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RIESZ AND PRE-RIESZ MONOIDS

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ABSTRACT. Call a directed partially ordered cancellative divisibility monoid M a Riesz monoid if for all $x, y_1, y_2 \geq 0$ in M, $x \leq y_1 + y_2 \Rightarrow x = x_1 + x_2$ where $0 \leq x_i \leq y_i$. We explore the necessary and sufficient conditions under which a Riesz monoid M with $M^+ = \{x \geq 0 | x \in M\} = M$ generates a Riesz group and indicate some applications. We call a directed p.o. monoid M Π-pre-Riesz if $M^+ = M$ and for all $x_1, x_2, ..., x_n \in M$, $glb(x_1, x_2, ..., x_n) = 0$ or there is $r \in \Pi$ such that $0 < r \leq x_1, x_2, ..., x_n$, for some subset Π of M. We explore examples of Π-pre-Riesz monoids of $*$-ideals of different types. We show for instance that if M is the monoid of nonzero (integral) ideals of a Noetherian domain D and Π the set of invertible ideals, M is Π-pre-Riesz if and only D is a Dedekind domain. We also study factorization in pre-Riesz monoids of a certain type and link it with factorization theory of ideals in an integral domain.

1. Introduction

By a monoid we shall mean a commutative semigroup with identity, denoted, $M =< M, *, e >$, where $*$ is the monoid operation and e is the identity of M under $*$. Also by a partially ordered (p.o.) monoid we shall mean a monoid M whose partial order $\leq$ is compatible with the operation defined on the monoid that is $x \leq y$ implies $z * x \leq z * y$, for all $x, y, z \in M$. We denote such monoids M by $M =< M, *, e, \leq>$. A p.o. monoid $< M, *, e, \leq>$ is called upper (resp., lower) directed if for each pair of elements $x, y \in M$ there is a $z \in M$ such that $z \geq x, y$ (resp., $z \leq x, y$). A monoid that is both upper and lower directed is called directed. A monoid $< M, *, e, \leq>$ is called a divisibility monoid if $x \leq y$ in M if and only if there is $a \in M^+ = \{m \in M | m \geq e\}$ such that $y = x * a$. An upper directed divisibility monoid is clearly directed. Finally we call a (p.o.) monoid cancellative if $(z * x \leq z * y \Rightarrow x \leq y)$ $z * x = z * y \Rightarrow x = y$ for all $x, y, z \in M$.

Call a directed p.o. monoid $M =< M, *, e, \leq >$ a Riesz monoid, if (a) M is a cancellative divisibility monoid and (b) every element x of M is primal i.e. for $y_1, y_2 \in M$, $x \leq y_1 * y_2 \Rightarrow x = x_1 * x_2$ such that $x_i \leq y_i$. It turns out that Riesz monoids have the so called Riesz interpolation property, and we indicate conditions under which a Riesz monoid generates a Riesz group. Since Riesz monoids have been considered as additive as well as multiplicative monoids, under different guises, with the same theory, we have taken $*$ to serve as the monoid operation. Call elements $x_1, ..., x_n$ in a monoid $M =< M, *, e, \leq >$ disjoint if $z \leq x_i$ implies $z \leq e$. We shall use $x_1 \wedge ... \wedge x_n = e$ to denote the fact that $x_1, ..., x_n$ are disjoint. Call a

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directed monoid M a pre-Riesz monoid if for each finite set of $x_1, ..., x_n$ in M^+, $x_1 \wedge ... \wedge x_n = e$ or there is $t \in M$ such that $e < t \leq x_1, ..., x_n$. One aim of this article is to study Riesz and pre-Riesz monoids, and how they can be used, and the other is to study factorization in pre-Riesz monoids and link it with the current work on factorization of ideals of an integral domains as in [4]. Our work involves star operations on integral domains and other tools, enough that we postpone giving the plan of the paper until after we have given sufficient information on star operations and related tools.

Briefly let D be an integral domain with quotient field K, unless stated otherwise we assume that $D \neq K$. Let $F(D)$ (resp., $f(D)$) be the set of nonzero fractional ideals (resp., nonzero finitely generated fractional ideals) of D. A star operation $\star$ on D is a function on $F(D)$ that satisfies the following properties for every $I, J \in F(D)$ and $0 \neq x \in K$:

(i) $(x)^\star = (x)$ and $(xI)^\star = xI^\star$,
(ii) $I \subseteq I^\star$, and $I^\star \subseteq J^\star$ whenever $I \subseteq J$, and
(iii) $(I^\star)^\star = I^\star$.

Now, an ideal $I \in F(D)$ is a $\star$-ideal if $I^\star = I$, so a principal ideal is a $\star$-ideal for every star operation $\star$. Moreover $I \in F(D)$ is called a $\star$-ideal of finite type if $I = J^\star$ for some $J \in f(D)$. It can be shown that (a) for every star operation $\star$ and $I, J \in F(D)$, $(IJ)^\star = (IJ^\star)^\star = (I^\star J^\star)^\star$, (the $\star$-multiplication), (b) $I +^\star J = (I + J)^\star = (I + J^\star)^\star = (I^\star + J^\star)^\star$ (the $\star$-sum) and (c) $(I^\star \cap J^\star)^\star = I^\star \cap J^\star$($\star$-intersection). A reader in need of a quick review of star operations may consult sections 32 and 34 of [17]. For our purposes we include the following.

