A question/Answer session on v-domains

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Let D be an integral domain, K the quotient field of D, $F(D)$ the set of nonzero fractional ideals of D and let $f(D) = \{A \in F(D) | A \text{ is finitely generated}\}$. D is called a v-domain if for all $A \in f(D)$, $(AA^{-1})^{-1} = D$. The v-domains are an interesting study because (a) they have appeared, often naturally, in several guises and contexts (b) even though they are simple to define they have some hard to see special properties and (c) while they seem to be everywhere they have received very little coverage in the literature. Most of the peculiarities mentioned in (b) above are either known or easy to prove but they are either hidden in some old books/journals or require some special treatment to provide an easy proof. The aim of these notes is to perform a brief survey of v-domains asking some questions that are either questions that may be asked about any domain of interest in multiplicative ideal theory or they are questions arising from the answers. The study of v-domains involves the use of the so called star operations. A reader who is not familiar with star operations may need to read the introduction to star operations at least from sections 32 and 34 of Gilmer's book [21], and/or from Halter-Koch's book [24], before entering the Q/A session.

Question 0. Give us a gist of what star operations are and how the v-domains are related to them.

Answer:

0.1. A star operation $*$ on an integral domain D is a map $* : F(D) \to F(D)$, given by $I \mapsto I^*$, such that the following conditions hold for each $0 \neq a \in K$ and for all $I, J \in F(R)$:

(i) $D^* = D$ and $(aI)^* = aI^*$;

(ii) $I \subseteq I^*$, and $I \subseteq J \Rightarrow I^* \subseteq J^*$;

(iii) $I^{**} = I^*$.

For standard material about star operations, see Sections 32 and 34 of [21], as mentioned above. For our purposes we note the following.

0.2. Given two ideals $I, J \in F(D)$ we have $(IJ)^* = (I^*J)^* = (I^*J^*)^*$ ($*$-multiplication) and we have $(I+J)^* = (I^*+J)^* = (I^*+J^*)^*$ ($*$-sum).

0.3. A nonzero fractional ideal I is a $*$-ideal if $I = I^*$ and it is $*$-finite if $I^* = J^*$ for some finitely generated ideal $J \in F(D)$. A star operation $*$ is of finite type if $I^* = \bigcup\{J^* : J \subseteq I \text{ and } J \text{ is finitely generated}\}$, for each $I \in F(D)$. To each star operation $*$, we can associate a star operation of finite type $*_f$, defined by $I^{*_f} = \bigcup\{J^* : J \subseteq I \text{ and } J \text{ is finitely generated}\}$, for each $I \in F(D)$. If I is a finitely generated ideal then $I^* = I^{*_f}$.

0.4. Several star operations can be defined on D. The trivial example of a star operation is the identity operation, called the d-operation, $I_d = I$ for each

$I \in F(D)$. Two nontrivial star operations which have been intensively studied in the literature are the v-operation and the t -operation. Recall that the v-closure of an ideal $I \in F(D)$ is $I_v = (I^{-1})^{-1}$, where for any $J \in F(D)$ we set $J^{-1} = (D: J) = \{x \in K : xJ \subseteq D\}$. Note that for any star operation $*$ we have $(I^*)^{-1} = I^{-1} = (I^{-1})^*$. A v-ideal is also called a divisorial ideal. It is easy to show that for each $I \in F(D)$ we have $I_v = \cap xD$ where $x \in K$ such that $I \subseteq xD$. The t -operation is the star operation of finite type associated to v. Thus $I = I_t$ if and only if, for every finite set $x_1, \ldots, x_n \in I$ we have $(x_1, \ldots, x_n)_t \subseteq I$. If $\{D_\alpha\}$ is a family of overrings of D (rings such that $D \subseteq D_\alpha \subseteq K$) and $D = \cap D_\alpha$, then, for all $I \in F(D)$, the association $I \mapsto I^* = \cap ID_\alpha$ is a star operation "induced" by $\{D_\alpha\}$.

0.5. If $*_1$ and $*_2$ are two star operations defined on D we say that $*_2$ is coarser than $*_1$ (notation $*_1 \leq *_2$) if for all $I \in F(D)$ we have $I^{*_1} \subseteq I^{*_2}$. If $*_1 \leq *_2$ then for each $I \in F(D)$ we have $(I^{*_1})^{*_2} = (I^{*_2})^{*_1} = I^{*_2}$. The v-operation is the coarsest of all star operations on R and the t-operation is coarsest among star operations of finite type. Using Zorns Lemma it is easy to show that if $*$ is a star operation of finite type and if $I \in F(D)$ is an integral $*$-ideal then there is an integral $*$-ideal M containing I such that M is a maximal ideal among integral $*$-ideals. We call such an ideal M a maximal $*$-ideal and it is easy to see that M is a prime ideal. It can also be shown that if $*$ is of finite character then $D = \cap D_M$ where M ranges over maximal $*$-ideals of D.

0.6. For any star operation $*$, the set of fractional $*$-ideals is a semigroup under the $*$-multiplication $(I, J) \mapsto (IJ)^*$, with unity D. An ideal $I \in F(D)$ is called $*$-invertible if I^* is invertible with respect to the $*$-multiplication, i.e., $(II^{-1})^* = D$. Clearly, as the v-operation is the coarsest every $*$-invertible ideal is v-invertible. If $*$ is a star operation of finite type, then a $*$-invertible ideal is $*$-finite. For concepts related to star invertibility the readers may consult my paper [49] and if need arises references there. For our purposes we note that D is a v-domain if every nonzero finitely generated ideal of D is v-invertible. Now we leave it to the reader to reconcile this definition with the one in the introduction above. As this is going to be mentioned later an integral domain D is a Prufer $*$-multiplication domain (P$*$MD), for a finite type star operation $*$, if every nonzero finitely generated fractional ideal of D is $*$-invertible. The P$*$MD's were introduced and studied by Houston, Malik and Mott in [27]. A PtMD is often called a PVMD, becuse that was the name given to it by Gilmer in [21]. Following [27] one can call D, for a general star operation $*$, a $*$-multiplication domain if every nonzero finitely generated ideal of D is $*$-invertible. These domains were first studied in a paper by Anderson, Mott and Zafrullah in [8], where they were characterized in Corollary 4.3. From the comments above it follows that these $*$-multiplication domains for a general star operation $*$ are v-domains.

Question 00. In the introduction you mentioned Halter-Koch's book which is about ideal systems. What is an ideal system and how is it related to $*$-operations and v-domains? Also what is multiplicative ideal theory?

Answer:

00.1. The $*$-operations were initially introduced for commutative rings, then they were adapted for commutative semigroups. The semigroup-theorists and partially ordered-group-theorists established that star operations have essentially to do with the multiplicative structure of an integral domain. So, suitable ideal systems were established and their study was named Multiplicative Ideal Theory. The theory of ideal systems is a deep theory, we shall delve into it only as much as is related to our topic. By a semigroup we shall mean a commutative semigroup.

