. But generally a t-f-potent domain is far from being a GCD domain. Before we delve into examples, let's prove a necessary result, by mimicking Theorem 4.12 of [20] and its proof. (We shall also use Theorem 4.21 of [20], in the proofs of results below.)

Proposition 3. *Let D be an integral domain and let L be an extension of the field of fractions K of D. Then each ideal I of $R = D + XL[X]$ is of the form $f(X)FR = f(X)(F + XL[X])$, where F is a nonzero D-submodule of L such that $f(0)F \subseteq D$ and $f(X) \in L[X]$. The finitely generated ideals of R are of the form $f(X)JR$, where J is a finitely generated D-submodule of L and $f(X) \in R$.*

Proof. First observe that a subset of R of the form $f(X)FR$, where $f(0)F \subseteq D$, is in fact an ideal of R. According to [21, Lemma 1.1], the following are equivalent for an ideal I of R: (1) I is such that $I \cap D \neq 0$, (2) $I \supseteq XL[X]$ and (3) $IL[X] =$

$L[X]$. Further if any of these hold, then $I = I \cap D + XL[X] = (I \cap D)R$ and taking $f = 1$, $F = I \cap D$ we have the stated form. Let's now consider the case when $IL[X] \neq L[X]$. In this case $IL[X] = f(X)L[X]$ where $f(X)$ is a variable polynomial of $L[X]$. Then there is a nonzero element $\alpha \in L$ such that $\alpha f(X) \in I$. Let $F = \{\alpha \in L \mid \alpha f(X) \in I\}$. Then F is a D-submodule of L. Since $F \neq 0$ and $f(X)F \subseteq I, I \supseteq f(X)FR = f(X)(F + XL[X])$. Now if $h(X) \in I$, then $h(X) = f(X)(\alpha_0 + ... + \alpha_n X^n)$, where $\alpha_0, , ..., \alpha_n \in L$ whence $h(X) = \alpha_0 f(X) + h\prime(X)$, where $h\prime(X) \in f(X)XL[X] \in I$. Hence $\alpha_0 \in F$ and $h(X) \in f(X)(F + XL[X])$. Thus $I = f(X)(F+XL[X]) = f(X)FR$, from which it also follows that $f(0)F \subseteq D$. Finally let I be finitely generated, then by the above we have $I = f(X)FR$ where F is a finitely generated D-submodule of L and $f(X) \in L[X]$. If $f(0) = 0$, then $f(X)$ is obviously in R. So let's consider $f(0) = h \neq 0$ and $F = (\alpha_1, \alpha_2, ..., \alpha_r)D$. Since $f(0)F \subseteq D$ we must have $h\alpha_i \in D$. But then $I = f(X)FR$ can be written as $I = \frac{f(X)}{h}(h\alpha_1, h\alpha_2, ..., h\alpha_r)R$ where $\frac{f(X)}{h} \in R$. □

(I was struggling with an earlier version of Proposition 3 and Prof. T. Dumitrescu's suggested improvement for it when I remembered Theorem 4.12 of [20]. I am thankful for his input.)

Lemma 3.5. *Let D be an integral domain and let L be an extension field of the field of fractions K of D. Then $d \in D \backslash (0)$ is a t-f-homogeneous element of D if and only if d is a t-f-homogeneous element of $D + XL[X]$.*

Proof. Let's first note that $D+XL[X]$ has the $D+M$ form. Thus if I is a nonzero ideal of D then $(I+XL[X])_v = I_v + XL[X] = I_v(D+XL[X])$, by [12, Proposition 2.4] and using this we can also conclude that $(I + XL[X])_t = I_t + XL[X] = I_t(D + XL[X])$. Now let d be a t-f-homogeneous element of D then dD is a t-f-homogeneous ideal, so any t-ideal of finite type, of D, containing dD is principal. Next consider $d \in D + XL[X]$. Any t-ideal of finite type F of R containing d intersects D and so has the form $(F \cap D) + XL[X]$, according to [21, Lemma 1.1]. Consequently F contains $dD+XL[X]$. We show that F is principal. For this let $F = (a_1+Xf_1(X), ..., a_n+Xf_n(X)_t = ((a_1, ..., a_n)+XL[X])_v = ((a_1, ..., a_n)_v+XL[X])$. But $((a_1, ..., a_n)_v + XL[X]) = F \supseteq dD + XL[X]$ forces $(a_1, ..., a_n)_v \supseteq dD$. Also dD being t-f-rigid, $(a_1, ..., a_n)_v$ must be principal whence F is principal Now note that according to [21], every prime ideal M of R that intersects D is of the form $M \cap D+XL[X]$ and using the above mentioned result of [12, Proposition 2.4] we can show that every maximal t-ideal M that intersects D is of the form $M \cap D+XL[X]$ where $M \cap D$ is a maximal t-ideal of D and that, conversely, if m is a maximal ideal of D then $m + XL[X]$ is a maximal ideal of R. Thus, finally, if m is the unique maximal t-ideal of D containing d then $m + XL[X]$ is a maximal t-ideal of R containing d and if N were another maximal t-ideal containing d then $N \cap D$ would be another maximal t-ideal of D containing d a contradiction. Thus d is a t-f-homogeneous ideal of R. □

Proposition 4. *Let D be an integral domain and let L be an extension field of the field of fractions K of D. Then D is t-potent if and only if $R = D + XL[X]$ is.*

Proof. Note that, according to [21, Lemma 1.2], every prime ideal P of R that is not comparable with $XL[X]$ contains an element of the form $1 + Xg(X)$, so must contain a prime element of the form $1 + Xg(X)$ and so must be a principal prime.

We next show that a finitely generated ideal $F \nsubseteq XL[X]$ of R is t-homogeneous if and only if F is of the form $A + XL[X]$, where A is a t-homogeneous ideal of D or generated by a prime power of the form $(1 + Xh(X))^n$, [20, Theorem 4.21]. Obviously if A is contained in a unique maximal t-ideal P of D then $A + XL[X]$ is contained in the maximal t-ideal $P + XL[X]$ and any maximal t-ideal that contains $A + XL[X]$ also contains $P + XL[X]$. Next, an ideal generated by a prime power is t-homogeneous anyway. Conversely let F be a finitely generated nonzero ideal of R. Then by Proposition 3, $F = f(X)(J + XL[X])$ where $f(X) \in L[X]$ as F is not contained in $XL[X]$, $f(0) = 1$ forcing J to be a finitely generated ideal of D. If in addition F has to be t-homogeneous then F is either contained in a prime ideal of the form $P + XL[X]$ or in a prime ideal incomparable with $XL[X]$. In the first case $F = J + XL[X]$ where J is a rigid ideal belonging to P and in the second case $F = f(X)R$, [20, Theorem 4.21]. □

Corollary 1. *Let D be an integral domain and let L be an extension field of the field of fractions K of D. Then D is t-f-potent if and only if $R = D + XL[X]$ is.*

Proof. Suppose that D is t-f-potent. As in the proof of Proposition 4 every maximal t-ideal P of R that is not comparable with $XL[X]$ contains an element of the form $1 + Xg(X)$, so must contain a prime element of the form $1 + Xg(X)$ and so must be a principal prime. Next the maximal t-ideals comparable with $XL[X]$ are of the form $P + XL[X]$ where P is a maximal t-ideal of D. Since D is t-f-potent P contains a t-f-rigid element, which is also a t-f-rigid element of R, by Lemma 3.5. So $P + XL[X]$ contains a t-f-rigid element of R. In sum, every maximal t-ideal of R contains a t-f-rigid element of R and so R is t-f-potent. Conversely suppose that R is t-f-potent. Then as for each maximal t-ideal M of D, $M + XL[X]$ is a maximal t-ideal, each M contains a t-f-rigid element of R and hence of D, by Lemma 3.5. Thus each maximal t-ideal of D contains a t-f-rigid element of D. □

Recall, from [4], that a GCD domain of finite t-character that is also of t-dimension 1 is termed as a generalized UFD (GUFD).

