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ON t-LINKED OVERRINGS

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1. Introduction

Let R be a (commutative integral) domain, with integral closure R', complete integral closure R^*, and quotient field K. As in [DHLZ], we say that an overring (R-subalgebra of K) T is <u>t-linked</u> (<u>over</u> R) if $I^{-1} = R$ for a nonzero finitely generated ideal I of R implies $(IT)^{-1} = T$. The t-linked concept was used in [DHLZ, Theorem 2.10 and Corollary 2.18] to find characterizations of certain classes of PVMD's analogous to characterizations of Prüfer domains due to Davis and Richman. According to [DHLZ, Corollary 2.3], R^* is t-linked over R for each domain R. It was shown in [DHLZ, Corollary 2.14(a)] that R' is t-linked over R if R is a Noetherian (or, more generally, a quasicoherent) domain, but [DHLZ] left open the question whether R' is t-linked over R for each domain R. The first major result of the present paper, Theorem 2.4, implies, i.a., that R' is t-linked over R if R' is a Prüfer domain (see Corollary 2.5). (The condition that R' be Prüfer has often been characterized, and many sufficient conditions for it are known. Just a few of these are cited in Section 2.) A number of related results (Theorem 2.4, Proposition 2.7) give new characterizations of R' being Prüfer. Another major result, Example 4.1, settles the open question from [DHLZ], by presenting a seminormal domain R such that R' is not t-linked over R. (Recall that a domain R is seminormal if R contains each element x in its quotient field with x^2 and x^3 in R. For seminormality see, e.g., [S].)

The work in Section 2 depends crucially on the class of domains R for which $R \subseteq T$ is t-linked for each overring T of R. We shall call these rings t-linkative. The class of t-linkative domains was introduced (but not named) in [DHLZ, Theorem 2.6], and Lemma 2.1 recalls several characterizations. Natural examples include arbitrary treed domains (in the sense of [D_1]), in particular, arbitrary Prüfer domains or domains of (Krull) dimension at most 1. It will be convenient to say that a domain R is super-t-linkative in case each overring of R is t-linkative. Corollary 2.5 establishes that R is super-t-linkative in case R' is a Prüfer domain. We characterize the super-t-linkative domains in the class of Noetherian domains (Proposition 3.13), in the class of one-dimensional domains

(Proposition 3.12) and in the class of pseudo-valuation domains (PVD's), in the sense of [HH] (Corollary 3.6).

Additional examples of t-linkative and super-t-linkative domains are found in Section 3, where the behavior of these properties is studied under certain pullback constructions. The major result of this section, Theorem 3.5, includes, in particular, the $D + M$ construction, in the sense of [BR].

All the rings in this paper are commutative with identity and all the ideals are supposed to be nonzero. An ideal I of a domain R is called unitary if $I_v = R$, equivalently if $(R:I) = R$. In addition to the notation introduced above, for any domain R with quotient field K and a polynomial f in K[X], we let c(f) denote the content of f over R, and we let $\dim_v(R)$ denote the valuative dimension of R. If $R \subseteq T$ are domains, we denote by $td_R(T)$ the transcendence degree of the quotient field of T over that of R; we say that T is an algebraic extension of R if $td_R(T) = 0$. As for background, we assume familiarity with [DHLZ] and associated t-theoretic notions. Unreferenced material is standard, typically as in $[G_1]$.

2. Prüfer integral closure and super-t-linkative domains

It is convenient to begin by recalling some characterizations of t-linkative domains.

LEMMA 2.1. ([DHLZ, Theorem 2.6]) For a domain R, the following six conditions are equivalent:

(1) Each overring of R is t-linked over R;

(2) Each valuation overring of R is t-linked over R;

(3) Each nonzero maximal ideal of R is a t-ideal;

(4) Each proper nonzero ideal I of R satisfies $I_t \neq R$;

(5) Each proper nonzero finitely generated ideal I of R satisfies $I_t \neq R$;

(6) Each t-invertible ideal of R is invertible.

Note that a domain R is t-linkative if and only if every factor ring of R by a principal ideal has the property that each proper finitely generated ideal is annihilated by a nonzero element (see, e.g., [CR]).

It is apparent that t-linkative domains are ubiquitous. Indeed, if P is an associated prime of a principal ideal of a domain R (as in the sense of [BH]), then R_P is t-linkative. As every domain is an intersection of such localizations, every domain is an intersection of t-linkative ones (see [BH], especially Proposition 4). Moreover, if R is any domain and $\underline{X}$ is a set of indeterminates over R, then the domain $R[\underline{X}]_{N_v}$, as defined in [K], is t-linkative by [K, Proposition 2.1 (2) and Corollary 2.3 (3)]. In particular, if R is a t-linkative domain, then the Nagata ring $R(\underline{X})$ is also t-linkative.

As promised in the introduction, we now consider the condition on a domain R that R′ be a Prüfer domain. Numerous characterizations of this property have been given (cf. [GH, Theorem 5] (see also [D_2, Corollary 5]), [ES, Theorem 2], [P, Proposition 2.26], [ADF, Theorem 2.7]); and several sufficient conditions for this property are known (cf. [G_2, Corollary 3], [AADH, Theorem 2.1]). We next give another useful characterization, by removing the irreducibility condition in [ADF, Theorem 2.7].