00.2. A nonempty set S is called a semigroup if there is a binary operation $\circ$ defined on S such that (a) S is closed under $\circ$ and (b) $\circ$ is associative (and for our puposes commutative). If S contains an identity e we call S a monoid. If there is an element 0 in S such that for all $x \in S$ we have $0 \circ x = x \circ 0 = 0$ we say that S has a zero element. Finally if for all a, x, y in S with $a \neq 0$, $a \circ x = a \circ y$ implies that $x = y$ we say that S is cancellative. In what follows we shall be working with commutative and cancellative monoids with or without zero. Have you noticed what is behind all this preparation? If D is an integral domain then considered as a semigroup under multiplication D is precisely a commutative, cancellative monoid with 0. In Halter-Koch's book [24] these monoids are considered. From now on when we say monoid, in the context of ideal systems, we mean a commutative cancellative monoid (with zero). There are many advantages in adopting this (ideal systems') approach, for example we can, deal with the divisibility, form the monoids of fractions and the groupoid of fractions of S in the same manner, avoiding 0 in the denominator, as the rings of fractions and the field of fractions of an integral domain D. We denote the groupoid of S by $q(S) = \{\frac{a}{b} : a, b \in S \text{ where } b \neq 0\}$. We can also define ideals and fractional ideals as follows.

00.3. Let a be an element of the monoid S. By aS we mean the set $\{a \circ s : s \in S\}$. We call a nonempty subset A of S an ideal if for all $a \in A$ we have $aS \subseteq A$. By a fractional ideal of S we mean a nonempty subset B of $q(S)$ such that xB is an ideal of S for some $x \in S \backslash \{0\}$. The book [24] does not consider ideals directly and deals with fractional subsets $\Phi(S)$ of $q(S)$. A fractional subset is a subset F of $q(S)$ such that xF is a subset of S for some $x \in S \backslash \{0\}$ ([24, page 121] and see the note in [24, Proposition 11.1, page 122]) An ideal system on S is a function $r : \Phi(S) \to \Phi(S)$, taking A to A_r such that for all $x \in q(S)$, $A, B \in \Phi(S)$ we have

(i) $A \cup \{0\} \subseteq A_r$

(ii) $A \subseteq B_r$ implies $A_r \subseteq B_r$

(iii) $xS \subseteq \{x\}_r$

(iv) $(xA)_r = xA_r$

00.4. Let $x \in A_r$ then $\{x\} \subseteq A_r$ and so by (ii) $\{x\}_r \subseteq A_r$. Now by (iii) $xS \subseteq A_r$. That is A_r is a fractional ideal of S. Next if you look up Proposition 2.1 on page 16 of [24] you will see that $(A_r)_r = A_r = (A \backslash \{0\})_r = (A \cup \{0\})_r$ and that if $A \subseteq B$ then $A_r \subseteq B_r$. Also note that if A is a fractional ideal of S then (i) gives $A \subseteq A_r$. Also by Proposition 2.1 (xiii) of [24] we have $\{1\}_r = 1S = S$ and so $S_r = S$. So if we were to restrict to fractional ideals an ideal system r is like a star operation r. On the other hand, as we have observed, A_r delivers an ideal (of the monoid) anyway, there is no harm in using the ideal systems for integral domains, as long as the ideal systems deliver ideals of the domains. Now a fractional subset A of S is said to be an r-ideal if $A_r = A$. Note that if I and J are any two fractional subsets of $q(S)$ (or fractional ideals) $IJ = \{ab : a \in I$ and $b \in J\}$. Now just like the star operations $(IJ)_r = (I_rJ)_r = (I_rJ_r)_r$, and you can call it the r-multiplication. There is an r-equivalent for $*$-sum too and that is $(I \cup J)_r = (I_r \cup J)_r = (I_r \cup J_r)_r$. A fractional r-ideal I of S is said to be r-finitely generated if $I = E_r$ where E is a finite (nonempty) subset of S. Given an ideal system r we can define an ideal system r, fin by $A_{r,fin} = \bigcup\{E_r$ where E varies over finite subsets of $A\}$. Now you see that r, fin is essentially like a star operation of finite type. Similarly you can define r-invertible by saying that $A \in \Phi(S)$ is r-invertible if $(AA^{-1})_r = S$. The upshot of this discussion is that with some imagination one can restate nearly every result or definition for star operations as a result or definition for ideal systems. For example given two ideal systems a and b on a monoid M we can say that b is coarser than a $(a \leq b)$ if for all $X \in \Phi(S)$ we have $X_a \subseteq X_b$.

Question 000. You said that there is no harm in using the ideal systems for integral domains as long as they deliver fractional ideals for domains (see 00.4), but (a) are there any ideals systems for domains that deliver fractional ideals for domains? (b) the ideal system allows the empty set ϕ, (c) in [24] at page 16 it is shown that $\{0\}_r = (0) = \phi_r$, the product of ideals in a semigroup is different from the product of ideals in a ring and the only equivalent of $*$-sum that you show is r-sum, which may not be an ideal in a domain. Could you explain?

Answer: A lot of good questions. There are indeed ideal systems definable on a domain as a monoid that deliver fractional ideals. These are called ideal systems induced by star operations. For this let me prepare a little. Let me introduce the d-system on an integral domain D. The d-system, as the usual r-system is defined $d : \Phi(D) \to \Phi(D)$ by $X \mapsto X_d = XD$ i.e. the (fractional) ideal of the domain D generated by X. It is easy to check that the d-system is an ideal system. Now let $*$ be a star operation on D and define $r(*) : \Phi(D) \to \Phi(D)$, by $X \mapsto (X_d)^*$ and verify that $r(*)$ is an ideal system of the domain D that delivers fractional ideals for fractional subsets. This ideal system $r(*)$ is called an ideal system induced by the star operation $*$. This is part (i) of Exercise 1 at page 33 in [24]. In the same exercise you will find the part that wants you to establish that $*$ is of finite character if and only if $r(*)$ is finitary. Now note that X being empty does not cause a problem because, as you point out $\phi_d = (0)$, and since in defining $*$-operations we throw out the ideal (0) we can always put $(0)^* = (0)$. There is of course the situation that you do not care about getting an ideal of the domain on applying an ideal system. In that case too there is no harm in using the language of ideal systems while dealing with integral domains. Now with all this let me bring in the following comment. In Chapter 17 of [24] Halter-Koch calls a monoid M an r-Prufer monoid, for a general ideal system r, if every r-finitely generated nonzero ideal of M is r-invertible. Note that if M is an integral domain then M being an r-Prufer monoid is the same as the domain in which every nonzero finitely generated ideal is $*$-invertible as studied in [8, Corollary 4.3]. In recognition of Halter-Koch's definition let us call an integral domain D a $*$-Prufer domain, for a general star operation $*$, if every nonzero finitely generated ideal of D is $*$-invertible. It is only fair to mention that most of the results in [24] on r-Prufer monoids are for finitary r. That makes the study of r-Prufer monoids in [24] or elsewhere, to-date, as good as the study of Prufer $*$-multiplication domains of [27].

Question 0000. Are there any ideal systems for general monoids that deliver (ring) ideals when used for an integral domain without the use of the d-system?

Answer: Certainly there are. I would mention two, the v-system and the t-system. It has been shown in Halter-Koch's [24, Theorem 11.4, page 125] that the function $v : \Phi(S) \to \Phi(S)$ defined by $X \mapsto X_v = \bigcap_{\substack{x \in q(S)\setminus\{0\} \\ X \subseteq xS}} xS$ is an ideal system. Later, on the same page, this system is called the v-system. Now if S were an integral domain X_v is an ideal of the domain S, becuse X_v is an intersection of fractional ideals of D. The t-system being the v, fin system gives a directed union of (ring) ideals and hence (ring) ideals.