Example 3.6. If D is a UFD (GUFD, Semirigid GCD domain) and L an extension of the quotient field K of D, then the ring $D + XL[X]$ is a t-f-potent domain.

The t-f-potent domains and their examples are nice but we must show that they have some useful properties. We start with the most striking property. Here let X be an indeterminate over D. A polynomial $f = \sum a_i X^i$ is called primitive if its content $A_f = (a_0, a_1, ..., a_n)$ generates a primitive ideal, i.e., $(a_0, a_1, ..., a_n) \subseteq aD$ implies a is a unit and super primitive if $(A_f)_v = D$. It is known that while a super primitive polynomial is primitive a primitive polynomial may not be super primitive, see e.g. Example 3.1 of [10]. A domain D is called a PSP domain if each primitive polynomial f over D is superprimitive, i.e. if $(A_f)_v = D$.

Proposition 5. *A t-f-potent domain D has the PSP property.*

Proof. Let $f = \sum a_i X^i$ be primitive i.e. $(a_0, a_1, ..., a_n) \subseteq aD$ implies a is a unit and consider the finitely generated ideal $(a_0, a_1, ..., a_n)$ in a t-f- potent domain$D$. Then $(a_0, a_1, ..., a_n)$ is contained in a maximal t-ideal M associated with a t-f-rigid element r (of course $M = M(rD)$) if and only if $(a_0, a_1, ..., a_n, r)_t = sD \neq D$. Since every maximal t-ideal of a t-f-potent domain is associated with a t-f-rigid element, we conclude that in a t-f-potent domain D, $f = \sum a_i X^i$ primitive implies that A_f

is contained in no maximal t-ideal of D; giving $(A_f)_v = D$ which means that each primitive polynomial f in a t-f-potent domain D is actually super primitive. □

Now PSP implies AP i.e. every atom is prime, see e.g. [10]. So, in a t-f-potent domain every atom is a prime. If it so happens that a t-f-potent domain has no prime elements then the t-f-potent domain in question is atomless. Recently atomless domains have been in demand. The atomless domains are also known as antimatter domains. Most of the examples of atomless domains that were constructed were the so-called pre-Schreier domains, i.e. domains in which every nonzero non unit a is primal (is such that $(a|xy$ implies $a = rs$ where $r|x$ and $s|y)$. One example (Example 2.11 [10]) was laboriously constructed in [10] and this example was atomless and not pre-Schreier, As we indicate below, it is easy to establish a method of telling whether a t-f-potent domain is pre-Schreier or not.

Cohn in [18] called an element c in an integral domain D primal if (in D) $c|a_1a_2$ implies $c = c_1c_2$ where $c_i|a_i$. Cohn [18] assumes that 0 is primal. We deviate slightly from this definition and call a nonzero element c of an integral domain D primal if $c|a_1a_2$, for all $a_1, a_2 \in D\backslash\{0\}$, implies $c = c_1c_2$ such that $c_i|a_i$. He called an integral domain D a Schreier domain if (a) every (nonzero) element of D is primal and (b) D is integrally closed. We have included nonzero in brackets because while he meant to include zero as a primal element, he mentioned that the group of divisibility of a Schreier domain is a Riesz group. Now the definition of the group of divisibility $G(D)(= \{\frac{a}{b}D : a, b \in D\backslash\{0\}\}$ ordered by reverse containment) of an integral domain D involves fractions of only nonzero elements of D, so it's permissible to restrict primal elements to be nonzero and to study domains whose nonzero elements are all primal. This is what McAdam and Rush did in [39]. In [45] integral domains whose nonzero elements are primal were called pre-Schreier. It turned out that pre-Schreier domains possess all the multiplicative properties of Schreier domains. So let's concentrate on the terminology introduced by Cohn as if it were actually introduced for pre-Schreier domains.

Cohn called an element c of a domain D completely primal if every factor of c is primal and proved, in Lemma 2.5 of [18] that the product of two completely primal elements is completely primal and stated in Theorem 2.6 a Nagata type result that can be rephrased as: Let D be integrally closed and let S be a multiplicative set generated by completely primal elements of D. If D_S is a Schreier domain then so is D. This result was analyzed in [10] and it was decided that the following version ([10, Theorem 4.4] of Cohn's Nagata type theorem works for pre-Schreier domains.

Theorem 3.7. *(Cohn's Theorem for pre-Schreier domains). Let D be an integral domain and S a multiplicative set of D. (i) If D is pre-Schreier, then so is D_S . (ii) (Nagata type theorem) If D_S is a pre-Schreier domain and S is the set generated by a set of completely primal elements of D, then D is a pre-Schreier domain.*

Now we have already established above that if r is a t-f-rigid element then $(r, x)_v$ is principal for each $x \in D\backslash\{0\}$. But then $(r, x)_v$ is principal for each $x \in D\backslash\{0\}$ if and only if $(r) \cap (x)$ is principal for each $x \in D\backslash\{0\}$. But then r is what was called in [8] an extractor. Indeed it was shown in [8] that an extractor is completely primal. Thus we have the following statement.

Corollary 2. *Let D be a t-f-potent domain. Then D is pre-Schreier if and only if D_S is pre-Schreier for some multiplicative set S that is the saturation of a set generated by some t-f rigid elements.*

(Proof. If D is pre-Schreier then D_S is pre-Schreier anyway. If on the other hand D_S is pre-Schreier and S is (the saturation of a set) multiplicatively generated by some t-f- rigid elements. Then by Theorem 3.7, D is pre-Schreier.)

One may note here that if D_S is not pre-Schreier for any multiplicative set S, then D is not pre-Schreier. So the decision making result of Cohn comes in demand only if D_S is pre-Schreier. Of course in the Corollary 2 situation, the saturation S of the multiplicative set generated by all the t-f-rigid elements of D, leading to: if D_S is not pre-Schreier then D is not pre-Schreier for sure and if D_S is pre-Schreier then D cannot escape being a pre- Schreier domain.

Example 3.8. Let $D = \cap_{i=1}^{i=n} V_i$ be a finite intersection of distinct non-discrete rank one valuation domains with quotient field $K = qf(D)$, X an indeterminate over D and let L be a proper field extension of K. Then (a) $D + XL[X]$ is a non-pre-Schreier, t-f-potent domain and (b) $D + XL[X]_{(X)}$ is an atomless non-pre-Schreier, t-f-potent domain.

Illustration: (a) It is well known that D is a Bezout domain with exactly n maximal ideals, M_i [36], with $V_i = D_{M_i}$. Thus $D = \cap D_{M_i}$ and each of M_i being a t-ideal must, each, contains a t-homogeneous ideal by Proposition 1. $D + XL[X]$ is t-f-potent by Corollary 1.

One more result that can be added needs introduction to a neat construction called the Nagata ring construction these days. This is how the construction goes.

