ON A RESULT OF GILMER

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The letter D denotes a commutative integral domain throughout. We say that D is locally X if D_M is X for each maximal ideal M of D.

In [**2**], Robert Gilmer proves: *if D is locally factorial and semi-quasi-local, then D is a* UFD. (Note that *factorial ring* is another name for a UFD.)

In this note we prove the following generalization of the above mentioned result of Gilmer.

THEOREM 1. *Let D be a locally factorial integral domain with the property that every non-zero non-unit of D is in a finite number of maximal ideals of D. Then D is a* UFD *if and only if every invertible ideal of D is principal.*

We begin by indicating the proofs of two lemmas on GCD domains.

LEMMA 1. *Let D be locally* GCD. *Then D is a* GCD *domain if and only if (i) $aD \cap bD$ is finitely generated for all a, b in D,* (ii) *every invertible ideal in D is principal.*

Note that D is a GCD domain if every two elements of D have a *greatest common divisor* (GCD). Further, it can be easily verified that D is a GCD domain if and only if $aD \cap bD$ is principal for all a, b in D.

Proof. If D is a GCD domain (i) is obvious and (ii) can be verified easily (*cf.* [**3**; p. 45, Ex. 15]). Conversely, let D satisfy the main hypothesis and (i) and (ii) above and let $a, b \in D$. Then for each maximal ideal M, $(aD \cap bD)D_M = aD_M \cap bD_M$ is principal because D_M is a GCD domain. Now, by (i) above and Theorem 62 of [**3**], $aD \cap bD$ is invertible, while by (ii) above it is principal. Finally, a, b being arbitrary, the result is obvious.

It is well known that $D = \bigcap D_M$, where M ranges over all maximal ideals of D. We call $\bigcap D_M$ the *local representation* of D. We say that the local representation of D is of *finite character* if each non-zero non-unit of D is in only finitely many maximal ideals of D.

LEMMA 2. *Let D be locally* GCD. *If the local representation of D is of finite character and if every invertible ideal of D is principal, then D is a* GCD *domain.*

Proof. Let a, b be an arbitrary pair of elements of D. We show that $aD \cap bD$ is finitely generated. For this we note that if ab is a non-zero non-unit, it belongs to finitely many maximal ideals and that $(aD \cap bD)D_M = aD_M \cap bD_M$ is principal for each maximal ideal M. To complete the proof it is sufficient to cite Lemma 37.3 of [**1**]. Stated for integral domains, this lemma reads as follows:

Let $x \in D$ such that x belongs to finitely many maximal ideals $M_1, M_2, \ldots, M_n$ of D. If A is an ideal of D such that A contains x and if AD_{M_i} is finitely generated for each i between 1 and n, then A is finitely generated.

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Proof of Theorem 1. Clearly D is a GCD domain. The proof consists in showing that D is a Krull domain (because D is UFD if and only if it is a Krull as well as a GCD domain). We note that every minimal prime P of D is contained in some maximal ideal M. So $D_P = (D_M)_{PD_M}$ is a discrete valuation ring. Further, since each of D_M is Krull (being a UFD), $D = \bigcap D_P$, where P ranges over all minimal primes of D. Finally, the finite character of the local representation of D implies that every element of D belongs to finitely many minimal primes of D.

It is easy to verify that if D is locally Krull then D is Krull if and only if every principal ideal of D has finitely many minimal primes. With this result and Lemma 1 in view, we can state the following result.

THEOREM 2. *Let D be locally factorial. Then D is factorial if and only if*

(1) *$aD \cap bD$ is finitely generated for all a, b in D,*

(2) *every invertible ideal, in D, is principal,*

(3) *every principal ideal of D has finitely many minimal primes.*

Remark 1. We note that Lemma 37.3 of **[1]** is a very strong result and it can be put to use in a number of ways. In the following, we write down two statements which are immediate consequences of this lemma.

I. *Let D be locally noetherian. Then D is noetherian if it has local representation of finite character.*

II. *Suppose that D has local representation of finite character. If D is locally a* GCD *domain of finite character and if every invertible ideal of D is principal, then D is a* GCD *domain of finite character.*

Remark 2. In view of Lemmas 1 and 2 one may ask, "*What behaviour of the elements of D does sufficiently indicate that D is locally GCD*?" The answer, in view of Lemma 1, is not very hard to find. Using Theorem 62 of [3], we can verify that if, for all a, b in D, $aD \cap bD$ is invertible then D is locally GCD. This condition, it may be noted, is necessary and sufficient in the following two cases:

(1) that in which D has local representation of finite character (*cf.* Lemma 2),

(2) that where $aD \cap bD$ is finitely generated for all $a, b \in D$ (*cf.* Lemma 1).

The following statement is an easy corollary of (2): *a noetherian domain is locally factorial if and only if $aD \cap bD$ is invertible for all a,b in D.*

References

1. R. W. Gilmer, *Multiplicative ideal theory* (Marcel Dekker, New York, 1972).
2. R. W. Gilmer, " A note on unique factorization ", *Delta (Waukesha)*, 3 (1972), 7–8.
3. I. Kaplansky, *Commutative rings* (Allyn and Bacon, Boston, 1970).

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