LEMMA 2.2. Let X be an indeterminate over a domain R. Let K be the quotient field of R. Then the following conditions are equivalent:

(1) For each nonzero $f \in R[X]$, there exists $g \in fK[X] \cap R[X]$ with $c(g) = R$;

(2) R' is a Prüfer domain.

Proof. According to [ADF, Theorem 2.7], R' is a Prüfer domain if and only if, for each maximal ideal M of R, no upper to zero in R[X] is contained in M[X]. By [G_1, Proposition 33.1 (1)], this last condition is equivalent to requiring that each upper to zero in R[X] contain a polynomial g such that $c(g) = R$. Now, the uppers to zero are the ideals $fK[X] \cap R[X]$ arising from irreducible $f \in K[X]$. Accordingly, (1) $\Rightarrow$ (2) follows trivially. Conversely, assume (2) and consider nonzero $f \in R[X]$. Factor $f = \prod f_i$, with each f_i irreducible in K[X]. Let $Q_i = f_iK[X] \cap R[X]$. Since Q_i is an upper to zero, (2) produces $g_i \in Q_i$ with $c(g_i) = R$. Put $g = \prod g_i$. Since the set of polynomials of unit content is multiplicatively closed, we obtain that $c(g) = R$. Moreover, $g \in fK[X] \cap R[X]$, yielding (1). □

Another characterization of Prüfer integral closure is given in Theorem 2.4 below. We first prove a proposition which is well known but is included here for lack of a convenient reference.

PROPOSITION 2.3. Let R be a domain with quotient field K and f a nonzero polynomial in R[X]. Then $c(f)_v = R \Leftrightarrow fK[X] \cap R[X] = fR[X]$.

Proof. (⇒) Consider $h \in K[X]$ such that $fh \in R[X]$. By the content formula [G_1, Corollary 28.3], there is an integer $m \geq 1$ such that $c(f)^{m+1}c(h) = c(f)^m c(fh) \subseteq R$, the inclusion holding since f, $fh \in R[X]$. Applying the v-operation and observing that $(c(f)^i)_v = R$ for all $i \geq 1$, we have $c(h)_v = c(fh)_v \subseteq R$. In particular, $c(h) \subseteq R$; that is, $h \in R[X]$. We conclude that $fK[X] \cap R[X] = fR[X]$.

(⇐) Assume that $fK[X] \cap R[X] = fR[X]$. We show that $c(f)^{-1} = R$. Accordingly, let $u \in K$ with $uc(f) \subseteq R$. Then $uf \in fK[X] \cap R[X] = fR[X]$; so $u \in R$, as desired. □

The statement of the next theorem depends on the following definition from [HZ]. A domain R is called a UMT-domain in case each upper to zero in R[X] is a maximal t-ideal.

THEOREM 2.4. For a domain R, the following conditions are equivalent:

(1) R is a t-linkative UMT-domain;

(2) R' is a Prüfer domain.

Proof. In view of Lemma 2.1, an appeal to [HZ, Theorem 3.5] yields both the implication (1) ⇒ (2) and the fact that (2) implies R is a UMT-domain. It remains only to show that (2) implies R is t-linkative. To this end, we consider a finitely generated unitary ideal I of R. Pick $f \in R[X]$ such that $c(f) = I$. Assuming (2), we infer from Lemma 2.2 that $c(g) = R$ for some $g \in fK[X] \cap R[X]$. Since $(c(f))_v = R$, we have by Proposition 2.3 that $fK[X] \cap R[X] = fR[X]$. Hence $g = fh$ for some polynomial h in R[X]. Thus $R = c(g) \subseteq c(f) = I \subseteq R$, so $I = R$. We conclude by Lemma 2.1 that R is t-linkative. □

COROLLARY 2.5. Let R be a domain such that R' is a Prüfer domain. Then R is super-t-linkative. In particular, R' is t-linked over R.

Proof. Each overring S of R inherits the property of having Prüfer integral closure (cf. [G_1, Theorem 26.1 (1)]). Thus S is t-linkative by Theorem 2.4, and so R is super-t-linkative. □

REMARK 2.6. (a) The converse of Corollary 2.5 is false. Indeed, an integrally closed super-t-linkative domain is not necessarily Prüfer. To see this, let X be an indeterminate over a field F, and consider a nontrivial valuation domain (V, M) of the form $V = F(X) + M$. Then $R = F + M$ is integrally closed, and by Theorem 3.5 (5) or Corollary 3.6 below, R is super-t-linkative. Nevertheless, R is not a Prüfer domain by [BG, Theorem 2.1 (i)]).

(b) In Corollary 2.5 we cannot replace "Prüfer" by "PVMD" since a PVMD is t-linkative if and only if it is Prüfer: if R is a t-linkative PVMD, then, by Lemma 2.1, each maximal ideal of R is a t-ideal; since R is a PVMD, we obtain that R_M is a valuation domain for each maximal (t-)ideal M, and so R is Prüfer.