Now recall that in 0.4 we mentioned that if $I \in F(D)$ we have $I_v = \bigcap_{\substack{x \in K\setminus\{0\} \\ I \subseteq xD}} xD$. The above remarks tell us that say for $x, y \in K\setminus\{0\}$ $(xD+yD)_v = \{x,y\}_v$. Let us call this property of the v- (and hence t-) operations the multi-role property. About the v-systems one must however be careful, in ring theory $(0)_v$ does not have a place nor does ϕ_v. So we can say that for a "fractional subset X of K", $X_v = (X)_v$ if $X \neq \phi$, $\{0\}$.

Indeed, in general if an ideal system r is such that when used for a domain, r is coarser than the d-system, r has the multi-role property.

Question 1. Where and in what context did the v-domains first appear?

Answer: The v-domains are precisely the integral domains for which the v-operation is aritrithmetisch brauchbar (a. b. for short). Recall that Krull [32] called a star operation $*$ an a. b. operation if for all $A \in f(D)$ and $B, C \in F(D)$, $(AB)^* \subseteq (AC)^*$ implies that $B^* \subseteq C^*$. I asked Robert Gilmer and Joe Mott about the origins of v-domains. They had the following to say:

We believe that Prufer's 1932 (in J. reine angew. Math. vol. 168, p. 1-36) paper is the first to discuss the concept in complete generality. In van der Waerden. paragraph 105 of volume 2 Modern Algebra, copyright 1950, there is the general abstract definition of v-ideal function and completely integrally closed rings. There is the notion of quasi-equality of ideals, and the observation that the classes of quasi-equal ideals form a group.

The original copyright of modern algebra was 1931 so van der Waerden. (or Artin

or Noether from whose lecture notes the book was derived) were before Prufer.

Krull's Idealtheorie (p. 121) says that van der Waerden treated special cases of v-ideals in "Zur produktzerlegung der ideale in ganz abgeschlossenen ringen" (on the product decomposition of ideals in integrally closed rings) Math. Ann. 101 (1929), p. 293-308. I. Arnold, "Ideale in kommutativen Halbgruppen" (ideals in commutative semigroups) Rec. Math. Soc. Math. Moscou vol. 36 (1929), p. 401-407, treats v-ideals in semigroups.

I don't have jaffard's book anymore but I would check that to see if there is an earlier reference.

we don't know who first used the words v-domain. [Joe Mott, personal e-mail]

From Joe Mott and Robert Gilmer we learn that the notion of a v-ideal was possibly known to Emil Artin and to Emmy Noether, before Prufer who dealt with the concept in complete generality in 1932, though we still do not know who came up with the name "v-domain". Recently Franz Halter-Koch reminded me of Lorenzen's work [33], who introduced v-ideals nearly ten years later than I. Arnold cited in the e-mail above. Evan Houston has also sent me a reference to a paper of Dieudonné [17]. The paper provides a clue to where v-domains came out as a separate class of rings, though they were not called v-domains there. I note that [17] has also been mentioned in Halter-Koch's book [24, page 216], where it is mentioned that [17] gives an example of a v-domain that is not a PVMD. So, to this date, we know that v-domains first showed up as a separate entity in [17]. (If any of the readers can add to this information they may write to me at the address given below, Muhammad.)

Question 2. What are the contexts in which v-domains show up?

Answer:

(2.1) As a generalization of Prufer domains: An integral domain D is a Prufer domain if every $A \in f(D)$ is invertible. Now an invertible ideal is a $*$-invertible $*$-ideal for any star operation $*$ and in fact, it is easy to establish that, if $*_1$ and $*_2$are two star operations such that for all $A \in F(D)$ $A^{*_1} \subseteq A^{*_2}$ any $*_1$-invertible ideal is also $*_2$-invertible. So a Prufer domain is a Prufer v-multiplication domain (PVMD)(every $A \in f(D)$ is t-invertible) which is in turn a v-domain. The picture can be refined if we recall another property of Prufer domains. An integral domain D is Prufer iff D_M is a valuation domain for each maximal ideal M of D. Griffin [22] showed that D is a PVMD iff D_M is a valuation domain for each maximal t-ideal M of D. Calling a valuation overring V of D *essential* if $V = D_P$ for some prime ideal P (which is invariably the center of V over D) and calling D *essential* if D is expressible as an intersection of its essential valuation overrings we note that a Prufer domain is essential and so is a PVMD (because $D = \cap D_M$ where M varies over maximal t-ideals of D (see 0.5)). In fact it is easy to see that every integral domain D that is locally essential is essential. Now add to this information the following well known result.

Proposition 1 *An essential domain is a v-domain.*

Proof. *Let $D = \bigcap_{\alpha\in I} D_{P_\alpha}$ where each D_{P_α} is a valuation domain with center P_α and let A be a nonzero finitely generated ideal of D and let ϖ be the star operation induced by $\{D_{P_\alpha}\}$ on D. Then $(AA^{-1})^{\varpi} = \bigcap_{\alpha\in I}(AA^{-1})D_{P_\alpha}$ $= \bigcap_{\alpha\in I}(AD_{P_\alpha})(A^{-1}D_{P_\alpha}) = \bigcap_{\alpha\in I}(AD_{P_\alpha})(AD_{P_\alpha})^{-1}$ (because A is f.g.) $=$ $\bigcap_{\alpha\in I} D_{P_\alpha}$ (because A is finitely generated and because each D_{P_α} is a valuation domain. This gives $(AA^{-1})^{\varpi} = D$ and hence $(AA^{-1})_v = D$.* ■

For an alternate proof of Proposition 1, and much more, the reader may consult [47, Corollary 3.2]. Halter-Koch has informed me that Proposition 1 follows from [24, Exercise 21.6 (page 244)(i)]. Indeed essential monoids can be and have been defined, in [24], and Proposition 1 does follow from Halter-Koch's exercise, but the result was already known for essential domains, see for instance, [48, Lemma 4.5]. If we closely look at [24, Exercise 21.6 (page 244)], we note that part (ii) of Exercise 21.6 of [24] follows from Lemma 8 of [45], one of the papers that set the ball rolling for multiplicative ideal theory, in recent days. On the other hand part (iii) of the same exercise was known to the authors of [27] for domains in the statement that a P$*$MD is a PVMD. It is indeed remarkable that all those results known for integral domains can be interpreted for monoids.

With this proposition at hand we have the following picture.

Prufer $\Rightarrow_1$ PVMD $\Rightarrow_2$Locally PVMD $\Rightarrow_3$ Essential such that every quotient ring is essential (such a domain is called a P-domain in [38]) $\Rightarrow_4$ Essential $\Rightarrow_5$ v-domain. (The P-domains were characterised in a somewhat special way by Papick in [42].)

(2.2) As a generalization of Bezout domains: D is Bezout iff every finitely generated ideal of D is principal, D is a GCD domain if and only if for every $A \in f(D)$ $A_t = A_v$ is principal, a Generalized GCD domain iff for each $A \in f(D)$ A_t is invertible Dan and David Anderson [2] and indeed D is a v-domain iff for each $A \in f(D)$, A_t is v-invertible. If we keep in mind the fact that a GCD domain is a PVMD we have the following addition to the existing picture:

Bezout $\Rightarrow_6$ GCD domain $\Rightarrow_7$ GGCD domain $\Rightarrow_8$ locally GCD domain $\Rightarrow_9$ locally PVMD ...