To each star operation $\star$ we can associate a star operation $\star_s$ defined by $I^{\star_s} = \bigcup\{ J^\star \mid J \subseteq I \text{ and } J \in f(D) \}$. A star operation $\star$ is said to be of finite type, or of finite character, if $I^\star = I^{\star_s}$ for all $I \in F(D)$. Indeed for each star operation $\star$, $\star_s$ is of finite character. Thus if $\star$ is of finite character $I \in F(D)$ is a $\star$-ideal if and only if for each finitely generated subideal J of I we have $J^\star \subseteq I$. Also it is easy to see that for a nonzero finitely generated ideal I we have $I^\star = I^{\star_s}$. For $I \in F(D)$, let $I_d = I$, $I^{-1} = (D :_K I) = \{ x \in K \mid xI \subseteq D \}$, $I_v = (I^{-1})^{-1}$, $I_t = \bigcup\{ J_v \mid J \subseteq I \text{ and } J \in f(D) \} = I_{v_s}$ and so the t-operation is an example of a star operation of finite character. Star operations of finite character will figure prominently in our discussion. Let I be an integral ideal such that $I^\star \neq D$, the definition of a $\star$-operation of finite type allows for the existence of a maximal integral $\star$-ideal containing I, via Zorn's Lemma. A maximal $\star$-ideal can be shown to be a prime ideal. Thus if D is a domain that is not a field and if $\star$-$Max(D)$ denotes the set of maximal $\star$-ideals of D then $\star$-$Max(D) \neq \phi$. A fractional ideal I is called $\star$-invertible if $(II^{-1})^\star = D$. It is well known that if I is $\star$-invertible for a finite character star operation $\star$ then $I^\star$ and I^{-1} are of finite type and that every $\star$-invertible $\star$-ideal is divisorial. An integral domain D is a PVMD if every nonzero finitely generated ideal of D is t-invertible. Denote the set of all $\star$-invertible fractional $\star$-ideals of D by $Inv_\star(D)$ and note that $Inv_\star(D)$ is a group under $\star$-multiplication. We also note that $Inv_\star(D)$ is a partially ordered group with order induced by $I \leq J \Leftrightarrow JI^{-1} \subseteq D$. For a star operation $\star$ of finite character we show that $Inv_\star(D)$ is a Riesz group if and only if every integral $\star$-invertible $\star$-ideal of D is a primal element of $Inv_\star(D)$ under the induced order mentioned above. Given two star operations μ and ρ we say that $\mu \leq \rho$, if $A^\mu \subseteq A^\rho$ for all $A \in F(D)$; equivalently $\mu \leq \rho$ if

$(A^{\mu})^{\rho} = (A^{\rho})^{\mu} = A^{\rho}$ for all $A \in F(D)$. Thus if $\mu \leq \rho$ every ρ-ideal is a μ-ideal. Generally $\star \leq v$ for every star operation $\star$.

Consider the set $I^{\star}(D)$ of integral $\star$-ideals of D and note that $I^{\star}(D)$ is a monoid under $\star$-multiplication $(\times^{\star})$, with $I \times^{\star} J = (IJ)^{\star}$, with D as its identity. Also if we define the partial order $\leq$ on $F^{\star}(D)$ by $A \leq B$ if and only if $A \subseteq B$, $I^{\star}(D)$ is a lattice under $\star$-sum $+^{\star}$ defined by $I +^{\star} J = (I+J)^{\star}$ and $\star$-intersection $\cap^{\star}$ defined by $I \cap^{\star} J = I \cap J$, with $\sup(I,J) = I +^{\star} J = (I+J)^{\star} = I \vee J$ and $\inf(I,J) = I \cap^{\star} J = I \cap J = I \wedge J$, for all $I, J \in F^{\star}(D)$. What is interesting is that the order on $F^{\star}(D)$ is compatible with $\times^{\star}$ and that makes $F^{\star}(D)$ a lattice ordered monoid, with the property that $\times^{\star}$ distributes over $\vee_{\alpha}$ for all α, i.e., $I \times^{\star} (\vee_{\alpha \in J} I_{\alpha}) = \vee_{\alpha \in J} I \times^{\star} I_{\alpha}$. It turns out that the set $\mathcal{L}_{\star}(D) = I^{\star}(D) \cup \{0\}$ is a complete multiplicative lattice with the least element 0 and the greatest element D,i.e., $\mathcal{L}_{\star}(D) =< I^{\star}(D) \cup \{0\}, +^{\star}, \times^{\star}, \vee, \wedge, D, 0 >$. The lattice is complete in that $\vee S = \vee\{s | s \in S\}$ and $\wedge S = \wedge\{s | s \in S\}$ exist for each subset S of $\mathcal{L}_{\star}(D)$. Because our work takes us to groups of divisibility, order on which is defined by reverse containment, we are forced to consider defining $A \leq B$ if and only if $A \supseteq B$. With this definition of order we get $\Gamma_{\star}(D) =< I^{\star}(D) \cup \{0\}, +^{\star}, \times^{\star}, \wedge, \vee, 0, D >$ and with this definition of order, 0 become the largest and D the smallest element of $\Gamma_{\star}(D)$. That's not all, $\sup(I,J) = I \cap J$, $\inf(I,J) = I +^{\star} J$ and $\times^{\star}$ distributes over $\wedge_{\alpha \in J}$, that is $I \times^{\star} (\wedge_{\alpha \in J} I_{\alpha}) = \wedge_{\alpha \in J} I \times^{\star} I_{\alpha}$ in $\Gamma_{\star}(D)$, if $\wedge_{\alpha \in J} I_{\alpha} \neq 0$. So, though $\vee S$ and $\wedge S$ exist for each subset S of $\Gamma_{\star}(D)$, it's not quite a multiplicative lattice. Let's call $\Gamma_{\star}(D)$ a reverse multiplicative lattice. Because 0 does not have a function in matters of divisibility we shall consider $\Delta_{\star}(D) = \Gamma_{\star}(D) \backslash \{0\}$, even at the cost of having to settle for $\vee S$ existing for finite subsets S of $I^{\star}(D)$. If we restrict our attention to the monoid $\varphi_{\star}(D)$ of nonzero integral $\star$-ideals of finite type under $\star$-multiplication with $I \leq J$ if and only if $I \supseteq J$, then $\varphi_{\star}(D)$ is a $\wedge$-semilattice with $I \times^{\star} (J \wedge K) = I \times^{\star} J \wedge I \times^{\star} K$.