On the other hand we do not know whether R' being a PVMD implies that R' is t-linked over R.

(c) The quasilocal case of Theorem 2.4 is worth mentioning. Thus let R be a quasilocal domain, not a field, whose maximal ideal is a t-ideal (so that R is t-linkative). Then R$'$ is Prüfer if and only if R is a UMT-domain.

By [HH], a domain R is a *pseudo-valuation domain* (for short, PVD) in case Spec(R) = Spec(V) as sets for some (canonically associated) valuation overring V of R. According to [AD_1, Proposition 2.6], PVD's may be characterized as the pullbacks of the form $V \times_L F$, where V is a valuation domain with residue field L and F is a subfield of L. In [AADH, Theorem 2.1], the property of having Prüfer integral closure was characterized using the PVD concept. A companion result is given next.

PROPOSITION 2.7. For a domain R, the following conditions are equivalent:

(1) Each integrally closed PVD overring of R is a valuation domain;

(2) R$'$ is a Prüfer domain.

Proof. (2) $\Rightarrow$ (1): Assume (2). Then each integrally closed PVD overring S of R is a quasilocal overring of the Prüfer domain R$'$, and so S is a valuation domain.

(1) $\Rightarrow$ (2): Without loss of generality, $R = R'$ is integrally closed and quasilocal, say with maximal ideal M. If the assertion fails, there is a valuation overring V of R with prime ideals $Q \subsetneq N$ of V such that $Q \cap R = M$ ($= N \cap R$) (cf. [G_1, Theorem 19.15 (1)$\Leftrightarrow$(7)]). Put $L = V_Q/Q$, the quotient field of V/Q; and let F denote the algebraic closure of R/M in L. By the above comments, the pullback $T = V_Q \times_L F$ is a PVD. Moreover, by a direct calculation or an appeal to [F, Corollary 1.5(5)], T is integrally closed. Notice also that V_Q is a dominating overring of T;

indeed, they have the same maximal ideal. Hence, to show that T is not a valuation domain (and thus complete the proof), it suffices to prove that $F \neq L$, that is, that L is not an algebraic extension of R/M. This follows from the fact that the domain V/Q is not a field and lies between R/M and L (alternatively, cf. Lemma 3.10 below). □

Proposition 2.8 (b) below is a generalization of Corollary 2.5.

PROPOSITION 2.8. Let R be a domain such that S' is a Prüfer domain for each proper overring S of R. Then:

(a) If R is integrally closed, then either R is a PVD or R is a Prüfer domain.

(b) R is super-t-linkative.

Proof. (a) Without loss of generality, R is not a Prüfer domain. Hence R is quasilocal. (Otherwise, for each maximal ideal M of R, R_M is a quasilocal integrally closed proper overring, hence a valuation domain by the hypothesis; this would contradict R not being a Prüfer domain.) By Proposition 2.7, R has an integrally closed PVD overring S which is not a valuation domain. Being quasilocal, S is not a Prüfer domain. Thus, by the hypothesis, S is not a proper overring of R; that is, R = S is a PVD.

(b) If R is not integrally closed, the hypothesis yields that $R' = (R')'$ is a Prüfer domain, whence R is super-t-linkative by Corollary 2.5. If R is integrally closed, then, by (a), R is treed and so it is t-linkative by [DHLZ, Corollary 2.7]. Moreover, Corollary 2.5 yields that each proper overring of R is t-linkative. Thus R is super-t-linkative. □

REMARK 2.9. (a) In view of Proposition 2.8(a), it seems important to note that not every PVD is a super-t-linkative domain. To see this, let X, Y be independent indeterminates over a field F, and consider a nontrivial valuation domain (V, M) of the form $V = F(X,Y)+M$. Then $R = F+M$ is a PVD since $R = V \times_{F(X,Y)} F$. However, by direct calculation or an appeal to Corollary 3.6 below, one sees readily that R is not super-t-linkative.

(b) The converse of Proposition 2.8(b) is false even for PVD's. To see this, let $k \subsetneq F$ be fields such that F is an algebraic extension of k. Let (V,M) be a nontrivial valuation domain of the form $V = F(X)+M$. Then $A = k+M$ is a PVD which is super-t-linkative (by Corollary 3.6 below, since $td_k(F(X)) = 1$), but $F+M$ is a proper integrally closed overring of A which is not a Prüfer domain.

3. Pullbacks and super-t-linkative domains

In view of the last two results of the previous section, it is natural to seek to characterize the PVD's that are also super-t-linkative. We do so in Corollary 3.6, after first studying the question for the more general context of pullbacks (Theorem 3.5 below). We begin with a useful lemma.

LEMMA 3.1. For a domain R, the following conditions are equivalent:

(1) R is super-t-linkative;

(2) R_M is super-t-linkative for each maximal ideal M of R;

(3) Each quasilocal overring of R is t-linkative.