(2.3) As a link between completely integrally closed (CIC) domains and integrally closed integral domains: Recall that D is CIC $\Leftrightarrow$ for all $A \in F(D)$ $[A :_K A] = D \Leftrightarrow$ for all divisorial $A \in F(D)$ $[A :_K A] \Leftrightarrow$ for all $A \in F(D)$ $(AA^{-1})_v = D$, Gilmer [21, Section 34]. In Bourbaki [12] an integral domain D is called regularly integrally closed iff for all $A \in f(D)$ $[A_v :_K A_v] = D$. It is easy to establish that a regularly integrally closed integral domain is a v-domain i.e. for all $A \in f(D)$ $(AA^{-1})^{-1} = D$ see, for instance [21, Section 34]. Regularly integrally closed integral domains make their appearance in the study of pseudo integrality, by Anderson, Houston and Zafrullah [9] where an element $x \in K$ is called pseudo integral over D if $x \in [A_v :_K A_v]$ for some $A \in f(D)$. The terms pseudo integral closure ($\widetilde{D} = \bigcup_{I \in f(D)} (I_v : I_v)$) and pseudo integrally closed ($D = \widetilde{D}$) are coined in the obvious fashion and it is clear that D is pseudo integrally closed iff for all $A \in f(D)$, $[A_v :_K A_v] = [A_v :_K A] = D$. Finally it is well known that D is integrally closed iff $[A :_K A] = D$ for all $A \in f(D)$. From these observations, it follows that D is CIC $\Rightarrow_{10}$ D is a v-domain $\Rightarrow_{11}$ D is integrally closed. (Note that Okabe and Matsuda [41] generalized pseudo integral closure to $*$-integral closure $D^* = \bigcup_{I \in f(D)} (I^* : I^*)$ and later Halter-Koch (a) denoted D^* by $cl^*(D)$ in [23], along with other results. In view of this notation, $\widetilde{D} = cl_v(D)$.)

Of these all except $\Rightarrow_3$are known to be irreversible. We leave the case of irreversibility of $\Rightarrow_3$ as an open question and proceed to give examples to show that all the other implications are irreversible.

Irreversibility of $\Rightarrow_1$: For $\Rightarrow_1$let D be a Prufer domain that is not a field. Then, as $D[X]$ is a PVMD iff D is [7, Corollary 3.3], we conclude that $D[X]$ is a PVMD that is not Prufer.

Irreversibility of $\Rightarrow_2$: For $\Rightarrow_2$ we know that every ring of fractions of a PVMD is again a PVMD. (The easiest proof of this fact can be given by noting that if I is t-invertible then so is IR_S where S is a multiplicative set of R [13, Lemma 2.6] for an alternative proof see Heinzer and Ohm [28]. That $\Rightarrow_2$ is not reversible has been shown by producing examples of locally PVMD's that are not PVMD's at several places: In [38] an example of a non PVMD essential domain due to Heinzer and Ohm [28] was shown to have the property that it was locally PVMD and hence a P-domain. Later [48] contained a method of constructing such examples. (Recently Fontana and Kabbaj [18] have studied essential domains and considered P-domains.)

Irreversibility of $\Rightarrow_3$: Open.

Irreversibility of $\Rightarrow_4$: The example to show that $\Rightarrow_4$ is not reversible was constructed by Heinzer in [26].

Irreversibility of $\Rightarrow_5$: To show that $\Rightarrow_5$ is not reversible let us note that by $\Rightarrow_{10}$a CIC domain is a v-domain and Nagata [39] and [40] has produced an example of a one dimensional quasilocal CIC domain that is not a valuation ring. This proves that a v-domain may not be essential. (It may be useful to have an example of a nonessential v-domain that is simpler than Nagata's example.)

Irreversibility of $\Rightarrow_6$: The case of $\Rightarrow_6$can be handled in the same manner as that of $\Rightarrow_1$

Irreversibility of $\Rightarrow_7$: For $\Rightarrow_7$we note that a Prufer domain is a generalized GCD domain [2] and that a Prufer domain D is a Bezout domain iff D is GCD. In fact according to Cohn [14] a Prufer domain D is Bezout iff D is a generalization of GCD domains called a Schreier domain. Briefly an integrally closed integral domain whose group of divisibility is a Riesz group is a Schreier domain [14].

Irreversibility of $\Rightarrow_8$: For the irreversibility of $\Rightarrow_8$note that D is a GGCD domain iff D is a PVMD that is a locally GCD domain [2] and as noted above there are examples in [48] of locally GCD domains that are not PVMD's.

Irreversibility of $\Rightarrow_9$: That $\Rightarrow_9$is irreversible is well known in that there do exist examples of Krull domains that are not locally factorial.

Irreversibility of $\Rightarrow_{10}$: By Theorem 4.42 of [15] $T = D + XK[X]$ is a v-domain iff D is, here K is the quotient field of D. If D is not equal to K then obviously T is an example of a v-domain that is not completely integrally closed. This establishes that $\Rightarrow_{10}$is not reversible. Recently, bringing to, a sort of, a close a lot of efforts to restate results of [15] in terms of general pullbacks Houston and Taylor [29] use some remarkable techniques to prove results related to v-domains, PVMD's, GCD domains and Bezout domains.

Irreversibility of $\Rightarrow_{11}$: With some introductory remarks we now establish the irreversibility of $\Rightarrow_{11}$. An integral domain D is called a Mori domain if D satisfies ACC on its nonzero integral divisorial ideals. According to Querre [43] D is a Mori domain iff for every nonzero integral ideal A of D there is a finitely generated ideal $B \subseteq A$ such that $A_v = B_v$. Thus if D is a Mori domain then D is CIC iff D is a v-domain. But a completely integrally closed Mori domain is a Krull domain, see for example Fossum [19, Theorem 3.6]. Now it can be shown that if $K \subseteq L$ is an extension of fields and if X is an indeterminate then $K + XL[X]$ is a Mori domain see, for example, Theorem 4.18 of Gabelli and Houston's [20] and references there. It is easy to see that the complete integral closure of $K + XL[X]$ is $L[X]$. Thus if $K \subsetneq L$ then $K + XL[X]$ is not completely integrally closed. So, there do exist Mori domains that are not Krull. Now consider $\widetilde{Q} + XR[X]$, where R is the field of real numbers and $\widetilde{Q}$ the algebraic closure of Q in R. That $\widetilde{Q} + XR[X]$ is integrally closed is easy to see using first principles (i.e. the definition of integrality). So $\widetilde{Q} + XR[X]$ can serve as an example of an integrally closed domain that is not a v-domain.

Question 3. You have told us that if D is a PVMD then every quotient ring of D is a PVMD. Is it true that if D is a v-domain and S a multiplicative set in D then D_S is a v-domain?