Let $*$ be a star operation on a domain D, let X be an indeterminate over D and Let $S_* = \{f \in D[X] | \ (A_f)^* = D\}$. Then the ring $D[X]_{S_*}$ is called the Nagata construction from D with reference to $*$ and is denoted by $Na(D, *)$. Indeed $Na(D, *) = Na(D, *_f)$

Proposition 6. *([35] Proposition 2.1.) Let $*$ be a star operation on R. Let $*_f$ be the finite type star operation induced by $*$. Let $S_* = \{f \in D[X] | (A_f)_* = D\}$. Then (1) $S_* = D[X] \backslash \cup_{M \in \Gamma} M[X]$ where Γ is the set of all maximal $*_f$-ideals of D. (Hence S_* is a saturated multiplicatively closed subset of $D[X]$.), (2) $\{M[X]_{S_*}\}$ is the set of all maximal ideals of $[DX]_{S_*}$.*

As pointed out in [26], proof of Part (1) of the following proposition has a minor flaw, in that for a general domain it uses a result ([27, 38.4]) that is stated for integrally closed domains. The fix offered in [26] is a new result and steeped in semistar operations. We offer, in the following, a simple change in the proof of [35, (1) Proposition 2.2.] to correct the flaw indicated above.

Proposition 7. *([35] Proposition 2.2.) Let T be a multiplicatively closed subset of $D[X]$ contained in $S_v = \{f \in D[X] | (A_f)_v = D\}$. Let I be a nonzero fractional ideal of D. Then (1) $(I[X]_T)^{-1} = I^{-1}[X]_T$, (2) $(I[X]_T)_v = I_v[X]_T$ and (3) $(I[X]_T)_t = I_t[X]_T$.*

(1) It is clear that $I^{-1}[X]_T \subseteq (I[X]_T)^{-1}$. Let $u \in (I[X]_T)^{-1}$. Since for any $a \in I \backslash \{0\}$ we have $(I[X]_T)^{-1} \subseteq a^{-1}D[X]_T \subseteq K[X]_T$ we may assume that $u = f/h$ with $f \in K[X]$ and $h \in T$. Then $f \in (I[X]_T)^{-1}$. Hence $fI[X]_T \subseteq D[X]_T$. Hence $bf \in D[X]_T$ for any $b \in I$. Now $bfg \in D[X]$ for some $g \in S_v$. So $(A_{bfg})_v \subseteq D$. By [40, Proposition 2.2.], $(A_{bfg})_v = (A_{bf}A_g)_v = (A_{bf})_v$, since $(A_g)_v = D$ and hence

v-invertible. Therefore $bA_f \subseteq (bA_f)_v = (A_{bf})_v \subseteq D$ for any $b \in I$ Hence$A_f \subseteq I^{-1}$. Hence $f \in I^{-1}[X]$ and $f/h \in I^{-1}[X]_T$. Therefore $(I[X]_T)^{-1} = I^{-1}[X]_T$.

Theorem 3.9. *([35], Theorem 2.4.) Let $*$ be a finite type star operation on D. Let I be a$F[X]_{S_v}$ nonzero ideal of D. Then I is $*$-invertible if and only if $I[X]_{S_*}$ is invertible.*

Theorem 3.10. *([35], Proposition 2.14.) Let $*$ be a star operation on D. Then any invertible ideal of $D[X]_{S_*}$ is principal.*

Thus we have the following corollary.

Corollary 3. *Let I be a t-ideal of finite type of D. Then I is t-invertible if and only if $I[X]_{S_v}$ is principal.*

Proof. If $I[X]_{S_v}$ is principal then $I[X]_{S_v}$ is invertible and so I is t-invertible by Theorem 3.9. Conversely let F be a finitely generated ideal such that $F_t = I$. Then F is t-invertible and so, by Theorem 3.9, is $F[X]_{S_v}$ invertible and hence principal by Theorem 3.10. But then $F[X]_{S_v} = (F[X]_{S_v})_t = I[X]_{S_v}$. □

Lemma 3.11. *Let I be a t-ideal of finite type of D. Then $I[X]_{S_v}$ is d-homogeneous if and only if I is t-homogeneous. Consequently $I[X]_{S_v}$ is t-f-rigid if and only if I is t-super homogeneous.*

Proof. Let I be a t-homogeneous ideal of D. That $I[X]_{S_v}$ is a t-ideal of finite type is an immediate consequence of Proposition 7. If M is the unique maximal t-ideal containing I, then at least $M[X]_{S_v} \supseteq I[X]_{S_v}$. Suppose that $\mathcal{N}$ is another maximal ideal of $D[X]_{S_v}$ containing $I[X]_{S_v}$. But by Proposition 6, $\mathcal{N} = N[X]_{S_v}$ for some maximal t-ideal N of D. But then $N = D \cap N[X]_{S_v} \supseteq D \cap I[X]_{S_v} \supseteq I$. This forces $N = M$ and consequently $N[X]_{S_v} = M[X]_{S_v}$ making $I[X]_{S_v}$ homogeneous.

Conversely if $I[X]_{S_v}$ is d-homogeneous contained in a unique $M[X]_{S_v}$,suppose that N is another maximal t-ideal containing I. Then again $N[X]_{S_v} \supseteq ID[X]_{S_v}$ which is d-homogeneous, a contradiction unless $N = M$.

The consequently part follows from Corollary 3. □

Let's all a domain $*$-f-r-potent if every maximal $*$-ideal of D contains a $*$-f-rigid element.

Proposition 8. *Let D be an integral domain with quotient field K, X an indeterminate over D and let $S_v = \{f \in D[X] | (A_f)_v = D\}$. Then (a) D is t-potent if and only if $D[X]_{S_v}$ is d-potent and (b) D is t-super potent if and only if $D[X]_{S_v}$ is d-f-r-potent*

Proof. (a) Suppose that D is potent Let $M[X]_{S_v}$ be a maximal ideal of $D[X]_{S_v}$ and let I be a t-homogeneous ideal contained in M. By Lemma 3.11, $I[X]_{S_v}$ is d-homogeneous, making $M[X]_{S_v}$ d-potent. Conversely suppose that $D[X]_{S_v}$ is d-potent and let M be a maximal t-ideal of D. Then $M[X]_{S_v}$ is a maximal ideal of $D[X]_{S_v}$ and so contains a d-homogeneous ideal $\mathcal{I} = (f_1, f_2, ..., f_n)D[X]_{S_v}$. Now let $I = (f_1, f_2, ..., f_n)$. Then $\mathcal{I} = ID[X]_{S_v}$ and $I \subseteq (A_I)_t[X]_{S_v} \subseteq M[X]_{S_v}$, since $M[X]_{S_v}$ is a t-ideal and $f_i \in M[X]_{S_v} \cap D[X]$. This gives $\mathcal{I} = ID[X]_{S_v} \subseteq (A_I)_t[X]_{S_v} \subseteq M[X]_{S_v}$ making $(A_I)_t[X]_{S_v}$ another homogeneous ideal, contained in $M[X]_{S_v}$ and containing $\mathcal{I}$. But then $(A_I)_t \subseteq M$ is a t-homogeneous ideal, by Lemma 3.11. (b) Use part (a) and Corollary 3. □

The other property that can be mentioned "off hand" is given in the following statement.

Theorem 3.12. *A t-f-potent domain of t-dimension one is a GCD domain of finite t-character.*

A domain of t-dimension one that is of finite t-character is called a weakly Krull domain. (D is weakly Krull if $D = \cap D_P$ where P ranges over a family $\mathcal{F}$ of height one prime ideals of D and each nonzero non unit of D belongs to at most a finite number of members of $\mathcal{F}$.) A weakly Krull domain D is dubbed in [11] as $*$-weakly Krull domain or as a type 1 $*$-SH domain. Here a $*$-homogeneous ideal I is said to be of type 1 if $M(I) = \sqrt{I^*}$ and D is a type 1 $*$-SH domain if every nonzero non unit of D is a $*$-product of finitely many $*$-homogeneous ideals of type 1. In the following lemma we set $* = t$.

