Pseudo Almost Integral Elements

D. D. Anderson[a] and Muhammed Zafrullah[b]

Abstract

Let D be an integral domain with quotient field K. We define an element $\alpha \in K$ to be *pseudo almost integral over* D if there is an infinite increasing sequence $\{s_i\}$ of natural numbers and a nonzero $c \in D$ with $c\alpha^{s_i} \in D$. We investigate when a pseudo almost integral element is almost integral or integral. We also determine the sequences $\{s_i\}$ with the property that for any domain D and $\alpha \in K$, whenever $c\alpha^{s_i} \in D$ for some nonzero $c \in D$, than α is actually almost integral over D.

Let D be an integral domain with integral closure $\bar{D}$ and quotient field K. Recall that $\alpha \in K$ is *almost integral over* D if there exists a nonzero $c \in D$ with $c\alpha^n \in D$ for all $n \geq 1$. The set D^* of all elements of K almost integral over D is an overring of D called the *complete integral closure of* D. It is easily checked that $\alpha \in K$ is almost integral over $D \Leftrightarrow D[\alpha]$ is a D-fractional ideal $\Leftrightarrow D[\alpha]$ is contained in a finitely generated D-submodule of K. Hence for D Noetherian, α is almost integral over D if and only if α is integral over D. We define $\alpha \in K$ to be *pseudo almost integral over* D if there exists a nonzero $c \in D$ with $c\alpha^n \in D$ for infinitely many natural numbers n. Let us denote the set of elements of K pseudo almost integral over D by $\tilde{D}$. Thus we have $\alpha \in K$ is integral over $D \Rightarrow \alpha$ is almost integral over $D \Rightarrow \alpha$ is pseudo almost integral over D; so $D \subseteq \bar{D} \subseteq D^* \subseteq \tilde{D}$. For results on almost integrity and complete integral closure, the reader is referred to Gilmer [1, Section 13].

Our interest in pseudo almost integral elements began with a query posed to the second author concerning Huneke's "definition" [4, Example 1.6.1] that for D Noetherian and $\alpha \in K$, α is almost integral over D if there is a nonzero $c \in D$ with $c\alpha^n \in D$ for infinitely many n. Using the fact that this definition of almost integrality agrees with the notion of integrality in the Noetherian case, Huneke remarks that for D Noetherian (with char $D = p > 0$), $(x)^* = x\bar{D} \cap D$ for each $x \in D$, where $(x)^*$ is the tight closure of (x). Using this it is easy to see that D is integrally closed if and only if each principal ideal of D is tightly closed. We first show that for Noetherian integral domains Huneke's definition agrees with the usual definition.

Proposition 1 *Let D be a Noetherian integral domain with quotient field K. If $\alpha \in K$ is pseudo almost integral over D, then α is integral over D. Thus $\bar{D} = D^* = \tilde{D}$.*

Proof. Suppose that α is pseudo almost integral over K. So there is a nonzero $c \in D$ with $c\alpha^{n_i} \in D$ where $\{n_i\}$ is an infinite increasing sequence of natural numbers. Consider the ideal $I = (c\alpha^{n_1}, c\alpha^{n_2}, \cdots)$. Since D is Noetherian, $I = (c\alpha^{n_1}, \cdots, c\alpha^{n_r})$ for $n_1 < n_2 < \cdots < n_r$. So $c\alpha^{n_{r+1}} = a_1 c\alpha^{n_1} + \cdots + a_r c\alpha^{n_r}, a_i \in D$, and hence $\alpha^{n_{r+1}} - a_r\alpha^{n_r} - \cdots - a_1\alpha^{n_1} = 0$. Thus α is integral over D. ■

There is an alternative way of proving Huneke's statement if you know that the integral closure of a Noetherian domain is a Krull domain and hence an intersection of rank-one discrete valuation domains. A domain D with quotient field K is *root closed* if for $\alpha \in K$ with $\alpha^n \in D$ for some $n \geq 1$, then $\alpha \in D$. Recall that the complete integral closure is integrally closed [1, Theorem 13.1] and hence is root closed.

Proposition 2 *Let D be an integral domain with quotient field K. Suppose that D is root closed. Then $\alpha \in K$ is pseudo almost integral over D if and only if α is almost integral over D.*

Proof. Suppose that $\alpha \in K$ is pseudo almost integral over D. So there is a nonzero $c \in D$ with $c\alpha^n \in D$ for infinitely many natural numbers n. Let m be a natural number. We show that $c\alpha^m \in D$; hence α is almost integral over D. Choose a natural number $n > m$ with $c\alpha^n \in D$. So $(c\alpha^m)^n = c^{n-m}(c\alpha^n)^m \in D$. Since D is root closed, $c\alpha^m \in D$. The other implication always holds. ■

We note that Proposition 2 in a slightly different form is given in [3, Proposition 1.5].