Having described the set of star ideals of a domain, at such length, we owe it to the reader to mention a recently studied generalization of Riesz groups and hence of Riesz monoids. In [36] a directed p.o. group $G =< G, *, e, \leq >$ was called a pre-Riesz group if for $e < x_1, x_2, ..., x_n \in G$, with $\mathcal{L}(\{x_1, x_2, ..., x_n\}) \neq \mathcal{L}(\{e\})$, there exists $r \in G$ such that $e < r \leq x_1, x_2, ..., x_n$. Here $\mathcal{L}(S)$ denotes the set of lower bounds of a subset S of G. Encouraged by our results on pre-Riesz groups we defined and studied pre-Riesz monoids in [27]. The definition boils down, in the commutative case, to: a pre-Riesz monoid is a directed p.o. monoid M, with least element e such that for any $x_1, x_2, ..., x_n \in M \backslash \{e\}$ $glb(x_1, x_2, ..., x_n) = e$ or there is $r \in M$ with $e < r \leq x_1, x_2, ..., x_n$, where glb stands for the greatest lower bound or inf.

Obviously if G is a pre-Riesz group, then G^{+} is a pre-Riesz monoid. It turns out that some very interesting examples of pre-Riesz monoids are afforded by $\Delta_{\star}(D)$ and its subsets. Being a lattice, with the least element D playing the role of the identity of the monoid we have: for all $D < A_1, A_2, ..., A_r \in \Delta_{\star}(D)$, $\inf(A_1, A_2, ..., A_r) \neq D$ implies $D < \inf(A_1, A_2, ..., A_r) \leq A_1, A_2, ..., A_r$. Frankly, while a pre-Riesz monoid seemed to have plenty of interesting features, $\Delta_{\star}(D)$ appears lackluster and devoid of teeth, as it were. To give $\Delta_{\star}(D)$ "teeth", designate a subset Π of proper nonzero $\star$-ideals of D and impose the condition that for all $D < A_1, A_2, ..., A_r \in \Delta_{\star}(D)$, with $\inf(A_1, A_2, ..., A_r) \neq D$ there is $A \in \Pi$ such that $A \leq A_1, A_2, ..., A_r$. Now as $\inf(A_1, A_2, ..., A_r)$ means $\star$-sum of A_i there seems to be no reason to repeat the A_i. Thus we call the monoid $\Delta_\star(D)$ a Π-Riesz monoid if for $A \in \Delta_\star(D)$, $A \neq D$ implies that there is $X \in \Pi$ such that $X \leq A$. We can denote this setup as $D = (D, \Delta_\star(D), \Pi_D)$ where Π_D stands for Π with reference to D. This designation gives rise to a number of possibilities. For instance Π_D can be the set of proper nonzero principal (invertible, t-invertible t-ideals, t-ideals of finite type, divisorial ideals) of D. If we are no longer dealing with groups related to the group of divisibility we can revert to the set up of $\mathcal{L}_\star(D)$ and our new setup can be $D = (D, \mathcal{L}_\star(D)\backslash\{0\}; \Pi_D)$ defined by $A \neq D$ implies that there is $X \in \Pi$ such that $A \subseteq X$. Indeed we can vary the star operation $\star$, setting $\star = d, t$ etc. and, keeping the choices for Π the same, we can vary D as Noetherian, Mori etc. Recall that D is called a Mori domain if D satisfies ACC on its integral divisorial ideals and that a Noetherian domain is Mori. We note that the $\wedge$-semilattice $\varphi_\star(D)$ is a pre-Riesz monoid under $\star$-multiplication, if we set $\Pi_D = \varphi_\star(D)$. Also a p.o. monoid M is called $\wedge$-smooth if whenever $x \wedge y$ exists in M we have $z * (x \wedge y) = z * x \wedge z * y$ and if $x \wedge (y \wedge z)$ or $(x \wedge y) \wedge z$ exists then $x \wedge (y \wedge z) = (x \wedge y) \wedge z$.

One may ask whether Riesz monoids satisfy the Riesz interpolation, as do Riesz groups. The answer is yes and can be readily verified as we shall show below. Having shown that, in section 2, we study the conditions under which a Riesz monoid generates a Riesz group. Also in section 2, we study $\star$-Schreier domains as integral domains D such that the group $Inv_\star(D)$ of $\star$-invertible $\star$-ideals of D, under $\star$-multiplication, is a Riesz group and show that D is $\star$-Schreier if and only if every integral $\star$-invertible $\star$-ideal of D is a primal element of $Inv_\star(D)$. In section 3, we study the Π-pre-Riesz monoids in their various guises and show for instance that a Mori domain D is a Krull domain (resp., a UFD) if and only if $D = (D, \mathcal{L}_t\backslash\{0\}, \Pi_D\}$ where Π_D consists of proper t-invertible t-ideals (resp., nonzero principal ideals) of D. Finally, in section 4, we call an element h in a $\wedge$-smooth pre-Riesz monoid M a homogeneous element if $h > e$ and for all $e < u, v \leq h$ we have $e < t \leq u, v$. We show that in a $\wedge$-smooth pre-Riesz monoid M a sum/product of finitely many homogeneous elements can be expressed as a finite product of mutually disjoint homogeneous elements, up to some permutation of the constituents. By calling a $\wedge$-smooth pre-Riesz monoid M virtually factorial if M is generated by its homogeneous elements and by observing that a homogeneous element of the $\wedge$-semilattice $\varphi_\star(D)$ is precisely a homogeneous ideal of D we show: Let $\star$ be a finite character star operation defined on an integral domain D. Then the monoid $\varphi_\star(D)$ of nonzero $\star$-ideals of finite type of D, under $\star$-multiplication, is a $\wedge$-smooth pre-Riesz monoid with order defined by $I \leq J$ if and only if $I \supseteq J$, for $I, J \in \varphi_\star(D)$. Moreover the following hold: (1) D is a $\star$-SH domain if and only if $\varphi_\star(D)$ is a v-factorial monoid, (2) If $h(\varphi_\star(D)) \neq \phi$, then $H_h(\varphi_\star(D))$ is a v-factorial monoid. For $\star$-SH domains, the reader may consult section 4 or [4].

2. **Riesz monoids**

We first show in this section that a Riesz monoid satisfies the $(2, 2)$ Riesz interpolation and wave hands about the (m, n) interpolation.