Lemma 3.13. *A t-f-potent weakly Krull domain is a type 1 t-f-SH domain.*

Proof. A weakly Krull domain is a type 1 t-SH domain. But then for every pair I, J of similar homogeneous ideals $I^n \subseteq J^*$ and $J^m \subseteq I^*$ for some positive integers m, n. So J is a t-f-homogeneous ideal if I is and vice versa. Thus in a t-f-potent weakly Krull domain the t-image of every t-homogeneous ideal is principal whence every nonzero non unit of D is expressible as a product of t-f-homogeneous elements which makes D a t-f-SH domain and hence a GCD domain. □

Proof. of Theorem 3.12 Use Theorem 5.3 of [33] for $* = t$ to decide that D is of finite t-character and of t-dimension one. Indeed, that makes D a weakly Krull domain that is t-f-potent. The proof would be complete once we apply Lemma 3.13 and note that a t-f-SH domain is a GCD domain and of course of finite t-character. □

Generally a domain that is t-f-potent and with t-dimension > 1, is not necessarily GCD nor of finite t-character.

Example 3.14. $D = Z + XL[[X]]$ where Z is the ring of integers and L is a proper extension of Q the ring of rational numbers. Indeed D is prime potent and two dimensional but neither of finite t-character nor a GCD domain.

There are some special cases, in which a t-f-potent domain is GCD of finite t-character.

i) If every nonzero prime ideal contains a t-f- homogeneous ideal. (Use (4) of Theorem 5 of [11]) along with the fact that D is a t-f-SH domain if and only if D is a t-SH domain with every t-homogeneous ideal t-f-homogeneous. Thus a t-f-potent domain of t-dim 1 is of finite character.

ii) If D is a t-f-potent PVMD of finite t-character that contains a set S multiplicatively generated by t-f-homogeneous elements of D and if D_S is a GCD domain then so is D.

I'd be doing a grave injustice if I don't mention the fact that before there was any modern day multiplicative ideal theory there were prime potent domains as Z the ring of integers and the rings of polynomials over them. It is also worth mentioning that there are three dimensional prime potent Prufer domains that are not Bezout. The examples that I have in mind are due to Loper [38]. These are non-Bezout Prufer domains whose maximal ideals are generated by principal primes.

4. $*$-FINITE IDEAL MONOIDS

In [42], we called a directed p.o. group G pre-Riesz if its positive elements satisfied the following property.

(pR): If $x_1, x_2, ..., x_n$ are strictly positive elements in G and x_i are such that there is at least one $g \in G$ with $g \nleq 0$ such that $g \leq x_1, x_2, ..., x_n$ then there is at least one $r \in G$ such that $0 < r \leq x_1, x_2, ..., x_n$.

By a basic element, in the above paper, we meant a strictly positive element $c \in G$ such that for every pair of strictly positive elements c_1, c_2 preceding c we have $r \in G$ such that $0 < r \leq c_1, c_2$.

Note that it is essentially the positive cone $G^+ = \{g \in G | g \geq 0\}$ of the pre-Riesz group that satisfies the (pR), but with reference to elements of its main group. So let's call a commutative p.o. monoid $M =< M, +, 0, \leq >$ a pre-Riesz monoid if M is upper directed and satisfies (pR'): For any finite set of strictly positive elements $x_1, x_2, ..., x_n \in M$, either $glb(x_1, x_2, ..., x_n) = 0$ or there is $r \in M$ such that $0 < r \leq x_1, x_2, ..., x_n$. Note that the '+' and '0' are mainly symbolic, standing in for the monoid operation and the identity. Note also that to avoid getting into trivialities we shall only consider non-trivial pre-Riesz monoids, i.e., ones that are different from $\{0\}$.

Here, of course, we do not require that $a \leq b \Leftrightarrow a + x = b$. The partial order may be pre-assigned but must be compatible with the binary operation of the monoid. Let's call M a divisibility p.o. monoid if in M $a \leq b \Leftrightarrow a + x = b$, for some $x \in M$.

A monoid M is said to have cancellation if $a + b = a + c$ implies $b = c$. Obviously if in a cancellation monoid with order defined as above we have $a + b \leq a + c$ then $b \leq c$.

Proposition 9. *Let $a, b \in M$ where M is a divisibility pre-Riesz monoid with cancellation. Then $lub(a, b) = a + b$ if and only if $glb(a, b) = 0$.*

Proof. Suppose that $lub(a, b) = a + b$ and let there be $r > 0$ such that $r \leq a, b$. Then $a = r + x$ and $b = r + y$ for some $x, y \in M$. Obviously, as $r > 0$, $x < a$ and $y < b$. Thus $r + x + y < a + b$ yet $r + x + y \geq a, b$ contradicting the assumption that $lub(a, b) = a + b$. Conversely suppose that $glb(a, b) = 0$ and let there be, by way of contradiction, r such that $r \geq a, b$ yet $r < a + b$. Then $r = a + x = b + y$ and $a + b = r + z$. Taking $r = a + x$ we have $a + b = a + x + z$. Cancelling a from both sides we get $b = x + z$. Similarly substituting for $r = b + y$ and cancelling b from both sides we get $a = y + z$. But then $z \leq a, b$ and hence $z = 0$ forcing $a = y$ and $b = x$ and $r = a + b$, a contradiction. □

If $glb(a, b)$ (resp., $lub(a, b)$) exists in a monoid M we denote it by $a \wedge b$ (resp., $a \vee b$)

Example 4.1. (1) If G is a Riesz group then as shown in [42, Proposition 3.1], G^+ is a pre-Riesz monoid. (2) Indeed G is a pre-Riesz group if and only if G^+ is a pre-Riesz monoid and indeed a pre-Riesz group can be regarded as a pre-Riesz monoid. (3) Let $*$ be a finite character star operation defined on a domain D and let Γ be the set of all proper $*$-ideals of finite type of D. Then $\Gamma \cup \{D\}$ is a pre-Riesz monoid under $*$-multiplication because $glb(A_1, A_2, ..., A_n) = D$ if and only if $(A_1, A_2, ..., A_n)^* = D$. Let's denote this monoid by $< \Gamma \cup \{D\}, *, D, \leq >$ and call it $*$-finite ideals monoid ($*$-FIM)

(This is because the $*$-product of finitely many members of Γ is again of finite type and this $*$-product is associative. Here the partial order is induced by reverse containment i.e. for $A, B \in \Gamma$, $A \leq B$ if and only if $A \supseteq B$ and of course $*$-multiplication is compatible with the order, i.e., for $A, B, C \in \Gamma$ with $A \leq B$ then $(AC)^* \leq (BC)^*$ (since $A \supseteq B$ implies that $(AC)^* \supseteq (BC)^*$).)

Let's call D $*$-coherent if for all $A, B \in \Gamma$ we have $A \cap B \in \Gamma$.

Proposition 10. *Let* $< \Gamma \cup \{D\}, *, D, \leq >$ *be a $*$-finite monoid (1) For all* $H, K \in \Gamma$, *(We have* $H \wedge K \in \Gamma$ *as* $(H, K)^*$ *and if* $H \cap K \in \Gamma$, *then* $H \vee K = H \cap K$.