Answer: No. If D and S are as given then D_S is not necessarily a v-domain. Heinzer [26] constructs an example of an essential domain D with a prime ideal P such that D_P is not essential. What is interesting is that an essential domain is a v-domain by Proposition 1 above and that D_P is a ring of the type $K + XL[X]_{(X)} = (K + XL[X])_{XL[X]}$where L is a field and K its subfield that is algebraically closed in L. Now $K + XL[X]_{(X)}$ is an integrally closed Mori domain and in the irreversibility of $\Rightarrow_{11}$we have seen that if the Mori domain D_P is a v-domain it must be a Krull domain and hence essential. (Note: Likewise if D is CIC then it may be that for some multiplicative set S D_S is not completely integrally closed. A well known example in this connection is the ring E of entire functions. For E is a completely integrally closed Bezout domain that is infinite dimensional (see the reference mentioned in Ex 20 page 148 of [21]) . Localizing E at one of the prime ideals of dimension greater than one would give a valuation domain of rank greater than one which is obviously not completely integrally closed. Now the well known examples of CIC domains with some quotient rings not CIC are all such that their quotient rings are at least v-domains and it would be instructive to see if we can find an example of a CIC domain whose quotient rings are not all v-domains.)

Question 4. It is well known that if $\{D_\alpha\}_{\alpha \in I}$ is a family of overrings of D with $D = \bigcap_{\alpha \in I} D_\alpha$ and if each D_α is a CIC domain then so is D see e.g. [21, Ex. 11, p 145]. If in the above statement "CIC domain" is replaced by "v-domain" is the statement still true?

Answer: The answer in general is no, because every integrally closed integral domain is expressible as an intersection of a family of its valuation overrings (see e.g. [21, Theorem 19.8]) and of course a valuation ring is a v-domain. If however each of D_α is a ring of fractions of D then the answer is yes. To establish this we need to prepare a little.

In [1] Dan Anderson shows that if $\{D_\alpha\}$ is a defining family of overrings of D and if $*$ is a star operation on D then the operation $'$ induced on $F(D)$ by $A' = \bigcap (AD_\alpha)^*$ is a star operation.

Proposition 2 *Let $\{D_\alpha\}_{\alpha \in I}$ be a family of quotient rings of D such that $D = \bigcap D_\alpha$. If each of D_α is a v-domain then so is D.*

Proof. Let σ be the star operation on D defined by $A \mapsto A^\sigma = \bigcap (AD_\alpha)_v$. To show that D is a v-domain it is sufficient to show that every nonzero finitely generated ideal is σ-invertible, for if $A \in f(D)$ and $(AA^{-1})^\sigma = D$ then applying the v-operation to both sides we get $(AA^{-1})_v = D$. Now $(AA^{-1})^\sigma = \bigcap ((AA^{-1})D_\sigma)_v = \bigcap ((AD_\sigma)(A^{-1}D_\sigma))_v = \bigcap ((AD_\sigma)(AD_\sigma)^{-1})_v$ (because D_α are quotient rings and A is f.g.) $= \bigcap D_\alpha$ (because each D_α is a v-domain) $= D$. ■

Corollary 3 *Let $\{P_\alpha\}$ be a family of prime ideals of D such that $D = \bigcap D_{P_\alpha}$. If $D_{P\alpha}$ for each α is a v-domain then so is D.*

This in turn leads to an interesting conclusion.

Corollary 4 *Let S be a multiplicative set in D. If for all prime ideals P of D such that P is maximal w.r.t. being disjoint from S, D_P is a v-domain then D_S is a v-domain.*

Question 5. Is it then enough to say, in view of Corollary 3, that if D is locally a v-domain then D is a v-domain?

Answer: Depends on what you mean by "locally a v-domain". If by "D is locally a v-domain" you mean that for every maximal ideal M of D, D_M is a v-domain then you are just right, by Proposition 2, yet if by "D is locally a v-domain" you mean that for every prime ideal P of D D_P is a v-domain you get much more in return. To indicate this we need to prove a statement and for that we need to recall that a prime ideal that is minimal over an ideal of the type $0 \neq (a) :_D (b) \neq D$ is called an associated prime of a principal ideal.

According to Brewer and Heinzer [11] if S is a multiplicative set of D then $D_S = \bigcap\{D_P|$ where P ranges over associated primes of principal ideals with $P \cap S = \phi$ }. Indeed if we let $S = \{1\}$ then we have $D = \bigcap D_P$ where P ranges over all associated primes of principal ideals of D. Using this terminology and the information at hand it is easy to prove the following result.

Proposition 5 *Let D be an integral domain. Then the following are equivalent: (1) D is a v-domain such that for every multiplicative set S, D_S is a v-domain, (2) For every nonzero prime ideal P of D, D_P is a v-domain, (3) For every associated prime Q of D, D_Q is a v-domain.*

In the same spirit we can make the following statement for CIC domains.

Proposition 6 . *Let D be an integral domain. Then the following are equivalent: (1) D is a CIC domain such that for every multiplicative set S, D_S is CIC, (2) For every nonzero prime ideal P of D, D_P is CIC, (3) For every associated prime Q of D, D_Q is CIC.*

Question 6. You have told us, in the proof of irreversibility of $\Rightarrow_1$, that $D[X]$ is a PVMD if and only if D is. Can we make a similar statement about v-domains?

Answer: Yes. Part (4) of Corollary 3.3 of [7] can be restated as: The following holds for $D[X]$ if and only if it holds for D : For every finitely generated nonzero fractional ideal A, A_v is v-invertible. Using the definition of v-invertibility and using the definition of $*$-multiplication one can easily show that for $A \in F(D)$, A is v-invertible if and only if A_v is v-invertible. So the above cited result says that every $A \in f(D)$ is v-invertible if and only if every $J \in F(D[X])$ is v-invertible. That is D is a v-domain if and only if $D[X]$ is. There is of course a much more interesting result in [9] in terms of pseudo integral closures.

Question 7. Is there some new material of interest on v-domains?

Answer: Certainly there is material of interest.

(I) Let $\mathfrak{M}_f$ denote the set of v-ideals of finite type of a domain D with quotient field K. Dieudonné in [17] says that Lorenzen [33] showed that the following two statements are equivalent: (a) for any three v-ideals $A, B, C \in \mathfrak{M}_f$, $(AB)_v = (AC)_v$ implies $B = C$ (and in Dieudonné's own words: autrement dit, tout élément de $\mathfrak{M}_f$ est régulier pour la multiplication.) So the "regular" here is totally diffenerent from what we mean by a regular element of a ring or of a semigroup. However the word "regular" is used in [12, VII.1.Ex. 30, page 554] without any explanation.

(b) Each $A \in \mathfrak{M}_f$, has the property that for each $x \in K$, $xA \subseteq A$ implies that $x \in D$.

Dieudonné again cites Lorenzen [33] who had shown that for a divisorial ideal A to be v-invertible it is necessary and sufficient that for all $x \in K$, $xA \subseteq A$ implies that $x \in D$. So, (a) and (b) above are equivalent to "every

v-ideal of finite type is v-invertible". This we now know to be the definition of a v-domain. And of course Dieudonné gave in [17] an example of a domain D every element of whose $\mathfrak{M}_f$ is invertible yet whose $\mathfrak{M}_f$ is not a group under v-multiplication. That is an example of a v-domain that is not a PVMD. When I came visiting the US, in early 1987, I did not know anything about the history of v-domains but I was curious about them. I gave talks at several universities and after each talk what I wanted to talk to my hosts about was v-domains. As a result of all those "consultations" we got the "A to Z" paper [3]. In this paper we gave some new characterizations for v-domains and for completely integrally closed (CIC) domains. These characterizations were then made into two schemata of characterizations in [8]. In the A to Z paper we also showed that D is a v-domain if D is integrally closed and for all $A_1, A_2, ..., A_n \in f(D)$ we have $(A_1 \cap A_2 \cap ... \cap A_n)_v = (A_1)_v \cap ... \cap (A_n)_v$, i.e. v-distributes over finite intersections of elements of $f(D)$. The converse of this result was proved by Matsuda and Okabe in [36] and recently Anderson and Clarke have continued the study of $*$-operations that distribute over finite intersections in [5] and in [6] the study of domains in which the v-operation distributes over intersections. In [6] the authors asked several questions. One of the questions (Question 3.22) can be stated as: Is it true that if D is a v-domain then $(A \cap B)_v = A_v \cap B_v$, for all $A, B \in F(D)$. Mimouni [37] has recently constructed a Prufer domain with two ideals A, B such that $(A \cap B)_v \neq A_v \cap B_v$.