Theorem 2.1. *TFAE for a commutative cancellation divisibility monoid M. (1) Every $x \in M^+$ is primal (2) For all $a, b, x, y \in M^+$ with $a, b \leq x, y$ there is z such that $a, b \leq z \leq x, y$. (3) For all $a, b, x_1, ..., x_n \in M^+$ with $a, b \leq x_i$ there is z such* *that $a, b \leq z \leq x_i$, $i = 1, ..., n$. (4) For all $a_i, b_j \in M^+$ with $a_i \leq b_j$ there is z such that $a_i \leq z \leq b_j$.*

Proof. (1) $\Rightarrow$ (2). Suppose every positive element of M is primal. Let $a, b \leq x, y$. Then $x = x_1 * a = x_2 * b$ and $y = y_1 * a = y_2 * b$..(1)

Since $x_1 * a = x_2 * b$, $b \leq x_1 * a$.

Also since b is primal $b = b_1 * b_2$ where $b_1 \leq x_1$ and $b_2 \leq a$........... (2)

Let $x_1 = x_1' * b_1$ and $a = a_1 * b_2$.

Then $x_1 * a = x_2 * b$ can be written as $x_1' * b_1 * a_1 * b_2 = x_2 * b$, or $x_1' * a_1 * b_1 * b_2 = x_2 * b$.

Noting that $b = b_1 * b_2$ and cancelling b from both sides of the previous equation we get

$x_1' * a_1 = x_2$. ....(3)

Since $a_1 * b_2 = a$ we have $a, b \leq a_1 * b$.............................(4)

Using the value of x_2 we have $a_1 * b \leq x$ ....(5)

(Note: $x = x_2 * b = (x_1' * a_1) * b$)

Now consider $y_1 * a = y_2 * b$.

Using $a = a_1 * b_2$ and $b = b_1 * b_2$ we have $y_1 * a_1 * b_2 = y_2 * b_1 * b_2$.

Cancelling b_2 from both sides we get $y_1 * a_1 = y_2 * b_1$.

So that $b_1 \leq y_1 * a_1$ and as b_1 is primal we have $b_1 = b_3 * b_4$ where $b_3 \leq y_1$ and $b_4 \leq a_1$.

Writing $y_1 = y_1' * b_3$ and $a_1 = a_1' * b_4$ we can express $y_1 * a_1 = y_2 * b_1$ as $y_1' * b_3 * a_1' * b_4 = y_2 * b_1$. Cancelling $b_1 = b_3 * b_4$ from both sides we get $y_2 = y_1' * a_1'$. This gives $y = y_2 * b = y_1' * a_1' * b = y_1 * a$. Now as $y_1' \leq y_1$ we get $y_1 = y_4 * y_1'$ which on substituting in $y_1' * a_1' * b = y_1 * a$ gives $y_1' * a_1' * b = y_4 * y_1' * a$ and cancelling y_1' we get $y_4 * a = a_1' * b$ and so $a \leq a_1' * b$. That is $a, b \leq a_1' * b$ and $a_1' * b \leq y$. But as $a_1' \leq a_1$ and $x_2 = x_1' * a_1$ we have $a_1' * b \leq x_2 * b = x$. So we have $z = a_1' * b$ such that $a, b \leq z \leq x, y$.

(2) $\Rightarrow$ (1). Let $a \leq b * c$.

Then as $a, b \leq b * c$, $a * b$ there is x such that $a, b \leq x \leq b * c$, $a * b$ ....(i)

Now as $a \leq x$ we have $x = x_1 * a$ ....(ii)

Also as $b \leq x$ we have $x = x_2 * b$.......................(iii)

Using (i) and (iii) $x_2 \leq a$ and $x_2 \leq c$. Now as $x_2 \leq a$, setting $a = x_3 * x_2$ we have from $x_1 * a = x_2 * b$, the equation $b = x_1 * x_3$. So $a \leq b * c$ implies that $a = x_2 * x_3$, with $x_2, x_3 \in M^+$ such that $x_3 \leq b$ and $x_2 \leq c$. □

Part (2) of Theorem 2.1 is also called the $(2, 2)$ Riesz Interpolation Property. With some effort one can show that indeed a Riesz monoid satisfies, and is characterized by, (m, n) interpolaion for all positive integral m and n.

((2) $\Rightarrow$ (3). Let $a, b \leq x_i$ for $i = 1, ..., n$, in M. If $n = 1$ we have $z = x_1$ and if $n = 2$ then by (2) there is a $z \in M$ such that $a, b \leq z \leq x_1, x_2$. Suppose that for all $2 \leq r \leq n$ we have a z_1 such that $a, b \leq z_r \leq x_1, x_2, ..., x_r$ and consider $a, b \leq x_1, x_2, ..., x_{n+1}$. By the iduction hypothesis there is z_n such that $a, b \leq z_n \leq x_1, x_2, ..., x_n$ and this gives $a, b \leq z_n, x_{n+1}$. Now by (2) we have a z_{n+1} such that $a, b \leq z_{n+1} \leq z_n, x_{n+1}$. But since $z_{n+1} \leq z_n \leq x_1, x_2, ..., x_n$ we have the result that $a, b \leq z_{n+1} \leq x_1, x_2, ..., x_{n+1}$. Thus, by induction, if $a, b \leq x_i$ for $i = 1, ..., n$, in M, the there is z in M such that $a, b \leq z \leq x_i$ for $i = 1, ..., n$.

(3) $\Rightarrow$ (4). Suppose $a_i, b_j \in M^+$ with $a_i \leq b_j$ for $i = 1, ..., m$, $j = 1, ..., n$. Let $m = 1$. Then $z = a_1$ will do the job and if $m = 2$, the statement of (3) gives a z such that $a_1, a_2 \leq z \leq b_1, b_2, ..., b_n$. Now suppose that for all $2 \leq r \leq n$ we have that $a_1, ..., a_r \leq b_1, b_2, ..., b_n$ implies the existence of z_r with $a_1, ..., a_r \leq z_r \leq$

$b_1, b_2, ..., b_n$. Consider $a_1, ..., a_r, a_{r+1} \leq b_1, b_2, ..., b_n$. But as there is z_r, by induction hypothesis, we have $z_r, a_{r+1} \leq b_1, b_2, ..., b_n$ and (3) bapplies. to give z_{r+1} such that $z_r, a_{r+1} \leq z_{r+1} \leq b_1, b_2, ..., b_n$, which completes the job because $a_1, ..., a_r \leq z_r$.