Corollary 3 *Let D be an integral domain with quotient field K. Suppose that $\bar{D}$ is completely integrally closed. Then $\alpha \in K$ is pseudo almost integral over D if and only if α is integral over D. Hence $\bar{D} = D^* = \tilde{D}$.*

Proof. Suppose $\alpha \in K$ is pseudo almost integral over D. Then α is pseudo almost integral over $\bar{D}$. But $\bar{D}$ is root closed, so α is almost integral $\bar{D}$ by Proposition 2. But by hypothesis $\bar{D}$ is completely integrally closed; so $\alpha \in \bar{D}$. The other implication always holds. ■

Suppose that D is Noetherian. Then $\bar{D}$ is a Krull domain and hence completely integrally closed. By Corollary 3, $\alpha \in K$ is pseudo almost integral over D if and only if α is integral over D. This provides another proof of Proposition 1.

Corollary 4 *Let D be an integral domain with quotient field K. Then $D^* \subseteq \tilde{D} \subseteq (D^*)^*$. Suppose that D^* is completely integrally closed. Then $\alpha \in K$ is pseudo almost integral over D if and only if α is almost integral over D.*

Proof. Let $\alpha \in K$ be pseudo almost integral over D. Then α is pseudo almost integral over D^*. So again by Proposition 2, α is almost integral over D^*. Thus $\tilde{D} \subseteq (D^*)^*$. Suppose D^* is completely integrally closed, then $\alpha \in (D^*)^* = D^*$. The other implication always holds. ■

It is well known that the complete integral closure of an integral domain need not be completely integrally closed, see for example [1, Exercise 3, page

144]. A case where the complete integral closure D^* is completely integrally closed is when D^* is an intersection of rank-one valuation domains such as in the Noetherian case. Thus by Corollary 3 (Corollary 4), if the (complete) integral closure of D is an intersection of rank-one valuation domains, an element that is pseudo almost integral over D is (almost) integral over D. However, it is again well known (with an example going back to Nakayama) that a completely integrally closed domain need not be an intersection of rank-one valuation domains. For an example, which also happens to be Bezout, see [1, Example 19.12].

The following corollary gives an alternative way to show that the complete integral closure of a domain D is not completely integrally closed.

Corollary 5 *Let D be an integral domain with quotient field K. If there is an element $\alpha \in K$ that is pseudo almost integral over D, but not almost integral over D, then D^* is not completely integrally closed.*

Proof. This follows immediately from Corollary 4. ∎

We next give an example of a pseudo almost integral element that is not almost integral. Example 6 appeared in a different context in [2, page 73] and [3]. There they called an element *spotty* if it was pseudo almost integral but not almost integral.

Example 6 *Let k be a field, t, y indeterminates over k, and $D = k[t, \{ty^{2^n}\}_{n=0}^{\infty}]$. Then y is pseudo almost integral over D, but not almost integral over D. Now y is in the quotient field of D and by definition y is pseudo almost integral over D. Suppose that y is almost integral over D. Then there is a nonzero $c \in D$ with $cy^n \in D$ for all $n \geq 1$. Since a sum of nonzero monomials $\sum \alpha_i t^{n_i} y^{m_i}, \alpha_i \in k$, with distinct (n_i, m_i) is in D if and only each $t^{n_i} y^{m_i}$ is in D, we can assume that c has the form $t^\alpha y^\beta$ where $t^\alpha y^\beta \in D$. Thus $t^\alpha y^\beta y^n \in D$ for all $n \geq 0$ or $t^\alpha y^n \in D$ for all $n \geq \beta$. But $ty \in D$ gives $t^\beta y^i \in D$ for $0 \leq i \leq \beta$. Hence $t^{\alpha+\beta} y^n \in D$ for all $n \geq 0$. So there is a fixed natural number M with $t^M y^n \in D$ for all $n \geq 0$. Now $t^M y^n \in D$ implies $t^M y^n = t^\gamma \left(ty^{2^{n_1}}\right)^{m_1} \cdots \left(ty^{2^{n_i}}\right)^{m_i} = t^{\gamma+m_1+\cdots+m_i} y^{m_1 2^{n_1}+\cdots+m_i 2^{n_i}}, m_i > 0$. Hence $M = \gamma + m_1 + \cdots + m_i$ so $m_1 + \cdots + m_i \leq M$. Thus each natural member n is a sum of at most M not necessarily distinct powers of 2. But since $2^s + 2^s = 2^{s+1}$, each natural number n is a sum of at most M distinct powers of 2. But this is a contradiction since $2^{M+1} - 1$ is not a sum of M distinct powers of 2.*