Proof. Indeed as $(H, K)^* \leq H, K$ (sine $(H, K)^* \supseteq H, K$) and if $A \leq H, K$ for some $A \in \Gamma$, (i.e., $A \supseteq H, K$) then $A \leq (H, K)^*$. Let's put it this way $(H, K)^*$ is standard for $\inf(H, K)$ and $H \cap K$, if it exists is standard for $\sup(H, K)$ in ideal theory and so it is here. □

So, a $*$-finite monoid is actually a semilattice. Now let M be a pre-Riesz monoid and $H \in M$. Call H homogeneous if for all $0 < R, S \leq H$ we have a $0 < t < R, S$. Obviously $0 < K \in M$ is not homogeneous if and only if there are $0 < R, S < K$ such that $\inf(R, S) = 0$. Let's call $X, Y \in M$ disjoint if $\inf(X, Y) = 0$ and note that if H is homogeneous then H cannot be non disjoint with two or more disjoint elements. Also if X, Y are disjoint and $0 < x < X$ then x and Y are disjoint, for if not then there is $0 < r < x, Y, X$ making X, Y non-disjoint.

Call a set S of homogeneous elements of a pre-Riesz monoid M an independent set if every pair of elements of S is disjoint. In notes of my work with Yang and a student of his [37], other, restricted, versions of the following were proved. As the notes are not made public yet and there is a significant difference of the notions involved, I include below some related results that are relevant to this write up.

Proposition 11. *Let S be an independent set of homogeneous elements, in a pre-Riesz monoid, satisfying a property (Q). Then S can be enlarged to a maximal independent set T of homogeneous elements satisfying (Q).*

Proof. Let $\Gamma = \{B | B \supseteq S$ is an independent set of homogeneous elements satisfying $(Q)\}$. Obviously $\Gamma \neq \phi$. Now let $\{B_\alpha\}$ be a chain of members of Γ and let $C = \cup B_\alpha$. Then $C \supseteq S$ and for any pair $x, y \in C$, x, y are in B_α for some α so elements of C are homogeneous, satisfy (Q) and are homogeneous. So, $C \in \Gamma$. Thus by Zorn's Lemma Γ must contain a maximal element and that is our T. □

We shall call a set S of mutually disjoint elements, of a monoid M, a maximal disjoint set if (as usual) no set T exists of mutually disjoint elements such that $M \supseteq T \supsetneq S$ and we shall call a set S of mutually disjoint elements of M order maximal if no element s of S can be replaced by two distinct predecessors to form a set $(S \backslash \{s\}) \cup \{x, y\}$ of mutually disjoint elements. A maximal set of disjoint homogeneous elements is obviously order maximal too, but a mere maximal set of mutually disjoint elements may not be, as we have seen in the case of ideals in a ring.

An order maximal independent set B of homogeneous elements of a pre-Riesz monoid M is called a basis if B is also an order maximal set of mutually disjoint elements.

Lemma 4.2. . *Let M be a pre-Riesz monoid. Then a non-empty subset S of M is a basis if and only if S is disjoint and* $(S \backslash \{s\}) \cup \{x, y\}$ *is non-disjoint for any* $s \in S$ *and for any* $\{x, y\} \subseteq (M \backslash S) \cup \{s\}$, *with* $x \neq y$.

Proof. Let S be a basis and suppose that for some $s \in S$, $(S\backslash\{s\}) \cup \{x, y\}$ is disjoint for some $\{x, y\} \subseteq (M\backslash S) \cup \{s\}$, with $x \neq y$. Then $x \wedge s \neq 0$ and $y \wedge s \neq 0$ because S is a maximal set of disjoint elements of M. Since M is pre-Riesz, there is $t \in M$ such that $0 < t \leq x, s$ and $u \in M$ such that $0 < u \leq y, s$. Next as s is homogeneous, there is $w \in M$ such that $0 < w \leq t, u, x, y$, a contradiction. Conversely, suppose that S is disjoint and satisfies the condition in the lemma. If $S \cup \{x\}$ is disjoint for some $x \in M\backslash S$, then for any $s \in S$, $(S\backslash\{s\}) \cup \{s, x\}$ is disjoint and $s \neq x$, a contradiction. Therefore, S is an maximal disjoint set. If $s \in S$ and s is not homogeneous, then there exists at least one pair of elements $0 < x, y < s$ such that $x \wedge y = 0$. But then $x, y \notin S, x \neq y$ and $(S\backslash\{s\}) \cup \{x, y\}$ is disjoint, a contradiction. Thus, S is a maximal disjoint set consisting of homogeneous elements, i.e., S is a basis. □

Theorem 4.3. *(cf* [37, Theorem 9]*) A pre-Riesz monoid M has a basis if and only if (P): each $0 < x \in M$ exceeds at least one homogeneous element . Every basis of M is an order maximal independent subset and every order maximal independent subset of M is a basis provided M has a basis.*

Proof. Let $S = \{0 < a_\gamma | \gamma \in \Gamma\}$ be a basis for M, and consider $0 < x \in M$. There exists $\gamma \in \Gamma$ such that $x \wedge a_\gamma \neq 0$ for otherwise S is not a maximal set of disjoint elements. This means that there is $0 < h \leq x, a_\gamma$ and h is homogeneous because a_γ is homogeneous and $h \in (0, a_\gamma]$. Thus, M satisfies (P). Conversely, suppose that M satisfies the property (P). Since M is non-trivial there is at least one homogeneous element and, by Proposition 11, there exists a maximal independent subset $T = \{0 < a_\gamma | \gamma \in \Gamma\}$ of M, assuming that (Q) means "no restriction". All we need show is that T is a maximal set of disjoint elements. Suppose on the contrary that there is an element $0 < x \in M\backslash T$ such that $x \wedge a_\gamma = 0$ for all $\gamma \in \Gamma$. But then by the property (P), x exceeds a homogeneous element h, and h is disjoint with a_γ for all $\gamma \in \Gamma$. Therefore, $T \cup \{h\} \supsetneq T$and $T \cup \{h\}$ is an independent subset of M, but this is contrary to our choice of T. □

Conrad's F-condition on a pre-Riesz monoid reads thus: Each strictly positive element x in a pre-Riesz monoid M is greater than at most a finite number of (mutually) disjoint positive elements.

Proposition 12. *If a pre-Riesz monoid M satisfies Conrad's F-condition, then M has a basis.*

Proof. Suppose that the condition holds but M has no basis. Then by Theorem 4.3, there is at least one $0 < y \in M$ such that no homogeneous element is contained in $\{x \in M : 0 < x \leq y\}$. Then there exist two disjoint elements x_1, y_1 with $0 < x_1, y_1 < y$ where none of x_1, y_1 exceeds a homogeneous element for otherwise y would. So, say, $0 < x_2, y_2 < x_1$ with $x_2 \wedge y_2 = 0$. Since $x_1 \wedge y_1 = 0$ and $y_2 < x_1$ we have $y_1 \wedge y_2 = 0$. Next $0 < x_3, y_3 < x_2$ with $x_3 \wedge y_3 = 0$. We can conclude that y_1, y_2, y_3 are mutually disjoint. Similarly producing x_i s, y_i s and using induction we can produce an infinite sequence $\{y_i\}$ of mutually disjoint elements less than y. Contradicting the assumption that M satisfies Conrad's F-condition. □

Corollary 4. *The following are equivalent for a pre-Riesz monoid M: (i) M satisfies Conrad's F-condition, (ii) Every strictly positive element exceeds at least one and at most a finite number of homogeneous elements that are mutually disjoint,* *(iii) M contains a subset Γ of strictly positive elements such that every strictly positive element of M exceeds at least one member of Γ and at most a finite number of mutually disjoint members of Γ.*