(4) $\Rightarrow$ (2). Obvious once we put $m = n = 2$.)

Call a subset S of a monoid M conic if $x * y = e$ implies $x = e = y$, for all $x, y \in S$. In a p.o. group G the sets G^+ and $-G^+$ are conic. If D is an integral domain then the set $m(D)$ of nonzero principal ideals of D is a monoid under multiplication, with identity D, ordered by $aD \leq bD \Leftrightarrow$ there is $c \in D$ such that $bD = acD \Leftrightarrow aD \supseteq bD$. The monoid $m(D)$ is cancellative too and in $m(D)$ $xDyD = D \Rightarrow xD = yD = D$. So, $m(D)$ is a divisibility cancellative conic monoid. The monoid $m(D)$ is of interest because of the manner in which it generates a group. We know how the field of quotients of a domain is formed as a set of ordered pairs, each pair representing an equivalence class with $(a, b) = (c, d) \Leftrightarrow da = bc$ and then we represent the pair (a, b), $b \in D \backslash \{0\}$ by $\frac{a}{b} = ab^{-1}$. Now the group of $m(D)$ gets the form $G(D) = \{\frac{a}{b}D | \frac{a}{b} \in qf(D) \backslash \{0\}\}$, ordered by $\frac{a}{b}D \leq \frac{c}{d}D \Leftrightarrow \frac{a}{b}D \supseteq \frac{c}{d}D \Leftrightarrow$ there is $hD \in m(D)$ such that $\frac{a}{b}DhD = \frac{a}{b}D$, so that $m(D)$ is the positive cone of $G(D)$. The group $G(D)$ gets the name group of divisibility of D (actually of $m(D)$). Now any divisibility monoid that is also a cancellative and conic monoid M, with least element e can be put through a similar process of forming equivalent classes of ordered pairs to get its group of divisibility like group $G(M) = \{a * b^{-1} | a, b \in M\}$ with $x \leq y$ in $G(M) \Leftrightarrow x * h = y$ for some $h \in M$. (Here $a * b^{-1}$ becomes $a - b$, if $* = +$.)

Corollary 1. *A Riesz Monoid M has the pre-Riesz property. Also M^+ is conic for a Riesz monoid M.*

Proof. Let $e \leq x, y$ in M and suppose that there is $g \in M$ such that g is not greater than or equal to e yet $g \leq x, y$, that is $e, g \leq x, y$. Then by the $(2, 2)$ interpolation property there is $r \in M$ such that $e, g \leq r \leq x, y$. But then $r > e$, as $r \geq e$ and $r \neq e$ because $r \geq g$. Next suppose $x, y \geq e$. If $x * y = e$ and say $x \neq e$, then we have $e, x \leq x, x * y$ and by the $(2, 2)$ interpolation there is r such that $e < r < x, x * y$ contradicting the fact that $x * y = e$. □

Well a p.o. monoid M is a p.o. group if every element of M has an inverse and obviously if a p.o. monoid is a Riesz monoid and a group, it is a Riesz group. This brings up the question: Let M be a Riesz monoid and M^+ the positive cone of it, will M^+ generate a Riesz group? As we shall be mostly concerned with monoids M with e the least element, i.e. $M = M^+$, we remodel the question as: Let M be a Riesz monoid with $M = M^+$ the positive cone of it, will M generate a Riesz group? The following result whose proof was indicated to me by G.M. Bergman, in an email, provides the answer. (To preserve an example of his frank conversational style, I haven't changed the arguments.)

Theorem 2.2. *Suppose M is a cancellative abelian monoid, which is "conical", i.e., no two nonidentity elements sum/multiply to e, and which we partially order by divisibility; and suppose every element of M is primal, namely, that with respect to the divisibility induced order, (1) $x \leq a * b => x = u * v$ such that $u \leq a$ and $v \leq b$. Then the group generated by M is a Riesz group.*

Proof. Let us rewrite (1) by translating all the inequalities into their divisibility statements; so that $x \leq a * b$ becomes $x * y = a * b$ for some y and $u \leq a$ becomes

$a = u * u_1$, and similarly for the last inequality; and finally, let us rename the elements more systematically; in particular, using a, b, c, d for the above x, y, a, b. Then we find that (1) becomes $a * b = c * d \Rightarrow a = a_1 * a_2, c = a_1 * b_1, d = a_2 * b_2$ for some $a_1, a_2, b_1, b_2 \in M$. Now if we substitute the three equations to the right of the "$\Rightarrow$" into the equation before the "$\Rightarrow$", and use cancellativity, we find that $b = b_1 * b_2$; so the full statement is (2) $a * b = c * d \Rightarrow a = a_1 * a_2,\ b = b_1 * b_2,\ c = a_1 * b_1,\ d = a_2 * b_2$, for some $a_1, a_2, b_1, b_2 \in M$. Now let G be the group generated by M, ordered so that M is the positive cone. We want to show G has the Riesz Interpolation Property. So suppose that in G we have $p, q \leq r, s$. We can write these inequalities as (3) $r = p * a, s = p * c, r = q * d, s = q * b$ where $a, b, c, d \in M$. Now the sum of the first and last equations gives a formula for $r * s$, and so does the sum of the second and third equations. Equating the results, and cancelling the summands $p * q$ on each side, we get an equation in $M : a * b = c * d$. Hence we can apply (2) to get decompositions of a, b, c, d, and substitute these into (3), getting (4) $r = p * a_1 * a_2, s = p * a_1 * b_1, r = q * a_2 * b_2, s = q * b_1 * b_2$. Equating the first and third equations (or if we prefer, the second and fourth) and cancelling the common term a_2 (respectively, the common term b_1), we get (whichever choice we have made) (5) $p * a_1 = q * b_2$.The element given by (5) is clearly $\geq p, q$, while from (4) (using whichever of the equations for r we prefer and whichever of the equations for s we prefer), we see that it is $\leq r, s$. So this is the element whose existence is required for the ((2, 2)) Riesz interpolation property for G. □