Let D be an integral domain with quotient field K. Note that for any finite set $n_1 < n_2 < \cdots < n_s$ of natural numbers and $\alpha \in K$, there exists a nonzero $c \in D$ with $c\alpha^{n_i} \in D$ for $i = 1, \cdots, s$. Indeed, if $\alpha = a/b$ where $a, b \in D$, we can take $c = b^{n_s}$. Thus if $\alpha \in K$ and there exists a nonzero $c' \in D$ with $c'\alpha^n \in D$ for $n \in N \setminus \{n_1, \cdots, n_s\}$, then α is almost integral over D. Let us call a nonempty subset $S \subseteq \mathbb{N}$ *almost integral producing*, or *aip* for short, if for each integral domain D and each $\alpha \in K$ with the property that there is a nonzero

$c \in D$ with $c\alpha^s \in D$ for all $s \in S$, then α is actually almost integral over D. An *aip sequence* is an increasing sequence $\{s_n\}_{n=1}^{\infty}$ of natural numbers such that $\{s_n | n \in \mathbb{N}\}$ is aip. We have just shown that a cofinite subset of $\mathbb{N}$ is aip, no finite subset of $\mathbb{N}$ is aip (for $n_1 < \cdots < n_s$, $2^{n_s}(\frac{1}{2})^{n_i} \in \mathbb{Z}$, but $1/2$ is not almost integral over $\mathbb{Z}$), and Example 6 shows that $\{2^n\}_{n=0}^{\infty}$ is not aip. We would like to determine all infinite aip subsets of $\mathbb{N}$, or equivalently, all aip sequences.

It is clear that if $S \subseteq T \subseteq \mathbb{N}$ and S is aip, then T is also aip. Also, let $S \subseteq \mathbb{N}$ and let F be a finite subset of $\mathbb{N}$, then S is aip if and only if $S - F$ is aip. One direction follows from the first sentence of this paragraph. For the other direction, suppose S is aip. We show that $S - F$ is aip. Let D be an integral domain and let $\alpha \in K$ with the property that there is a nonzero $c \in D$ with $c\alpha^s \in D$ for $s \in S - F$. Since F is finite, there is a nonzero $c' \in D$ with $c'\alpha^f \in D$ for each $f \in F$. Then $(cc')\alpha^s \in D$ for all $s \in (S-F) \cup F = S$. Since S is aip, α is almost integral over D.

We next give a characterization of aip sequences.

Theorem 7 *Let $\{s_n\}_{n=1}^{\infty}$ be an increasing sequence of natural numbers. Then the following are equivalent.*

(1) *$\{s_n\}_{n=1}^{\infty}$ is an aip sequence.*

(2) *For $D = \mathbb{Z}_2[t, ty, \{ty^{s_n}\}_{n=1}^{\infty}]$, t, y indeterminates over $\mathbb{Z}_2$, y is almost integral over D.*

(3) *There exists a natural number M so that any natural number n can be written in the form $n = j + s_{i_1} + \cdots + s_{i_k}$ where $j \geq 0$, $s_{i_1}, \cdots, s_{i_k}$ are not necessarily distinct, and $j + k \leq M$.*

Proof. (1)$\Rightarrow$(2) Note that y is in the quotient field of D. Since $\{s_n\}_{n=1}^{\infty}$ is an aip sequence, $\{1\} \cup \{s_n | n \geq 1\}$ is an aip set. Thus y is almost integral over D. (2)$\Rightarrow$(3) This is similar to Example 6. Since y is almost integral over D, as in Example 6 there is a natural number M so that $t^M y^n \in D$ for all $n \geq 0$. Thus $t^M y^n = t^{\alpha} (ty)^j (ty^{s_{i_1}}) \cdots (ty^{s_{i_k}}) = t^{\alpha+j+k} y^{j+s_{i_1}+\cdots+s_{i_k}}$ where $s_{i_1}, \cdots, s_{i_k}$ are not necessarily distinct. Now $M = \alpha + j + k$, so $j + k \leq M$, and $n = j + s_{i_1} + \cdots + s_{i_k}$. (3)$\Rightarrow$(1) Let D be an integral domain with quotient field K. Let $\alpha \in K$ be such that $c\alpha^{s_i} \in D$ for each i and some $0 \neq c \in D$. We need to show that α is almost integral over D. Let $n \geq 1$ and write $n = j + s_{i_1} + \cdots + s_{i_k}$ where $j + k \leq M$. Choose $0 \neq c' \in D$ with $c'\alpha^i \in D$ for $1 \leq i \leq M$. Then $(c'c^M)\alpha^n = c'\alpha^j c^{M-k} c\alpha^{s_{i_1}} \cdots c\alpha^{s_{i_k}} \in D$. ∎