Proof. $(1) \Rightarrow (2)$ is trivial, and it is easy to show $(2) \Rightarrow (3)$. To prove $(3) \Rightarrow (1)$, assume (3), and consider any overring T of R. If N is a maximal ideal of T, then NT_N is a t-ideal of T_N, by (3). It is well known (and easy to show) that t-ideals of T_N contract to t-ideals of T, whence N is a t-ideal of T. Thus T is t-linkative, yielding (1). □

REMARK 3.2. Although super-t-linkativity is a local property, the same cannot be said for t-linkativity. While it is true that a domain which is locally t-linkative is itself t-linkative, the converse is false. To see this, consider the ring D in [MZ_1, Example 2.1]. Using the description of the maximal ideals of D given there, it is easy to show that this ring is a t-linkative domain having a maximal ideal M such that D_M is not t-linkative.

PROPOSITION 3.3. Let $D \subseteq E$ be domains.

(a) Let D be a field. Then every domain between D and E is t-linkative $\Leftrightarrow td_D(E) \leq 1$.

(b) If D is not a field and every domain between D and E is t-linkative, then E is an algebraic extension of D.

Proof. If D is not a field and if $x \in E$ is transcendental over D, then D[x] is not t-linkative (since for any nonzero nonunit $a \in D$, (a,x) is a proper unitary ideal of D[x]), proving (b).

Suppose D is a field. If $tr_D(E) \geq 2$, let x and y be two D-algebraically independent elements in E. Then (as in the above argument), the domain

$D[x,y] = D[y][x]$ is not t-linkative. This proves the direction ($\Rightarrow$) of (a). To prove ($\Leftarrow$) , note that in case $td_D(E) \leq 1$, the Krull-Akizuki theorem yields that any ring between D and E is one-dimensional, and is therefore t-linkative by [DHLZ, Corollary 2.7]. □

LEMMA 3.4. (cf. [BG, Theorem 3.1]). Let T be a domain, M a maximal ideal of T such that T_M is a valuation domain, and R a subring of T containing M. Then every quasilocal overring of R either contains T or lies between $R + MT_M$ and T_M.

Proof. Let (S,N) be a quasilocal overring of R. If $MS = S$, then S is an overring of T because $T \subseteq TS = TMS = MS = S$. If $MS \neq S$, then $MS \subseteq N$, and each element of $1 + M$ is a unit in S. Since T_M is the quotient ring of T with respect to the multiplicative subset $1 + M$, it follows that $MT_M \subseteq S$, and so S is an overring of $R + MT_M$. If S is not contained in T_M, then, since T_M is a valuation domain, there exists a nonzero element m in M such that $1/m \in S$. Thus $T_M = (1/m)mT_M \subseteq (1/m)MT_M \subseteq S$. □

THEOREM 3.5. Let T be a domain, M a maximal ideal of T and R a subring of T containing M. Set $D = R/M$ and $L = T/M$. Let K and F be the quotient fields of R and D respectively. Then:

(1) If R is t-linkative, then so is D.

(2) If R is t-linkative and M is a t-ideal of T, then T is t-linkative.

(3) If D and T are t-linkative domains, then so is R.

(4) Assume that T is t-linkative. Then each domain between R and T

is t-linkative $\Leftrightarrow$ each domain between D and L is t-linkative. If, furthermore, D is a field, then this property is equivalent to $td_D(L) \leq 1$.

(5) Assume that T is super-t-linkative and that T_M is a valuation domain. Then R is super-t-linkative $\Leftrightarrow$ each domain between D and L is t-linkative. If, furthermore, D is a field, then this condition is equivalent to the condition $td_D(L) \leq 1$.

(6) Assume that D is not a field and $td_K(L) \geq 1$. Then R is not super-t-linkative.

Proof. (1) Let J be a proper finitely generated ideal of D. Thus $J = (I+M)/M$ for some finitely generated ideal I of R. As $I \not\subseteq M$, we have $IT + M = T$. Hence, there is an element m in M such that $IT + Tm = T$. Let $I_0 = I + Rm$. Then I_0 is a finitely generated ideal of R, $I_0T = T$, and since $(I_0 + M)/M = J$, I_0 is a proper ideal of R. Since R is t-linkative, there is an element x in $K \setminus R$ such that $xI_0 \subseteq R$. We have $xT = xI_0T \subseteq T$; so $x \in T$. Thus $x + M$ is an element of $L = T/M$, and $x + M \in J^{-1} \setminus D$. It follows that D is t-linkative.

(2) Let J be a finitely generated proper ideal of T; we shall show that $J^{-1} \neq T$. If $J \subseteq M$, then $J^{-1} \neq T$ since M is a t-ideal. We may therefore assume $J \not\subseteq M$. Thus $J + M = T$, and so J contains an element of the form $1 + m$, where $m \in M$. Let $\{t_1, \dots, t_k\}$ be a set of generators for J. Let $S = \{1 + m, mt_1, \dots, mt_k\}$. Since $mt_i \equiv -t_i \bmod (T(1 + m))$ for all i, we see that S is a set of generators for J contained in R. Since R is t-linkative, there exists an element x in $K \setminus R$ such that $xS \subseteq R$. If $x \in T$, then $xm \in M$, so $x = x(1 + m) - xm \in R$, a contradiction. It follows that $x \notin T$. Hence T is t-linkative.