Let $\mathcal{I}_\star(D)$ be the set of integral $\star$-invertible $\star$-ideals and note that $\mathcal{I}_\star(D)$ is a monoid under $\star$-multiplication. Note that $\mathcal{I}_\star(D)$ is partially ordered by $I \leq J$ if and only if $I \supseteq J$. Indeed $J \subseteq I$ if and only if $(JI^{-1})^\star = H \subseteq D$, if and only if $J = (IH)^\star$, and as J, I are $\star$-invertible, H is $\star$-invertible and integral. Thus in $\mathcal{I}_\star(D)$, $I \leq J \Leftrightarrow J = (IH)^\star$ for some $H \in \mathcal{I}_\star(D)$. In other words $\mathcal{I}_\star(D)$ is a divisibility p.o. monoid. Because $\mathcal{I}_\star(D)$ involves only $\star$-invertible $\star$-ideals, it is cancellative too. Finally $\mathcal{I}_\star(D)$ is directed because of the definition of order. That $Inv_\star(D)$ is generated by $\mathcal{I}_\star(D)$ follows from the fact that every fractionary ideal of D can be written in the form A/d where $A \in F(D)$ and $d \in D\backslash\{0\}$. Finally, the partial order in $Inv_\star(D)$ gets induced by $\mathcal{I}_\star(D)$ in that for $I, J \in Inv_\star(D)$ we have $I \leq J \Leftrightarrow J \subseteq I \Leftrightarrow (JI^{-1})^\star \in \mathcal{I}_\star(D)$. Call $I \in \mathcal{I}_\star(D)$ $\star$-primal if for all $J, K \in \mathcal{I}_\star(D)$ $I \leq (JK)^\star$ we have $I = (I_1 I_2)^\star$ where $I_1^\star \leq J$ and $I_2^\star \leq K$. Call D $\star$-Schreier, for a star operation $\star$ of finite character, if every integral $\star$-invertible $\star$-ideal of D is primal.

Remark 2.3. For the group-theoretic version of Theorem 2.2, see (4) of Theorem 2.2 of [15]. However, Theorem 2.2 is not a repeat in that the environment is different. For instance, here we had to prove that a Riesz monoid is conic, while in the p.o. group environment G^+ being conic is a given.

Proposition 1. *Let $\star$ be a finite character star operation defined on D. Then D is a $\star$-Schreier domain if and only if $Inv_\star(D)$ is a Riesz group under $\star$-multiplication and order defined by $A \leq B \Leftrightarrow A \supseteq B$.*

Proof. Suppose that D is $\star$-Schreier, as defined above. That is each $I \in \mathcal{I}_\star(D)$ is primal. The notion of $\star$-Schreier suggests that we define $\leq$ by $A \leq B \Leftrightarrow A \supseteq B$. Then as for each pair of integral ideals I, J, $(IJ)^\star = D \Rightarrow J^\star = I^\star = D$, the same holds for members of $\mathcal{I}_\star(D)$ which are all integral $\star$-ideals. So $(IJ)^\star = D \Rightarrow I = J = D$. and so $\mathcal{I}_\star(D)$ is conic. Of course $\mathcal{I}_\star(D)$ is cancellative by the choice of ideals and by the definition of order $\mathcal{I}_\star(D)$ is a divisibility monoid. So by Theorem 2.2 $\mathcal{I}_\star(D)$ generates a Riesz group and by the above considerations $Inv_\star(D)$ is generated by $\mathcal{I}_\star(D)$. Consequently $Inv_\star(D)$ is a Riesz group. Conversely if $Inv_\star(D)$ is a Riesz group, with that order defined on it, then $\mathcal{I}_\star(D)$ is the positive cone of the Riesz group $Inv_\star(D)$ and so each element of $\mathcal{I}_\star(D)$ must be primal. □

Proposition 1 brings together a number of notions studied at different times. The first was quasi-Schreier, study started in [10] and completed in [3]. The target in these papers was studying $\mathcal{I}_d(D)$, i.e. the monoid of invertible integral ideals of D, when $Inv_d(D)$ is a Riesz group. Another study targeting $\mathcal{I}_t(D)$, i.e. the monoid of t-invertible integral t-ideals of D, for study along the same lines as above appeared in [12].

Now let's step back and require that every $\star$-invertible $\star$-ideal of D is principal. Then in Proposition 1, $\mathcal{I}_\star(D)$ is the monoid of nonzero principal integral ideals, each of which is primal and the Riesz group $Inv_\star(D)$ consists just of principal fractional ideals of D, and hence the group of divisibility of D. But then D is what was dubbed as a pre-Schreier domain in [38]. Let us finally note that an integrally closed pre-Schreier domain was initially introduced by Cohn in [6], as a Schreier ring. Cohn did say in [6] that the group of divisibility of a Schreier domain was a Riesz group. Now, looking at it from this angle, one can say that D is a Schreier domain if $D\backslash\{0\}$ is a multiplicative Riesz monoid. But the story does not end here. In [6], Cohn proved that if D is a Schreier ring and X an indeterminate over D, then so is $D[X]$. But then there was a lot of activity on monoid domains with Matsuda, at the forefront, converting a lot of results on polynomial rings to monoid rings. He, more than, translated Cohn's result into the language of monoid rings by saying in [29] that $D[X,S]$ is a Schreier domain if and only if D and $K[X;S]$ are Schreier domains where S is an integrally closed additive Schreier semigroup (actually monoid), the idea of an additive (pre-) Schreier monoid was born. Let me mention for those who need to refresh their memory that a monoid M is integrally closed if for $h \in q(M) = \{m_1 {*} m_2^{-1} | m_i \in M\}$, and for n a natural number, $nh \in M^+$ implies $h \in M^+$. In other words, M is integrally closed if, whenever $nx \geq ny$ for some positive integfer n we have $x \geq y$. Now it so happens that the description of an additive (pre-) Schreier monoid provided in [29] matches our description of Riesz monoids. Hence the general treatment of Riesz monoids, using $*$ for the binary operation, so that additive Riesz monoids are well represented. Finally a word about monoid domains. Given a commutative additive monoid M and a commutative ring R we can form a set $R[X;M] = \{\sum_{i=1}^{k} r_i X^{m_i} | r_i \in R \text{ and } m_i \in M\}$. Defining addition and multiplication as we do for polynomial addition and multiplication using $X^{m_1}X^{m_2} = X^{m_1+m_2}$ we can make it a ring. Now according to Theorem 8.1 of Gilmer's book [18] that $D[X;M]$ is a domain if and only if D is a domain and M is a grading (additive, canellative and torsionfree) monoid.