(II) Section 2 of [35] contains some characterizations of v-domains in terms of polynomials. The following results may be of interest.

Theorem 7 *([35, Theorem 2.5]) Suppose that D is an integrally closed domain and let $V_D = \{f \in D[X] : A_f$ is v-invertible. Then the following are equivalent: (1) D is a v-domain; (2) $V_D = D[X]\backslash\{0\}$; (3) $D[X]_{V_D}$ is a field; (4) Each nonzero element $\alpha \in K$ satisfies a polynomial $f \in D[X]$ such that A_f is v-invertible.*

(A_f denotes the (fractional) ideal generated by the coefficients of $f \in K[X]$, A_f is also denoted by $c(f)$.)

In the course of proving Theorem 7 we needed the following lemma.

Proposition 8 *([35, Lemma 2.6]). An integral domain D is a v-domain if and only if every nonzero fractional ideal with two generators is v-invertible.*

(III) In [10, page 171] a domain D is said to have a divisor theory if there is a factorial semigroup $\mathcal{D}$ and a semigroup homomorphism $(.) : D\backslash\{0\} \to \mathcal{D}$ given by $a \mapsto (a)$ such that

(D1) $(\alpha) \mid (\beta)$ in $\mathcal{D}$ if and only if $\alpha \mid \beta$ in D for $\alpha, \beta \in D\backslash\{0\}$.

(D2) $\mathfrak{g} \mid (\alpha)$ and $\mathfrak{g} \mid (\beta) \Rightarrow \mathfrak{g} \mid (\alpha \pm \beta)$ for $\alpha, \beta \in D\backslash\{0\}$ with $\alpha \pm \beta \neq 0$ and $\mathfrak{g} \in \mathcal{D}$.

(D3) $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$ if and only if $\mathfrak{a} = \mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b} \in \mathcal{D}$ with $\overline{\mathfrak{a}} = \{\mu \in D\backslash\{0\} : \mathfrak{a} \mid (\mu)\} \cup \{0\}$.

Taking the above definition as a starting point and noting that (D2) is unnecessary ([44]) Lucius [34] called D a domain with GCD-theory if there is a GCD-monoid $\mathcal{G}$ and a semigroup homomorphism $(.) : D\backslash\{0\} \to \mathcal{G}$ given by $\alpha \mapsto (\alpha)$ such that the following conditions hold:

(G1) $(\alpha) \mid (\beta)$ in $\mathcal{G}$ if and only if $\alpha \mid \beta$ in D for $\alpha, \beta \in D\backslash\{0\}$.

(G2) $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$ if and only if $\mathfrak{a} = \mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b} \in \mathcal{G}$ with $\overline{\mathfrak{a}} = \{\mu \in D\backslash\{0\} : \mathfrak{a} \mid (\mu)\} \cup \{0\}$.

The GCD-monoid $\mathcal{G}$ is called the divisor semigroup and $q(\mathcal{G})$ the divisor group of the GCD-theory (.). The elements of $\mathcal{G}$ (of $q(\mathcal{G})$) are called integral (resp. fractional) divisors, elements of the form (α), for $\alpha \in D\backslash\{0\}$, are called principal divisors. It is shown in Note 2.2 of [34] that the extension a GCD-theory $(.) : D\backslash\{0\} \to \mathcal{G}$ to a group homomorphism $(.)' : K\backslash\{0\} \to q(\mathcal{G})$ has the following properties:

(1) $(\alpha)' \mid (\beta)'$ with respect to $\mathcal{G}$ if and only if $\alpha \mid \beta$ with respect to D for $\alpha, \beta \in K\backslash\{0\}$.

(2) $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$ if and only if $\mathfrak{a} = \mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b} \in q(\mathcal{G})$ with $\overline{\mathfrak{a}} = \{\mu \in K\backslash\{0\} : \mathfrak{a} \mid (\mu)'\} \cup \{0\}$, (the division is w.r.t. $\mathcal{G}$). To clarify the role of $\overline{\mathfrak{a}}$ we have the following theorem.

Theorem 9 *([34, Theorem 2.5]) Let D be a domain with GCD-theory $(.) : D\backslash\{0\} \to \mathcal{G}$, $\mathfrak{a}$ any divisor and $\{\alpha_i\}_{i\in I}$ a family of principal divisors with $\mathfrak{a} = GCD(\{\alpha_i\}_{i\in I})$. Then $\overline{\mathfrak{a}} = (\{\alpha_i\}_{i\in I})_v = (\overline{\mathfrak{a}})$.*

As a part consequence of Theorem 9 we have the characterization of a v-domain as the domain with GCD-theory.

Theorem 10 *([34, Theorem and Definition 2.9]) For a domain D the following conditions are equivalent: (1) D is a ring with GCD-theory, (2) D is a v-domain.*

((1) $\Rightarrow$ (2) is a consequence of Theorem 9 (as given in Corollary 2.8 of [34]) and for (2) $\Rightarrow$ (1) a GCD-monoid is constructed, via Kronecker function rings, and a GCD-theory is demonstrated. The other important, related, theorem that comes from [34] is the following.

Theorem 11 *([34, Theorem 3.1]). Let D be an integrally closed domain with field of fractions K and let T be the integral closure of D in an algebraic extension L/K. Then T is a domain with GCD-theory if and only if D is a ring with GCD-theory.*

The notion of a GCD-theory being more in the domain of monoid theory the above mentioned results have been given a monoid treatment. In terms of monoids, a part of Theorem 11 appears as a Corollary to [25, Theorem 3.6]. Theorem 3.6 of [25] being somewhat important we copy it below.

Theorem 12 *([25, Theorem 3.6]). Let D be integrally closed, $K = q(D)$, L/K an algebraic field extension and T the integral closure of D in L. Then $cl_v(T)$ is the integral closure of $cl_v(D)$ in L, and $K_v(T)$ is the integral closure of $K_v(D)$ in $L(X)$.*

(Note: Here $cl_v(D)$ is the v-integral closure of [41] or $\widetilde{D}$ of [9] and, for D integrally closed, $K_v(D) = \{\frac{f}{g} : f, g \in D[X] \text{ and } (A_f)_v \subseteq (A_g)_v\}$). Note that in [25] Halter-Koch developes the theory of Kronecker function rings axiomatically.)

Theorem 10 and the related theory actually made its way into [24], in the form of some exercises. Below I describe them in case you are interested.