Corollary 8 *Let $\{s_n\}_{n=1}^{\infty}$ be an increasing sequence of natural numbers such that $\{s_{n+1} - s_n\}_{n=1}^{\infty}$ is bounded. Then $\{s_n\}_{n=1}^{\infty}$ is an aip sequence. Thus if $\{s_n\}_{n=1}^{\infty}$ is an arithmetic sequence, then $\{s_n\}_{n=1}^{\infty}$ is an aip sequence.*

Proof. Let $M = \max\{s_{n+1} - s_n\}$. Let $n \geq 1$, so for some i, $s_i \leq n \leq s_{i+1}$. Then $n = (n - s_i) + s_i$ where $(n - s_i) + 1 \leq M$. By Theorem 7, $\{s_n\}_{n=1}^{\infty}$ is an aip sequence. ∎

The corollary has also been observed in a slightly different form by Jim Coykendall. The converse of the corollary is false. For let $k \geq 2$, then there is a natural number M_k so that each natural number is a sum of at most M_k kth powers. Thus by Theorem 7 $\{n^k\}_{n=1}^{\infty}$ is an aip sequence. For a second example, consider $\left\{\frac{n(n+1)}{2}\right\}_{n=1}^{\infty}$, the sequence of triangular numbers. Since each natural number is a sum of at most three triangular numbers, $\left\{\frac{n(n+1)}{2}\right\}_{n=1}^{\infty}$ is an aip sequence. We next show that an aip sequence is bounded above by a polynomial. This gives another proof of Example 6.

Theorem 9 *Let $\{s_n\}_{n=1}^{\infty}$ be an aip sequence. Then there is a polynomial $p(x)$ with integer coefficients such that $s_n \leq p(n)$ for all $n \geq 1$.*

Proof. By Theorem 7, each natural number $m < s_{n+1}$ has the form $m = j + s_{i_1} + \cdots + s_{i_k}$ where $j \geq 0$, $i_k \leq n$, $s_{i_1}, \cdots, s_{i_k}$ are not necessarily distinct and $j + k \leq M$ for some fixed M. Since $\{s_{i_1}, \cdots, s_{i_k}\} \subseteq \{s_1, \cdots, s_n\}$, there are at most M choices for j and $1 + n + n^2 + \cdots + n^M$ choices for $s_{i_1} + \cdots + s_{i_k}$. Thus there at most $M\left(1 + n + \cdots + n^M\right)$ natural numbers less than s_{n+1}, so $s_{n+1} \leq M\left(1 + n + \cdots + n^M\right) + 1 \leq M\left(1 + (n+1) + \cdots + (n+1)^M\right)$. The result follows. ■

The reader may wish to speculate on the converse of Theorem 9. Note that the special case of $s_n = p(n) = n^k$ while true is a rather deep result.

Acknowledgment. The first author wishes to acknowledge the support service provided by the Obermann Center for Advanced Studies at the University of Iowa. We would like to thank David Anderson, Jim Coykendall, and Evan Houston for several helpful comments.

References

[1] R. Gilmer, *Multiplicative Ideal Theory*, Queen's Papers in Pure and Applied Mathematics, No. 90, Queen's University, Kingston, Ontario, 1992.

[2] E. Hamann, E. Houston, and J. L. Johnson, *Properties of uppers to zero in $R[x]$*, Pacific J. Math. 135 (1998), 56–79.

[3] E. Houston, *Prime t-ideals in $R[X]$*, *Commutative Ring Theory*, 163–170, Lecture Note Pure Appl. Math., vol. 153, Marcel Dekker, Inc., New York, 1994.

[4] C. Huneke, *Tight Closure and Its Applications*, CBMS Regional Conference Series in Mathematics, Number 88, American Mathematical Society, Providence, RI, 1996.

[a] Department of Mathematics
The University of Iowa
Iowa City, IA 52242 USA
dan-anderson@uiowa.edu

[b] 57 Colgate St.
Pocatello, ID 83201 USA
mzafrullah@usa.net