(3) Let I be a proper finitely generated ideal of R. If $I + M \neq R$, then $(I + M)/M$ is a proper ideal of $D = R/M$. Since $F \subseteq T/M$ and D is t-linkative, there exists an element t in $T \setminus R$ such that $tI \subseteq R$. Thus $I^{-1} \neq R$.

Now assume that $I + M = R$. The assumption $IT = T$ leads to the contradiction $M \subseteq I$. Thus $IT \neq T$. Since T is t-linkative, there exists an element x in $L \setminus T$ such that $xI \subseteq T$. If $xM \subseteq T$, we obtain that $x \in x(I+M) \subseteq T$, a contradiction. Hence $xM \not\subseteq T$. Pick an element m in M such that $xm \notin R$. Thus $(xm)I = (xI)m \subseteq Tm \subseteq M \subseteq R$, and so $xm \in I^{-1}$. We conclude that R is t-linkative.

(4) The equivalence $\Leftrightarrow$ in (4) follows from (1) and (3). For the second part, use Proposition 3.3 (a).

(5) We have $D = R/M = (R + MT_M)/MT_M$ and $L = T_M/MT_M$; so, by (4), each domain between D and L is t-linkative if and only if each domain between $R + MT_M$ and T_M is t-linkative. This condition clearly holds if R is super-t-linkative. For the converse, we obtain by Lemma 3.4 that any quasilocal overring of R is t-linkative; so by Lemma 3.1 we conclude that R is super-t-linkative. In case D is a field, we use Proposition 3.3 (a) for the final assertion.

(6) By Proposition 3.3 (b), there exists a domain E between D and L which is not t-linkative. Let $E = S/M$, where S is a ring between R and T. By (1), S is not t-linkative, and so R is not super-t-linkative. □

As a particular case of Theorem 3.5 (5) we obtain

<u>**COROLLARY 3.6**</u>. Let R be a PVD, not a field, with maximal ideal M and canonically associated valuation overring V. Put $F = R/M$ and $L = V/M$. Then R is super-t-linkative if and only if $td_F(L) \leq 1$. □

<u>**COROLLARY 3.7**</u>. Let R be a super-t-linkative domain, T an overring of R and Q a prime ideal of T. Then $td_{R/Q \cap R}(T/Q) \leq 1$. If $Q \cap R$ is not maximal in R, then T/Q is algebraic over $R/Q \cap R$.

<u>Proof.</u> We have that $R/Q \cap R = (R + QT_Q)/QT_Q$ is contained in T_Q/QT_Q. In order to conclude the proof, apply Theorem 3.5 (4) and (6) with R and T replaced by $R + QT_Q$ and T_Q, respectively. □

Obviously, an overring of a super-t-linkative domain is super-t-linkative. A less obvious closure property of the class of super-t-linkative domains is given by the next corollary.

<u>**COROLLARY 3.8**</u>. If P is a prime ideal of a super-t-linkative domain R, then R/P is super-t-linkative.

<u>Proof.</u> Since the quotient field of R/P is R_P/PR_P, any overring E of R/P is of the form $E = S/PR_P$, where S is an overring of $R + PR_P$ contained in R_P. Since S is a t-linkative domain, we may apply Theorem 3.5 (1) for R, T, M and D replaced by S, R_P, PR_P and E, respectively to obtain that E is t-linkative. Thus R/P is super-t-linkative. □

<u>**REMARK 3.9**</u>. (a) The assumption that T_M is a valuation domain used in Theorem 3.5 (5) holds in the classical $D + M$ construction (as in [G_1], [BG]).

(b) The analysis of the super-t-linkative property given in the previous theorem is less satisfactory than that of the t-linkative property because we lack an answer to the following question: If domains $D \subseteq D_1$ are nonfields such that D_1 is algebraic over D and D is super-t-linkative, must D_1 also be super-t-linkative? (cf. Proposition 3.3 above).

(c) In the statement of Theorem 3.5 (2), the hypothesis that M be a t-ideal of T cannot be inferred from the fact that R is t-linkative even if D is a field. To see this, let $T = L[X,Y]_{(X,Y)} = L + M$, where $M = (X,Y)T$; and put $R = K + M$, where K is a proper subfield of the field L. Then R is a quasilocal domain whose maximal ideal M is divisorial (hence a t-ideal), whence R is t-linkative. However, T is not t-linkative, since M is not a t-ideal of T.

(d) Combining Theorem 3.5 with the results of [MZ_2] produces several interesting examples of "unruly" Hilbert domains. For instance, we see via [MZ_2, Corollary 7] and Theorem 3.5 that if D is a PID with infinite prime spectrum and L is the quotient field of D, then $D + XL[X]$ is a two-dimensional, non-Noetherian, Bézout, Hilbert super-t-linkative domain in which each maximal ideal is principal.