Let H be a monoid. A monoid homomorphism $\varphi : H \to D$ is called a divisor homomorphism if $a, b \in H$ and $\varphi(a) \mid_D \varphi(b)$ imply that $a \mid_H b$. The GCD-theory is outlined in [24, Exercise 18.10, page 206] in terms of divisor homomorphisms and Exercise 19.6 at page 217 of [24] then wants us to prove that a monoid H is a v-Prufer monoid if and only if H possesses a GCD-theory.

(IV) There is a paper by Evan Houston and Zafrullah [31]. It talks about UMV-domains (uppers to zero are maximal v-ideals). Recall that if X is an indeterminate over an integral domain D and if P is a prime ideal of $D[X]$ such that $P \cap D = 0$ then P is an upper to zero. Recall also that an integral ideal maximal w.r.t. being a v-ideal is a maximal v-ideal. Maximal v-ideals are not a common sight. There are integral domains, such as a non-discrete rank one valuation domain, that do not have any maximal v-ideal. However a maximal v-ideal if it exists can be shown to be a prime. In any case in [31] we prove (in Theorem 3.3) the following result.

Proposition 13 *The following are equivalent for an integral domain D: (1) D is a v-domain, (2) D is an integrally closed UMV-domain, (3) D is integrally closed and every upper to zero in $D[X]$ is v-invertible, (4) D is integrally closed and every upper to zero $P = fK[X] \cap D[X]$ with f a linear polynomial is v-invertible.*

It would be unfair to leave you with this characterization of v-domains without giving a hint about where the idea came from. Zafrullah [46] proved the following Theorem.

Theorem 14 *([[46, Proposition 4]). Let D be an integrally closed integral domain, let X be an indeterminate over D and let $S = \{f \in D[X] : (A_f)_v = D\}$ then D is a PVMD if and only if for any prime ideal P of $D[X]$ with $P \cap D = (0)$ we have $P \cap S \neq \phi$.*

The proof used very basic properties of polynomial rings. In [46] it was also shown (in Lemma 7) that if D is a PVMD then every upper to zero in $D[X]$ is a maximal t-ideal. (A maximal t-ideal is an integral ideal maximal w.r.t. being a t-ideal and it too is necessarily prime, as mentioned in 0.5. Unlike maximal v-ideals the maximal t-ideals are everywhere, in that every t-ideal is contained in at least one maximal t-ideal.)

Around the same time Houston, Malik and Mott published [27]. In [27] the authors came up with a much better result (Proposition 2.6), using the $*$-operations much more efficiently. Briefly, this result said that an integrally closed integral domain D is a PVMD if and only if every upper to zero in $D[X]$ is a maximal t-ideal. It turned out that integral domains D such that their

Uppers to zero in $D[X]$ are Maximal "T"-ideals had an independent life. In [30] Houston and Zafrullah studying t-invertibility proved the following result.

Theorem 15 *([30, Theorem 1.4]). Let P be an upper to zero in $D[X]$. The following statements are equivalent: (1) P is a maximal t-ideal, (2) P is t-invertible, (3) $c(P)_t = D$. (In this case it is easy to produce $g \in P$ such that $(c(g))_v = D$.)*

Based on this result one can see that the following statement was a precursor to Theorem 3.3 of [31].

Theorem 16 *The following are equivalent for an integral domain D: (1) D is a PVMD, (2) D is an integrally closed UMT-domain, (3) D is integrally closed and every upper to zero in $D[X]$ is t-invertible, (4) D is integrally closed and every upper to zero $P = fK[X] \cap D[X]$ with f a linear polynomial is t-invertible.*

(V) Although the paper [29] is not about v-domains in particular, but it does have a few good results on v-domains. As the paper is essentially about pullbacks of a special kind we start with a description of that kind of pullbacks. Let I be a nonzero ideal of a domain T, $\varphi : T \to E = T/I$ the natural projection, and let D be an integral domain contained in E. Then $R = \varphi^{-1}(D)$ is the integral domain arising from the following pullback of canonical homomorphisms:

$$\begin{array}{ccc} R & \rightarrow & D \\ \downarrow & & \downarrow \\ T & \rightarrow & T/I = E \end{array}$$

This pullback is termed as the pullback of type $\square$. The first result of interest is the following.

Proposition 17 *([29, Lemma 2.1]) In a pullback of type $\square$, if R is a v-domain, then I is prime t-ideal of both R and T, $qf(D) = qf(E)$, R_I is a valuation domain, and $R_I = T_I$. Moreover $(I : I) = I^{-1} = (I_v : I_v)$.*

Next for an extension of domains $R \subseteq T$ call T v-linked (respectively t-linked) over R if whenever J is a nonzero (respectively finitely generated) ideal of R with $J^{-1} = R$ we have $(JT)^{-1} = T$. (The t-linked extensions were used in [16] by Dobbs, Houston, Lucas and Zafrullah in the study of PVMD's.) It is clear that "v-linked" implies "t-linked". Now we have already seen in the answer to Question 3 that a ring of fractions of a v-domain may not be a v-domain, so a t-linked overring of a v-domain may not be a v-domain, but when it comes to a v-linked overring we get a differen story.

Proposition 18 *([29, Lemma 2.4]) If R is a v -domain and T is a v-linked overring of R, then T is a v-domain. (Note that by an overring of D we mean a ring R with $D \subseteq R \subseteq K$.)*

Using Propositions 17 and 18 along with the fact that if I is a nonzero ideal of D then the ring $(I_v : I_v)$ is v-linked over D. Houston and Taylor prove the following result.

Proposition 19 *([29, Proposition 2.5]) In a pullback of type $\square$, if $T = (I : I)$ and R is a v-domain (respectively a PVMD), Then T is a v-domain (respectively a PVMD).*

There are of course a lot of other goodies in [29], but I will let you read them on your own, as they do not concern v-domains, directly.

(VI) The other important paper is unpublished and I cannot give you much from it. This paper is written by D.D. Anderson, D.F. Anderson, M. Fontana and myself [4]. This paper classifies the integral domains that come under the umbrella of v-domains. It is shown in the paper for instance that the theory of $*$-Prufer domains and hence of v-domains runs along pretty much the same lines as the theory of Prufer domains. We show for instance that, to keep the discussion less heavy, D is a $*$-Prufer domain if and only if the sum of two $*$-invertible ideals is again $*$-invertible and that D is a $*$-Prufer domain if and only if $((A \cap B)(A + B))^* = (AB)^*$ for all $A, B \in F(D)$. The first of these results is remarkable in that while an invertible (t-invertible t-) ideal is finitely generated (is of finite type) a v-invertible v-ideal may not be of finite type. For the second it is interesting to note that for $* = d$, $((A \cap B)(A + B))^* = (AB)^*$ for all $A, B \in F(D)$ is a (known) characterization of Prufer domains. For $* = t$, $((A \cap B)(A + B))^* = (AB)^*$ for all $A, B \in F(D)$ is a (known) characterization of PVMD's and for $* = v$, $((A \cap B)(A + B))^* = (AB)^*$ for all $A, B \in F(D)$ is a new characterization of v-domains. In [4] we also study $*$-completely integrally closed domains ($*$-CICD's) as the domains D such that each $A \in F(D)$ is $*$-invertible. Let me also refer you to a question left open in [47], on page 1910, at the conclusion of the proof of Corollary 3.2. with the claim that [4] may answer (parts of) that question negatively.

Franz Halter-Koch has shown a great deal of interest in [4]. He has promised that he would produce in the language of monoids further work that will contain the results of [4]. I am thankful to Franz Halter-Koch, Evan Houston, Robert Gilmer and Joe Mott for taking the time to read and comment on earlier versions of these notes.