3. Pre-Riesz monoids

If we adopt the language of $\mathcal{L}_\star(D)$ that takes $A \leq B$ if and only if $A \subseteq B$, the p.o. monoid $\mathcal{L}_\star(D)\backslash\{0\}$ is a pre-Riesz monoid because $A_1, A_2, ..., A_r \subseteq \sup(A_1, A_2, ..., A_r) \subsetneq D$, for all $A_1, A_2, ..., A_r \in \mathcal{L}_\star(D)\backslash\{0\}$ with $\sup(A_1, A_2, ..., A_r) \neq D$. Further if we relax it to: for all $A_1, ..A_r \in \mathcal{L}_\star(D)\backslash\{0\}$, $(A_1, A_2, ..., A_r)^\star \neq D$ $\Rightarrow \exists D \neq A \in \mathcal{L}_\star(D)\backslash\{0\}$ such that $A_i \subseteq A$ we still get a pre-Riesz monoid. If this doesn't sound very exciting let us designate a non-empty subset Π_D of $\mathcal{L}_\star(D)$, say

Π_D is the set of proper nonzero principal ideals of D. Now impose on $\mathcal{L}_\star(D)\backslash\{0\}$ the condition: $\forall A \in \mathcal{L}_\star(D)\backslash\{0\}$ $A \neq D \Rightarrow \exists\pi \in \Pi_D$ with $A \subseteq \pi$. Denote this domain by $D = (D, \mathcal{L}_\star(D), \Pi_D)$. Clearly this satisfies: $\forall A_1, ..A_r \in \mathcal{L}_\star(D)\backslash\{0\}$, $(A_1, A_2, ..., A_r)^\star \neq D \Rightarrow \exists\pi \in \Pi_D$ such that $A_i \subseteq \pi$ and so $\mathcal{L}_\star(D)\backslash\{0\}$ is a pre-Riesz monoid. Now, as we have seen in the introduction above, this translates to: every proper $\star$-ideal of D is contained in a proper principal ideal of D. In other words, if A is a $\star$-ideal that is not contained in any principal ideal then $A = D$. This too may not be very exciting at a first glance, even though every maximal $\star$-ideal of D is principal as a result. Next, take a pair of coprime elements x, y of D. Then $(x, y)^\star = D$. Thus, at least, for every pair x, y of coprime elements in D we have $(x, y)^\star = D$. Now let d be an irreducible element of D and suppose that $d|ab$ for some $a, b \in D$. If $d \nmid a$ and $d \nmid b$, then $D = ((d, a)^\star(d, b)^\star)^\star = (d^2, da, db, ab)^\star \subseteq dD$ a contradiction, because $d|ab$. Thus an irreducible element is a prime in $D = (D, \mathcal{L}_\star(D)\backslash\{0\}, \Pi)$, for any star operation $\star$. Now for $\star = d$ the identity operation $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi)$ is a domain in which every proper nonzero ideal is contained in a principal ideal, something stronger than what Cohn [6] called a pre-Bezout domain. $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi)$ is even stronger than what was called a special pre-Bezout, or spre-Bezout domain in [11]. Similarly if $D = (D, \mathcal{L}_v(D), \Pi)$, then D is something stronger than a PSP-domain (every primitive polynomial over D is super-primitive), also discussed in [11]. Can we find domains that satisfy these properties? Yes indeed!

Example 3.1. Let Z, Q denote the ring of integers and its quotient field respectively and let X be an indeterminate over Q, then the ring $D = Z + XQ[X]$ is $(D, \mathcal{L}_d(D)\backslash\{0\}, \Pi)$ where Π is the set of proper principal nonzero ideals of D.

Illustration: According to [8, Theorem 4.21] the prime ideals of D are of the form $pZ+XQ[X], XQ[X]$ and height one principal primes of the form $f(X)D$ where $f(X)$ is irreducible in $K[X]$ and $f(0) = 1$. Also, according to [8, Proposition 4.12], a general ideal of D is of the form $I = f(X)(F+XQ[X])$ where F is a Z-submodule of Q such that $f(0)F \subseteq Z$. If $f(0) = 0$, $f(X)$ is of the form $Xg(X)$ where $g(X) \in Q[X]$ and so for each $z \in Z$ we have $I \subseteq zZ$. If on the other hand $f(0) = d \neq 0$. Then as $f(0)F \subseteq Z$ we have that dF is an ideal of Z. So $I = f_1(X)d(F + XQ[X])$ where $f_1(X) \in D$ such that $f_1(0) = 1$ and $I = f_1(X)(dF + XQ[X])$. If dF is a proper ideal we need look no further for proper principal ideals containing I. But if $dF = Z$ and I is proper then $f_1(X)$ must be variable and a product of powers of primes of D. This "analysis" also establishes that if no principal ideals of D contain $I = f(X)(F + XQ[X])$, then I must be equal to D.

Now enough with this motivational chit chat. Let's state/prove some results.

A number of statements can be made in this connection, some examples are given below. Yet considering the situation that we are faced up with, some observations are in order.

Observation Suppose that $P =< P, \leq >$ is a non empty poset with the properties that every element of P precedes a maximal element of P and suppose that a non empty set Π is chosen from P by some rule. Then every maximal element of P is in Π if and only if every element of P is required to precede some element of Π.

This, somewhat simple observation may, in some instances, have some interesting consequences.