(e) In case R and T are domains, nonfields, with the same prime spectrum and such that $T \nsubseteq R$, then R is t-linkative. Indeed, in this case, by [AD_2, Proposition 2.1], R and T are quasilocal with a common maximal ideal, say M. Thus for t in $T \setminus R$, we have $t \in (R:M)$, and so M is a divisorial ideal of R and R is t-linkative.

<u>LEMMA</u> <u>3.10</u>. Let $R \subseteq T$ be domains, P a prime ideal of R and $Q_0 \subset Q_1 \subset \cdots \subset Q_n$ a strictly ascending chain of prime ideals in T lying over P. Then $n \leq td_{R/P}(T/Q_0)$.

Proof. Let $S = R \setminus P$. Replacing R by R_P/PR_P, T by T_S/Q_0T_S and Q_i by Q_iT_S/Q_0T_S for $0 \leq i \leq n$, we reduce to the case that R is a field and $P = (0)$. Now we easily reduce to the case that T is a finitely generated R-algebra and in this case the lemma is well known (cf. [N, Theorem 14.5]). □

THEOREM 3.11. If R is a finite-dimensional super-t-linkative domain, then $\dim_v(R) \leq 1 + \dim(R)$.

Proof. Set $d = \dim(R)$. Let T be an overring of R and $Q_0 \subset Q_1 \subset \cdots \subset Q_n$ be a strictly ascending chain of prime ideals in T. Put $P_i = Q_i \cap R$ for all i. If P_i is not maximal in R for some $i < n$, then T/Q_i is algebraic over R/P_i by Corollary 3.7, and so $P_i \subsetneq P_{i+1}$ by Lemma 3.10. In particular, if P_n is not maximal in R, then we have $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$, whence $n \leq d$.

Now assume that P_n is maximal in R, and choose r minimal such that $P_r = P_n$. Since the $n - r + 1$ ideals Q_j for $r \leq j \leq n$ lie over P_r and since by Corollary 3.7, $td_{R/P_r}(T/Q_r) \leq 1$, we obtain that $n - r \leq 1$ using Lemma 3.10. For $i < r$, P_i is not maximal in R, and so we have a strictly ascending chain $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_r$; thus $r \leq d$ and so $n \leq 1+d$. We conclude by [G_1,Theorem 30.9] that $\dim_v(R) \leq 1 + \dim(R)$. □

Referring to Theorem 3.11, recall that for any $1 \leq m \leq n$, there exists a domain R with $\dim(R) = m$ and $\dim_v(R) = n$ (cf. [G1, Proposition 30.16]).

For the next proposition, we recall that the pseudo-radical of a domain R is the intersection of all its nonzero prime ideals.

PROPOSITION 3.12. Let R be a one-dimensional domain. Then R is super-t-linkative $\Leftrightarrow \dim_v(R) \leq 2$.

Proof ($\Rightarrow$) is the case $\dim(R) = 1$ of Theorem 3.11.

($\Leftarrow$) Let T be an overring of R which is different from its quotient field. It is enough to show that T_Q is t-linkative for a given maximal ideal Q of T. Replacing T by T_Q and R by $R_{Q \cap R}$, we may assume that both T and R are quasilocal. Let N be the maximal ideal of T and M the maximal ideal of R. Clearly M is contained in the pseudo-radical of T; in particular $M \subseteq N$. Let I be a proper finitely generated ideal of T, and let $\{t_1, \ldots, t_n\}$ be a set of generators for I. Let $S = R[t_1, \ldots, t_n]_{N \cap R[t_1, \ldots, t_n]}$, $\overline{S} = S/MS$, $k = R/M = (R + MS)/MS$, and $\overline{s} = s + MS$ for each element $s \in S$. The ring $\overline{S}$ is a localization of the Noetherian ring $k[\overline{t_1}, \ldots, \overline{t_n}]$ and so is Noetherian. Moreover, $\dim(S) \leq 2$, and so $\dim(\overline{S}) \leq 1$. Since $\overline{S}$ is Noetherian, quasilocal and of dimension at most 1, it follows that $\sqrt{\overline{S}\overline{t_1} + \cdots + \overline{S}\overline{t_n}} = \sqrt{\overline{S}\overline{t}}$ for some nonzero element t in T. As M is contained in the pseudo-radical of S, we have $t_i \in \sqrt{St}$ for $1 \leq i \leq n$. Since $S \subseteq T$, we get $I \subseteq \sqrt{Tt}$. Let P be a prime t-ideal of T containing the principal ideal Tt. Then $I \subseteq P$, whence $I_t \neq T$. It follows that T is t-linkative, and so R is super-t-linkative. □

The previous proposition implies that a one-dimensional Noetherian domain is super-t-linkative. This yields half of the the next result. We also include a brief alternate proof.

PROPOSITION 3.13. A Noetherian domain R is super-t-linkative if and only if $\dim(R) \leq 1$.

Proof. If $\dim(R) \leq 1$ and S is an overring of R, then the Krull-Akizuki Theorem yields $\dim(S) \leq 1$, whence S is t-linkative by [DHLZ, Corollary 2.7], and so R is super-t-linkative.