Proof. (i) $\Rightarrow$ (ii) Conrad's F-condition, via Proposition 12, implies that every strictly positive element x exceeds at least one homogeneous element say h. The set $\{h\}$ is an independent set of $((Q))$homogeneous elements preceding x and by Proposition 11, $\{h\}$ can be expanded to a maximal independent set T of elements preceding x. But again by Conrad's F-condition, T must be finite. For (ii))$\Rightarrow$(i), suppose that (ii) holds yet M does not satisfy (i). Then there is $0 < x \in M$ that exceeds an infinite sequence $\{x_i\}$ of mutually disjoint strictly positive elements of M. Now each of x_i exceeds at least one homogeneous element h_i. Since $\{x_i\}$ are mutually disjoint, $\{h_i\}$ are mutually disjoint, which causes a contradiction. Whence, we have the conclusion. (ii)$\Rightarrow$ (iii) Take $\Gamma = \{h | h$ is a homogeneous element of $M\}$, then every positive element exceeds at least one member of Γ and at most a finite number. (iii)$\Rightarrow$ (i) Suppose that the given condition holds but Conrad's F-condition doesn't. That means there is some element $y > 0$ such that y is greater than an infinite number of mutually disjoint elements $\{y_\alpha\}$ of M. By (iii) each y_α exceeds a member z_α of Γ. As y_α are mutually disjoint, making y exceed an infinite number of mutually disjoint members of Γ, a contradiction. □

Corollary 5. *(Corollary to Corollary 4) Let D be an integral domain, $*$ a finite character star operation on D and let Γ be a set of proper, nonzero, $*$-ideals of finite type of D such that every proper nonzero finite type $*$-ideal of D is contained in some member of Γ . Then D is of finite $*$-character if and only if every nonzero finitely generated ideal I of D with $I^* \neq D$ is contained in at least one and at most a finite number of mutually $*$-comaximal members of Γ.*

Proof. We know that $M = \{A^* |\ A^* \neq D$ is a $*$-ideal of finite type of $D\} \cup \{D\}$ is a pre-Riesz monoid under $*$-multiplication and the set Γ can just be regarded as a subset of M and the theorem requires every strictly positive member of M exceeds at least one member of Γ and at most a finite number of mutually disjoint members of Γ. Now this means, according to Corollary 4, that every element A^* exceeds at least one basic element and at most a finite number of basic elements of M. Now take an element A^* in M and let h be a basic element of Γ containing A^*. Then, by Proposition 11, there is at least one maximal set S of mutually disjoint basic elements containing A^* and each $h \in S$ exceeds some member of Γ giving a maximal set T of basic elements in Γ and containing A^* Now this translates to: If the condition is satisfied, then or every $*$-ideal of finite type A there is a maximal set T of homogeneous $*$-ideals containing A and by the condition, T is finite. Now let $|T| = n$ and recall that if $T = \{H_1, ..., H_n\}$ then each of the H_i determines a unique maximal $*$-ideal $M(H_i)$. To show that $T' = \{M(H_1), ..., M(H_n)\}$ contains all the maximal $*$-ideals containing A^* assume that there is a maximal $*$-ideal $N \notin T'$ and containing A^*. Then there is $x \in N\backslash(\cup M(H_i)$. But then xD is $*$-comaximal with H_i for each i and hence $(x, A)^* \subseteq N$ is $*$-comaximal with each H_i which translates to: $(x, A)^*$ is disjoint with each basic element H_i. But then $(x, A)^*$ exceeds a basic element K which must be disjoint with each of H_i, killing the maximality of T. The converse is obvious because if there is an infinite number of mutually $*$-comaximal members of Γ then D cannot be of finite $*$-character because a maximal $*$-ideal cannot afford mutually $*$-comaximal ideals. □

Finally, it's important to mention that not all p.o. monoids are pre-Riesz monoids. According to Proposition 4.2 of [42] The group of divisibility $G(D)$ of a domain D is pre-Riesz if and only if (P): for all $x_1, x_2, ..., x_n \in D\backslash\{0\}$, $(x_1, x_2, ..., x_n)_v = D$ or $(x_1, x_2, ..., x_n) \subseteq rD$ for some non unit $r \in D$. As we can readily see, a domain satisfying (P) above is a domain satisfying the PSP property and in a PSP domain every atom is a prime. Thus an atomic domain (every nonzero non unit is expressible as a product of atoms) with PSP property is a UFD. Thus, say, if D is a non UFD Noetherian domain then $G(D)$ is not pre-Riesz. It may be noted that the set of principal ideals is under multiplication is a submonoid of $\Gamma \cup \{D\}$.

4.1. **Riesz monoids.** First off let's note that when we say "monoid" we mean a commutative monoid. Now call a directed p.o. monoid $M =< M, +, 0, \leq >$ a sub-Riesz monoid, if every element x of M is primal i.e. for $y_1, y_2 \in M$, $x \leq y_1 + y_2$ $\Rightarrow x = x_1 + x_2$ such that $x_i \leq y_i$ and a Riesz monoid if M is also divisibility and cancellative.

One may ask whether Riesz monoids satisfy the Riesz interpolation, as do Riesz groups. The answer is yes and can be readily checked as we show below. Note that by M^+ we mean the set $\{x \in M | x \geq 0\}$

Theorem 4.4. *TFAE for a commutative cancellation divisibility monoid M. (1) Every $0 \leq x \in M$ is primal (2) For all $a, b, x, y \in M^+$ with $a, b \leq x, y$ there is z such that $a, b \leq z \leq x, y$, (3) For all $a, b, x_1, x_2, ..., x_n \in M^+$ with $a, b \leq x_1, x_2, ..., x_n$ there exists z such that $a, b \leq z \leq x_1, x_2, ..., x_n$, (4) For all $a_1, a_2, ..., a_n, b_1, b_2, ..., b_m \in M^+$ with $a_1, a_2, ..., a_n \leq b_1, b_2, ..., b_m$ there exists d such that $a_1, a_2, ..., a_n \leq d \leq b_1, b_2, ..., b_m$.*

Proof. (1) $\Rightarrow$ (2) Let every positive element of M be primal.

Let $a, b \leq x, y$. Then $x = x_1 + a = x_2 + b$ and $y = y_1 + a = y_2 + b$......(1)

Since $x_1 + a = x_2 + b$, $b \leq x_1 + a$ and since b is primal $b = b_1 + b_2$ where $b_1 \leq x_1$ and $b_2 \leq a$. (2)

Let $x_1 = x_1' + b_1$ and $a = a_1 + b_2$. Then $x_1 + a = x_2 + b$ can be written as $x_1' + b_1 + a_1 + b_2 = x_2 + b$, or $x_1' + a_1 + b_1 + b_2 = x_2 + b$. Noting that $b = b_1 + b_2$ and cancelling b from both sides we get $x_1' + a_1 = x_2$. ......(3)

Since $a_1 + b_2 = a$ we have $a, b \leq a_1 + b$........(4)

Using the value of x_2 we have $a_1 + b \leq x$. (Note: $x = x_2 + b = (x_1' + a_1) + b$) ... (5)

Now consider $y_1 + a = y_2 + b$. Using $a = a_1 + b_2$ and $b = b_1 + b_2$ we have $y_1 + a_1 + b_2 = y_2 + b_1 + b_2$. Cancelling b_2 from both sides we get $y_1 + a_1 = y_2 + b_1$. So that $b_1 \leq y_1 + a_1$ and as b_1 is primal we have $b_1 = b_3 + b_4$ where $b_3 \leq y_1$ and $b_4 \leq a_1$. Writing $y_1 = y_1' + b_3$ and $a_1 = a_1' + b_4$ we can express $y_1 + a_1 = y_2 + b_1$ as $y_1' + b_3 + a_1' + b_4 = y_2 + b_1$. Cancelling $b_1 = b_3 + b_4$ from both sides we get $y_2 = y_1' + a_1'$. This gives $y = y_2 + b = y_1' + a_1' + b = y_1 + a$. Now as $y_1' \leq y_1$ we get $y_1 = y_4 + y_1'$ which on substituting in $y_1' + a_1' + b = y_1 + a$ gives $y_1' + a_1' + b = y_4 + y_1' + a$ and cancelling y_1' we get $y_4 + a = a_1' + b$ and so $a \leq a_1' + b$. That is $a, b \leq a_1' + b$ and $a_1' + b \leq y$. But as $a_1' \leq a_1$ and $x_2 = x_1' + a_1$ we have $a_1' + b \leq x_2 + b = x$. So we have $z = a_1' + b$ such that $a, b \leq z \leq x, y$.