I have handled some questions to this point and I promise to add some more material as it comes to me. If you disagree with some of the remarks or if you have an improvement in mind, or if you find an error do please write to me at zafrullah@lohar.com.

Muhammad

References

[1] D.D. Anderson, Star operations induced by overrings,Comm. Algebra 16(1988), 2535-2553.

[2] D.D. Anderson and D.F. Anderson, Generalized GCD domains, Comment. Math. Univ. St. Pauli, 28(1979), 215-221.

[3] D.D. Anderson, D.F. Anderson, D. Costa, D. Dobbs, J. Mott and M. Zafrullah, Some characterizations of v-domains and related properties, Coll. Math. 58(1989), 1-9.

[4] D.D. Anderson, D.F. Anderson, M. Fontana and M. Zafrullah, On v-domains and star operations, preprint.

[5] D.D. Anderson and S. Clarke, Star-operations that distribute over finite intersections, 33(2005), 2263-2274.

[6] D.D. Anderson and S. Clarke, When the v-operation distributes over intersections, Comm. Algebra 34 (12)(2006), 4327–4337.

[7] D.D. Anderson, D.J. Kwak and M. Zafrullah, On agreeable domains, Comm. Algebra 23 (13) (1995), 4861-4883.

[8] D.D. Anderson, J. Mott and M. Zafrullah, Some quotient based characterizations of domains of multiplicative ideal theory ,Bollettino U.M.I. (7) 3-B (1989), 455-476.

[9] D.F. Anderson, E. Houston and M. Zafrullah, Pseudo-integrality, Canad. Math. Bull. 34(1)(1991), 15-22.

[10] S.I. Borevich and Shafarevich, Number Theory, Translated from the Russian by Newcomb Greenleaf. Pure and Applied Mathematics, Vol. 20 Academic Press, New York-London 1966.

[11] J. Brewer and W. Heinzer, Associated primes of principal ideals, Duke Math. J. 41(1974), 1-7.

[12] N. Bourbaki, Commutative Algebra, Addison-Wesley, Reading Mass., 1972.

[13] A. Bouvier and M. Zafrullah, On some class groups of an integral domain, Bull. Soc. Math. Grece. 29(1988), 45-59.

[14] P.M. Cohn, Bezout rings and their subrings, Proc. Camb. Phil. Soc. 64(1968), 251-264.

[15] D. Costa, J. Mott and M. Zafrullah, The construction $D + XD_S[X]$, J. Algebra 53((1978), 423-439.

[16] D. Dobbs, E. Houston, T. Lucas and M. Zafrullah, t-linked overrings and Prufer v-multiplication domains, Comm. Algebra, 17(1989), 2835-2852.

[17] J. Dieudonné, Sur la theorié de la divisibilité, Bull. Soc. Math. France 69(1941), 133-144.

[18] M. Fontana and S. Kabbaj, Essential domains and two conjectures in dimension theory, Proc. Amer. Math. Soc. 132 (2004) 2529-2535.

[19] R. Fossum, The divisor class group of a Krull domain, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 74. Springer-Verlag, New York-Heidelberg, 1973.

[20] S. Gabelli and E. Houston, Coherentlike conditions in pullbacks, Michigan Math. J. 44(1987), 99-123.

[21] R. Gilmer, Multiplicative Ideal Theory, Marcel Dekker, 1972.

[22] M. Griffin, Some results on v-multiplication rings, Canad J. Math. 19(1967), 710-722.

[23] F. Halter-Koch, Generalized integral closures, pages 340-358, in Factorization in integral domains (D.D. Anderson, ed.) Marcel Dekker 1997.

[24] F. Halter-Koch, Ideal systems. An introduction to multiplicative ideal theory. Monographs and Textbooks in Pure and Applied Mathematics, 211. Marcel Dekker, Inc., New York, 1998.

[25] F. Halter-Koch, Kronecker function rings and generalized integral closures, Comm. Algebra 31(1)(2003), 45-59.

[26] W. Heinzer, An essential integral domain with a nonessential localization. Canad. J. Math. 33 (1981), no. 2, 400–403.

[27] E. Houston, S. Malik and J. Mott, Characterizations of $*$-multiplication domains, Canad. Math. Bull. (1984), 48-52

[28] W. Heinzer and J. Ohm, An essential ring which is not a v-multiplication ring. Canad. J. Math. 25 (1973), 856–861.

[29] E. Houston and R. Taylor, Arithmetic properties in pullbacks. J. Algebra 310 (2007), no. 1, 235–260.

[30] E. Houston and M. Zafrullah, On t-invertibility II, Comm. Algebra 17(8)(1989), 1955-1969.

[31] E. Houston and M. Zafrullah, UMV-domains, Arithmetical properties of commutative rings and monoids, 304–315, Lect. Notes Pure Appl. Math., 241, Chapman & Hall/CRC, Boca Raton, FL, 2005.

[32] W. Krull, Beitr¨age zur Arithmetik kommutativer Integritatsbereiche, I - II. Math. Z. 41 (1936), 545–577; 665–679.

[33] P. Lorenzen, Abstrakte Begrundung der multiplicativen Idealtheorie, Math. Z. 45(1939), 533-553.

[34] F. Lucius, Rings with a theory of greatest common divisors, Manuscripta Math. 95(1998), 117-136.

[35] J. Mott, B. Nashier and M. Zafrullah, Contents of polynomials and invertibility, Comm. Algebra 18 (5)(1990), 1569–1583.

[36] R. Matsuda and A. Okabe, On an AACDMZ question, Math. J. Okayama Univ. 35(1993), 41-43.

[37] A. Mimouni, Note on the star operations over polynomial rings, preprint, ArXiv:math/0702142v1[Math.AC] 6 Feb 2007.

[38] J. Mott and M. Zafrullah, On Prufer v-multiplication domains, Manuscripta Math. 35(1981), 1-26.

[39] M. Nagata, On Krull's conjecture concerning valuation rings, Nagoya Math. J. 4(1952), 29-33.

[40] M. Nagata, Nagoya Math. J. 9(1955), 209-212.

[41] A. Okabe and R. Matsuda, Star operations and generalized integral closures. Bull. Fac. Sci. Ibaraki Univ. Ser. A No. 24 (1992), 7–13.

[42] I. Papick, Super-primitive elements, Pacific J. Math. 105, no. 1 (1983), 217–226.

[43] J. Querre, Sur une propriete des anneaux de Krull, Bull. Sc. Math. 2^{e} serie 95(1971), 341-354.

[44] L. Skula, Divisorentheorie einer halbgruppe, Math. Z. 114(1970), 113-120.

[45] M.Zafrullah, On finite conductor domains, Manuscripta Math. 24(1978), 191-204.

[46] M. Zafrullah, Some polynomial characterizations of Prufer v-multiplication domains, J. Pure Appl. Algebra 32(1984), 231-237.

[47] M. Zafrullah, On a property of pre-Schreier domains, Comm. Algebra 15(1987), 1895-1920.

[48] M. Zafrullah, The $D + XD_S[X]$ construction from GCD domains, J. Pure Appl. Algebra 50(1988), 93-107.

[49] M.Zafrullah, Putting t-invertibility to use, Non-Noetherian commutative ring theory, 429–457, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000.