If the converse fails, there exists a Noetherian super-t-linkative domain such that $\dim(R) \geq 2$. By the Mori-Nagata Theorem, R' is a Krull domain. Thus each t-prime of R' has height 1. Since $\dim(R') = \dim(R) > 1$, R' has a maximal ideal of height > 1 which fails to be a t-ideal. This contradicts the fact that R' is t-linkative. □

The next result is also related to low-dimensional domains.

PROPOSITION 3.14. Let (R, M) be a quasilocal domain such that $\dim(R) = 2 = \dim_v(R)$ and R is not t-linkative. Then:

(a) R^* is a completely integrally closed PVMD.

(b) R' is t-linked over R if and only if $R' = R^*$.

Proof. (a) If $P \in \mathrm{Spec}(R)$ has height 1, then, by [ABDFK, Proposition 1.11], we have $\dim(R_P[X]) = 2$. By [G_1, Proposition 30.14], $(R_P)' = R'_{R \setminus P}$ is a Prüfer domain.

Since M is not a t-ideal and each height 1 prime is a t-ideal, the set of t-primes of R is just $\{P \in \mathrm{Spec}(R)\colon \mathrm{ht}(P) = 1\}$. By [DHLZ, Proposition 2.13(b)], $T = \cap R'_{R \setminus P}$ is the smallest integrally closed t-linked overring of R. Since R^*, the complete integral closure of R, is t-linked over R [DHLZ, Corollary 2.3], $T \subseteq R^*$. We claim that $T = R^*$. Indeed, we have seen that each $R'_{R\setminus P}$ is a one-dimensional Prüfer domain, and so T is an intersection of completely integrally closed domains, whence T is completely integrally closed. Hence $R^* \subseteq T^* = T$, and so $T = R^*$, as claimed.

It remains only to observe that R^* is a PVMD, namely that $A = (R^*)_Q$ is a valuation domain for each t-prime Q of R^*. Put $P = Q \cap R$ and $B = R_P$. Since R^* is t-linked over R, [DHLZ, Proposition 2.1] yields $P_t \neq R$. Hence $P \neq M$ and so, by the above comments, B' is a Prüfer domain. As an overring of a Prüfer domain, A' is also a Prüfer domain. But A is integrally closed (since R^* is), and so $A = A'$ is a quasilocal Prüfer (that is, valuation) domain.

(b) By [DHLZ, Corollary 2.3], we need only attend to the "only if" assertion. If R' is t-linked over R, then [DHLZ, Proposition 2.13(a)] yields $R' = \cap \{R'_{R\setminus P}$: P a t-prime of R$\}$; as shown in the proof of (a), this intersection is R^*, as desired. □

Referring to Proposition 3.14(b) above, we do not know an example of a two-dimensional domain R for which R' is not t-linked over R (see Remark 4.4 below).

<u>REMARK</u> <u>3.15</u>. So far our examples of super-t-linkative domains have been treed. However, not every super-t-linkative domain is treed. Indeed, R' being a Prüfer domain does not imply that R is treed: see, for instance, [P, Example 2.28].

Note that a quasilocal domain R with all its overrings treed is super-t-linkative (since any treed domain is t-linkative by [DHLZ], Corollary 2.7 (d)]). Nevertheless, by [D_3, Examples 2.3], R need not be a PVD (or even a going-down domain, in the sense of [D_1]). Moreover, in [D_3] similar constructions lead to two super-t-linkative treed domains, one of them with all its overrings being treed and the other one having a nontreed overring.

4. An integral closure that is not t-linked

We have seen that R' is t-linked over R if R is a Noetherian (or, more generally, a quasicoherent) domain [DHLZ, Corollary 2.14(a)] or if R' is a Prüfer domain (in Corollary 2.5). There are many other natural reasons why one might ask whether R' is t-linked over R for *each* domain R (cf. [DHLZ, Proposition 2.13(b) and comments on page 2850]). The next result answers this question in the negative.

EXAMPLE 4.1. There exists a domain R such that R' is not t-linked over R.

Proof. Let D be a domain of characteristic $\neq 2$. Let Y_1, Y_2, X be algebraically independent indeterminates over D. Put $A = D[Y_1, Y_2]$ and let I be the ideal of A[X] generated by the elements Y_1 and $XY_2 - 2$. Then $R = A + XI$ has the asserted property. Indeed, we shall show that if $J = Y_1R + Y_2R$, then $J^{-1} = R$ but $(JR')^{-1} \neq R'$.

To show $J^{-1} = R$, we consider $f \in J^{-1}$ and seek to show $f \in R$. Since $(A[X]:J) = A[X]$, we get $f \in A[X]$. Write $f = f_0 + Xf_1$, with f_0 in A and f_1 in A[X]. Since $f_1 Y_2 \in I$, it suffices to prove that I is a prime ideal of A[X] which does not contain Y_2. This, in turn, follows from the $D[Y_2]$-algebra isomorphisms $A[X]/I \cong D[Y_2, X]/(XY_2 - 2) \cong D[Y_2, 2/Y_2]$. Thus $f_1 \in I$, $f \in R$, and $J^{-1} = R$.