(2) $\Rightarrow$ (1). Let $a \leq b + c$.

Then as $a, b \leq b + c$, $a + b$ there is x such that $a, b \leq x \leq b + c$, $a + b$ ....(i)

Now as $a \leq x$ we have $x = x_1 + a$ ....(ii)
and as $b \leq x$ we have $x = x_2 + b$..........(iii)
Using (i) and (iii) $x_2 \leq a$ and $x_2 \leq c$. Now as $x_2 \leq a$, setting $a = x_3 + x_2$ we have from $x_1 + a = x_2 + b$, the equation $b = x_1 + x_3$. So $a \leq b + c$ implies that $a = x_2 a + x_3$, with $x_2, x_3 \in M^+$ such that $x_3 \leq b$ and $x_2 \leq c$.

(2) $\Rightarrow$ (3). Let $a, b \leq x_1, x_2, ..., x_n$. If $n = 2$ we have the result by (2). So suppose that $n > 2$ and suppose that for all $x_1, x_2, ..., x_{n-1}$ the statement is true. Then for $a, b \leq x_1, x_2, ..., x_{n-1}$ there is a d_1 such that $a, b \leq d_1 \leq x_1, x_2, ..., x_{n-1}$. But then for d_1, x_n there is d with $a, b \leq d \leq d_1, x_n$. But this d satisfies $a, b \leq d \leq x_1, x_2, ..., x_n$.

(3) $\Rightarrow$ (4). Let $a_1, a_2, ..., a_n \leq b_1, b_2, ..., b_m$. Then $a_1, a_2 \leq b_1, b_2, ..., b_m$ and so there is a d_1 such that $a_1, a_2 \leq d_1 \leq b_1, b_2, ..., b_m$. Now $d_1, a_3, ..., a_n \leq b_1, b_2, ..., b_m$ and induction on n completes the job. (4) $\Rightarrow$ (2). Obvious because (2) is a special case of (4). $\square$

Part (2) of Theorrem 4.4 is also called $(2,2)$ Riesz interpolation Property and (4) is (n, m) interpolaion for positive integral n and m.

Call a subset S of a monoid M conic if $x + y = 0$ implies $x = 0 = y$, for all $x, y \in S$. In a p.o. group G the sets G^+ and $-G^+$ are conic. If D is an integral domain then the set $m(D)$ of nonzero principal ideals of D is a monoid under multiplication, with identity D, ordered by $aD \leq bD \Leftrightarrow$ there is $c \in D$ such that $bD = acD \Leftrightarrow aD \supseteq bD$. The monoid $m(D)$ is cancellative too and in $m(D)$ $xDyD = 1 \Rightarrow xD = yD = 1$. So, $m(D)$ is a divisibility cancellative conic monoid. The monoid $m(D)$ is of interest because of the manner it generates a group. We know how the field of quotients of a domain is formed as a set of ordered pairs eah pair representing an equivalence class with $(a, b) = (c, d) \Leftrightarrow da = bc$ and then we represent the pair (a, b), $b \in D\backslash\{0\}$ by $\frac{a}{b} = ab^{-1}$. Now the group of $m(D)$ gets the form $G(D) = \{\frac{a}{b}D | \frac{a}{b} \in qf(D)\backslash\{0\}\}$, ordered by $\frac{a}{b}D \leq \frac{c}{d}D \Leftrightarrow \frac{a}{b}D \supseteq \frac{c}{d}D \Leftrightarrow$ there is $hD \in m(D)$ such that $\frac{a}{b}DhD = \frac{a}{b}D$, so that $m(D)$ is the positive cone of $G(D)$. The group $G(D)$ gets the name group of divisibility of D (actually of $m(D)$). Now any divisibility monoid that is also a cancellative and conic monoid M, with least element 0 can be put through a similar process of forming equivalent classes of ordered pairs to get group of divisibility like group $G(M) = \{a - b | a, b \in M\}$ with $x \leq y$ in $G(M) \Leftrightarrow x + h = y$ for some $h \in M$.

Corollary 6. *A Riesz Monoid M has the pre-Riesz property. Also M^+ is conic for a Riesz monoid M.*

Proof. Let $0 \leq x, y$ in M and suppose that there is $g \in M$ such that g is not greater than or equal to 0 yet $g \leq x, y$, that is $0, g \leq x, y$. Then by the $(2,2)$ interpolation property there is $r \in M$ such that $0, g \leq r \leq x, y$. But then $r > 0$, as $r \geq 0$ and $r \neq 0$ because $r \geq g$. Next suppose $x, y \geq 0$. If $x + y = 0$ and say $x \neq 0$, then we have $0, x \leq x, x + y$ and by the $(2,2)$ interpolation there is r such that $0 < x, x + y$ contradicting the fact that $x + y = 0$. $\square$

Well a p.o. monoid M is a p.o. group if every element of M has an inverse and obviously if a p.o. monoid is a Riesz monoid and a group it is a Riesz group. This brings up the question: Let M be a Riesz monoid and M^+ the positive cone of it, will M^+ generate a Riesz group? As we shall be mostly concerned with monoids M with 0 the least element, i.e. $M = M^+$ we remodel the question as: Let M be a Riesz monoid with $M^+ = M$ the positive cone of it, will M generate a Riesz group? The following result whose proof was indicated to me by G.M. Bergman, in an email, provides the answer.

Theorem 4.5. *Suppose M is a cancellative abelian monoid, which is "conical", i.e., no two nonidentity elements sum to 0, and which we partially order by divisibility; and suppose every element of M is primal, namely, that with respect the divisibility order, (1) $x \leq a+b => x = u+v$ such that $u \leq a$ and $v \leq b$. Then the group generated by M is a Riesz group.*

Proof. Let us rewrite (1) by translating all the inequalities into their divisibility statements; so that $x \leq a+b$ becomes $x+y = a+b$ for some y and $u \leq a$ becomes $a = u+u\prime$, and similarly for the last inequality; and finally, let us rename the elements more systematically; in particular, using a, b, c, d for the abovex, y, a, b. Then we find that (1) becomes $a+b = c+d \Rightarrow a = a\prime + a", c = a\prime + b\prime, d = a" + b"$ for some $a\prime, a", b\prime, b" \in M$. Now if we substitute the three equations to the right of the "$\Rightarrow$" into the equation before the "$\Rightarrow$", and use cancellativity, we find that $b = b\prime + b"$; so the full statement is (2) $a+b = c+d \Rightarrow a = a\prime + a",\ b = b\prime + b",\ c = a\prime + b\prime,\ d = a" + b"$, for some $a\prime, a", b\prime, b" \in M$. Now let G be the group generated by M, ordered so that M is the positive cone. We want to show G has the Riesz Interpolation property. So suppose that in G we have $p, q \leq r, s$. We can write these inequalities as (3) $r = p+a, s = p+c, r = q+d, s = q+b$ where $a, b, c, d \in M$. Now the sum of the first and last equations gives a formula for $r+s$, and so does the sum of the second and third equations.Equating the results, and cancelling the summands $p+q$ on each side, we get an equation in $M : a+b = c+d$. Hence we can apply (2) to get decompositions of a, b, c, d, and substitute these into (3), getting (4) $r = p+a\prime+a", s = p+a\prime+b\prime, r = q+a"+b", s = q+b\prime+b"$. Equating the first and third equations (or if we prefer, the second and fourth) and cancelling the common term $a"$ (respectively, the common term $b\prime$), we get (whichever choice we have made) (5) $p+a\prime = q+b"$.The element given by (5) is clearly $\geq p, q$, while from (4) (using whichever of the equations for r we prefer and whichever of the equations for s we prefer), we see that it is $\leq r, s$. So this is the element whose existence is required for the $((2,2))$ Riesz interpolation property for G. $\square$

A fractional ideal I is called $*$-invertible if $(II^{-1})^* = D$. It is well known that if I is $*$-invertible for a finite character star operation $*$ then I^* and I^{-1} are of finite type. Denote the set of all $*$-invertible fractional $*$-ideals of D by $Inv_*(D)$ and note that given an integral ideal I it is possible that I cannot always be expressed as a product of integral ideals. So when we talk about an integral $*$-

invertible $*$-ideal we are talking about the end result and not how it is expressed. Let $\mathcal{I}_*(D)$ be the set of integral $*$-invertible $*$-ideals and note that $\mathcal{I}_*(D)$ is a monoid under $*$-multiplication. Note that $\mathcal{I}_*(D)$ can be partially ordered by $I \leq J$ if and only if $I \supseteq J$. Indeed $J \subseteq I$ if and only if $(JI^{-1})^* = H \subseteq D$, if and only if $J = (IH)^*$, and as J, I are $*$-invertible, H is $*$-invertible and integral. Thus in $\mathcal{I}_*(D)$, $I \leq J \Leftrightarrow J = (IH)^*$ for some $H \in \mathcal{I}_*(D)$. In other words $\mathcal{I}_*(D)$ is a divisibility p.o. monoid. Because $\mathcal{I}_*(D)$ involves only $*$-invertible $*$-ideals, it is cancellative too. Finally $\mathcal{I}_*(D)$ is directed because of the definition of order. That $Inv_*(D)$ is generated by $\mathcal{I}_*(D)$ follows from the fact that every fractionary ideal of D can be written in the form A/d where $A \in F(D)$ and $d \in D\backslash\{0\}$. Finally, the partial order in $Inv_*(D)$ gets induced by $\mathcal{I}_*(D)$ in that for $I, J \in Inv_*(D)$

we have $I \leq J \Leftrightarrow J \subseteq I \Leftrightarrow (JI^{-1})^* \in \mathcal{I}_*(D)$. Call $I \in \mathcal{I}_*(D)$ $*$-primal if for all $J, K \in \mathcal{I}_*(D)$ $I \leq (JK)^*$ we have $I = (I_1I_2)^*$ where $I_1^* \leq J$ and $I_2^* \leq K$. Call D $*$-Schreier, for star operation $*$ of finite character, if every integral $*$-invertible $*$-ideal of D is $*$-primal.

Proposition 13. *Let $*$ be a finite character star operation defined on D. Then D is a $*$-Schreier domain if and only if $Inv_*(D)$ is a Riesz group under $*$-multiplication and order defined by $A \leq B \Leftrightarrow A \supseteq B$.*

Proof. Suppose that D is $*$-Schreier, as defined above. That is each $I \in \mathcal{I}_*(D)$ is primal. The notion of $*$-Schreier suggests that we define $\leq$ by $A \leq B \Leftrightarrow A \supseteq B$. Then as for each pair of integral ideals IJ, $(IJ)^* = D \Rightarrow J^* = I^* = D$, the same holds for members of $\mathcal{I}_*(D)$ which are all $*$-ideals. So $(IJ)^* = D \Rightarrow I = J = D$. and so $\mathcal{I}_*(D)$ is conic. Of course $\mathcal{I}_*(D)$ is cancellative by the choice of ideals and by the definition of ordr $\mathcal{I}_*(D)$ is a divisibility monoid. So by Theorem 4.5 $\mathcal{I}_*(D)$ generates a Riesz group and by the above considerations $Inv_*(D)$ is generated by $\mathcal{I}_*(D)$. Consequently $Inv_*(D)$ is a Riesz group. Conversely if $Inv_*(D)$ is a Riesz group, with that order defined on it, then $\mathcal{I}_*(D)$ is the positive cone of the Riesz group $Inv_*(D)$ and so each element of $\mathcal{I}_*(D)$ must be primal. $\square$

Proposition 13 brings together a number of notions studied at different times. The first was quasi-Schreier, study started in [23] and completed in [6]. The target in these papers was studying $\mathcal{I}_d(D)$, i.e. the monoid of invertible integral ideals of D, when $Inv_*(D)$ is a Riesz group. Another study targeting $\mathcal{I}_t(D)$, i.e. the monoid of t-invertible integral t-ideals of D, for study along the same lines as above appeared in [25].

Now let's step back and require that every $*$-invertible $*$-ideal of D be principal. Then in Proposition 13, $\mathcal{I}_*(D)$ is the monoid of principal ideals, each of which is primal and the Riesz group $Inv_*(D)$ of consists just of principal fractional ideals of D, and hence the group of divisibility of D. It is well known that if $*$ is of finite type each $*$-invertible $*$-ideal is a t-invertible t-ideal ([46]) and that in a pre-Schreier domain each t-invertible t-ideal is principal ([45, Theorem 3.6]). So we have the following corollary.

Corollary 7. *Let D be $*$-Schreier for any star operation $*$ of finite character. Then D is pre-Schreier if and only if each element of $\mathcal{I}_*(D)$ is principal.*

Proof. Suppose that eah member of $\mathcal{I}_*(D)$ is principal then in $\mathcal{I}_*(D)$. Then for $a, b, c \in D \backslash \{0\}$ we have $aD, bD, cD \in \mathcal{I}_*(D)$ and for $a|bc$ in D would be $aD \leq bDcD$ and in $\mathcal{I}_*(D)$ we must have $aD = (I_1I_2)^*$ where $I_1 \leq bD$ and $I_2 \leq cD$. But I_i being in $\mathcal{I}_*(D)$ must be principal. So,say, $I_i = a_iD$. But this gives $a = a_1a_2$ and $a_1D \leq bD, a_2D \leq cD$ gives $a_1|b, a_2|c$. In sum for all $a, b, c \in D\backslash\{0\}$ $a|bc \Rightarrow a = a_1a_2$ where $a_1|b$ and $a_2|c$ which is a way of saying that every nonzero element of D is primal. Conversely as indicated earlier D being pre-Schreier makes each $*$-invertible $*$-ideal of D principal and consequently all members of $\mathcal{I}_*(D)$ principal. $\square$

This brings us to the last item on the "agenda". In 1998, Professor Halter-Koch wrote a book, [31] and restated all the then known conepts of multiplicative ideal theory for monoids, in terms of ideal systems, except for one, he did not include a Schreier monoid nor a pre-Schreier monoid. Provided below is one of the missing definitions.

Definition 4.6. A conic, cancellative divisibility monoid $< M, \bullet, 1, \leq >$is a pre-Schreier or a Riesz monoid if every element of M is primal.

To end it all let's note, as Professor Halter-Koch would have, that an integral domain D all nonzero elements of whose multiplicative monoid are primal is pre-Schreier if $\leq$ is replaced by $|$.

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