Next, we claim that $X \in (JR')^{-1} \setminus R'$. Let $g = XY_2 - 1$. We have $g \in R'$ because $g^2 = XY_2(XY_2 - 2) + 1 \in XY_2 I + A \subseteq R$. Hence $XY_2 \in R'$. Of course, $XY_1 \in R \subseteq R'$, and so $X \in (JR')^{-1}$. To complete the proof of the claim, suppose, on the contrary, that $X \in R'$. Putting $Y_1 = 0$ and $Y_2 = 2/X$ in the integral equation over R satisfied by X, we obtain the contradiction that $X \in D[2/X]'$. Thus $X \notin R'$, the claim is established and, in particular, $(JR')^{-1} \neq R'$. □

Note that we may localize the ring R at the prime ideal $Q = XI + Y_1 R + Y_2 R$ in order to obtain a quasilocal domain R_Q such that $(R_Q)'$ is not t-linked over R_Q.

REMARK 4.2. In the previous example, if D is seminormal, then R is seminormal. Indeed, since I is a prime ideal of A, we may invoke the following lemma (with B = A).

LEMMA 4.3. Let $A \subseteq B$ be seminormal domains and I a radical ideal of B[X] such that X is not a zero-divisor mod I. Then the domain $D = A + XI$ is seminormal.

Proof. Let f be an element in the quotient field of D such that f^2 and f^3 are in D. We shall show that $f \in D$. Since the domain B[X] is seminormal by [BCM, Proposition 1 (a)], $f \in B[X]$. Write $f = f_0 + f_1X$, where $f_0 \in B$ and $f_1 \in B[X]$. Set $X = 0$ to obtain that $f_0{}^2$ and $f_0{}^3$ are in A, whence $f_0 \in A$. We may assume that $f_1 \neq 0$. Let $J = If_1{}^{-1} \cap B[X]$; then J is an ideal of B[X]. Since $f^2 = f_0{}^2 + 2f_0f_1X + f_1{}^2X^2$, we have $2f_0f_1 + f_1{}^2X \in I$, and so $2f_0 + f_1X \in J$; that is, $f_1X \equiv -2f_0 (\text{mod}\, J)$. By a similar calculation, using the fact that $f^3 \in D$, we obtain that $3f_0{}^2 + 3f_0f_1X + f_1{}^2X^2 \in J$. Since $f_1X \equiv -2f_0\ (\text{mod}\, J)$, we obtain that $3f_0{}^2 + 3f_0(-2f_0) + 4f_0{}^2 = f_0{}^2 \in J$, and so $f_1{}^2X^2 \in J$, that is, $f_1{}^3X^2 \in I$. Since X is not a zero-divisor mod I and I is radical, we obtain that $f_1 \in I$. We conclude that $f \in D$ and D is seminormal. □

REMARK 4.4. If D is a Noetherian domain of finite Krull dimension d, then in Example 4.1 we have $\dim(R) = d + 3$. Indeed, $D \subseteq R \subseteq D[Y_1, Y_2, X]$, whence $\dim(R) \leq d + 3$. (It is enough to show that $\dim(R_0) \leq d + 3$ for any ring R_0 between D and $D[Y_1, Y_2, X]$ which is finitely generated over D. But this follows because $\text{td}_D(R_0) \leq d + 3$.) On the other hand, if M is a maximal ideal of D, then we have $MR + XI \subsetneq MR + XI + Y_1R \subsetneq MR + XI + Y_1R + Y_2R$, a strictly ascending chain of 3 prime ideals in R. Thus $\dim(R) \geq \dim(D) + 3$, and so $\dim(R) = d + 3$ as claimed.

Arguing as above, we see that if D is of infinite Krull dimension, then R is also infinite dimensional.

The integral closure of a one-dimensional domain is t-linked over it by [DHLZ, Corollary 2.7]; and we have just seen that this conclusion does not hold in general for domains of any finite dimension ≥ 3. Thus just the two-dimensional case remains open.

REMARK 4.5. In Example 4.1, $R' = R[A', XY_2]$. Indeed, as shown in the proof of Example 4.1, we have $R[A', XY_2] \subseteq R'$. It is easy to show that $A'[X] = R[A', XY_2] + D'[X]$. Since $D' \subseteq R' \subseteq A'[X]$, we obtain that, if $R' \neq R[A', XY_2]$, then some polynomial f in $D'[X]$ of positive degree belongs to R'. Let F be the quotient field of D. Denote by T the domain constructed in Example 4.1 with D replaced by F; thus $R \subseteq T$. It follows that $f \in T$ and so $X \in T'$, contradicting the proof of Example 4.1.

Since $R' = R[A', XY_2 - 1]$ and $(XY_2 - 1)^2 \in R$, we see that if D is integrally closed, then R' equals the total root closure of R (in the sense of [ADR]). Thus the total root closure of R is not necessarily t-linked over R. Nevertheless, we do not know whether the seminormalization of a domain S is necessarily t-linked over S.

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