Lemma 3.2. *(1) If $\star \leq \rho$ are two star operations, where $\star$ is of finite type, and if every maximal $\star$-ideal of D is a ρ-ideal, then the set $\Pi_D(\rho\text{-}inv))$ of ρ-invertible ρ-ideals coincides with the set $\Pi_D(\star\text{-}inv))$ of $\star$-invertible $\star$-ideals of D, (2) Suppose that $\star \leq \rho$ are two star operations, where $\star$ is of finite type. Then $D = (D, \mathcal{L}_\star(D)\backslash\{0\}, \Pi_D(\rho\text{-}inv))$ for $\Pi_D(\rho\text{-}inv)$ the set of proper ρ-invertible ρ-ideals of D, if and only if every maximal $\star$-ideal of D is a ρ-invertible ρ-ideal of D.*

Proof. (1) Suppose every maximal $\star$-ideal is a ρ-ideal and let I be a ρ-invertible ideal. Claim that I is $\star$-invertible. For if not, then $(II^{-1})^\star \subseteq P$ for some maximal $\star$-ideal P. But then, as P is a ρ-ideal, $((II^{-1})^\star)^\rho \subseteq P$. Also as $((II^{-1})^\star)^\rho = (II^{-1})^\rho = D$ because $\rho \geq \star$, a contradiction. Next as a ρ-ideal is a $\star$-ideal we conclude that a ρ-invertible ρ-ideal is a $\star$-invertible $\star$-ideal. Thus $\Pi_D(\rho\text{-}inv)) \subseteq \Pi_D(\star\text{-}inv))$. For the reverse containment note that as $\star \leq \rho$, $(II^{-1})^\star = D \Rightarrow (II^{-1})^\rho = D$, i.e, $\star$-invertible is ρ-invertible. Also as a $\star$-invertible $\star$-ideal is a v-ideal and hence a ρ-ideal, we have the conclusion.

(2) Suppose $D = (D, \mathcal{L}_\star(D)\backslash\{0\}, \Pi_D(\rho\text{-}inv))$. Then every maximal $\star$-ideal M of D is contained in a ρ-invertible ρ-ideal of D and hence must be a ρ-invertible ρ-ideal. Conversely suppose that every maximal $\star$-ideal of D is a ρ-invertible ρ-ideal. By (1), $\Pi_D(\rho\text{-}inv) = \Pi_D(\star\text{-}inv)$. Thus $D = (D, \mathcal{L}_\star(D)\backslash\{0\}, \Pi_D(\rho\text{-}inv))$. □

Proposition 2. *(1) Let $\star > d$ be a finite type star operation defined on D. Then $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi_D)$ for Π_D the set of proper nonzero principal ideals (resp., $\star$-invertible $\star$-ideals, $\star$-ideals of finite type, $\star$-ideals, divisorial ideals) if and only if every maximal ideal of D is a principal ideal (resp., invertible ideal, $\star$-ideal of finite type, divisorial ideal) of D. (2) An integral domain D is $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi_D)$ for Π_D the set of proper nonzero principal ideals (resp., t-invertible t-ideals, t-ideals, t-ideals of finite type, divisorial ideals) if and only if every maximal ideal of D is a principal ideal (resp., invertible ideal,, t-ideal, t-ideal of finite type, divisorial ideal), (3) An integral domain D is $D = (D, \mathcal{L}_t(D), \Pi_D)$ for Π_D the set of proper nonzero proper principal ideals (resp.,invertible ideals, t-invertible t-ideals, t-ideals of finite type, divisorial ideals) if and only if every maximal t-ideal of D is a principal ideal (resp., invertible ideal, t-invertible t-ideal, t-ideal of finite type, divisorial ideal) of D.*

Proof. In the presence of Observation 3 and Lemma 3.2, it appears totally unnecessary to repeat the arguments required for the proofs of (1) and (2). However, we take up selective cases from (1). Suppose $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi_D)$ for Π_D the set of proper $\star$-ideals ($\star$-ideals of finite type, divisorial ideals). Then every maximal ideal is a $\star$-ideal ($\star$-ideal of finite type, divisorial ideal), by the condition. So $Max(D) \subseteq \Pi_D$ the set of proper $\star$-ideals ($\star$-ideals of finite type, divisorial ideals). Whence we can say that every proper ideal of D is contained in a proper $\star$-ideal ($\star$-ideal of finite type, divisorial ideal) of D. Now (2) is just a special case of (1). For (3), let $D = (D, \mathcal{L}_t(D), \Pi_D)$ for Π_D the set of proper nonzero principal ideals (resp.,invertible ideals, t-invertible t-ideals, t-ideals of finite type, divisorial ideals). Then, by the condition every maximal t-ideal of D is principal (resp., invertible, a t-invertible t-ideal, a t-ideal of finite type, divisorial ideal). Thus maximal t-ideals of D are contained in Π_D the set of proper nonzero principal ideals (resp.,invertible ideals, t-invertible t-ideals, t-ideals of finite type, divisorial ideals) and on can say that every proper t-ideal is contained in a nonzero proper principal ideal (resp.,invertible ideal, t-invertible t-ideal, t-ideal of finite type, divisorial ideal) □

Note that in case of (2) every maximal ideal being a t-ideal of finite type ensures that every maximal t-ideal of D is actually a maximal ideal. Indeed if we suppose that P is a maximal t-ideal that is not maximal then P is contained in a maximal ideal, say M, but M is aleady a t-ideal.

When it comes to the usual extensions of domains $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi_D)$ for suitable $\Pi_D s$ present interesting scenarios. We restrict to the star operations that are defined for the extensions considered.

Proposition 3. *(1) Let $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi_D)$ for Π_D the set of proper nonzero principal ideals (resp., t-invertible t-ideals, t-ideals, t-ideals of finite type, divisorial ideals), let X be an indeterminate over D and let $R = D[X]$. Then it never is the case that $R = (R, \mathcal{L}_d(R)\backslash\{0\}, \Pi_R)$ for Π_R the set of proper nonzero principal ideals (resp., t-invertible t-ideals, t-ideals, t-ideals of finite type, divisorial ideals) and (2) Let $D = (D, \mathcal{L}_t(D), \Pi_D)$ for Π_D the set of proper nonzero principal ideals (resp., t-invertible t-ideals, t-ideals of finite type, divisorial ideals), let X be an indeterminate over D and let $R = D[X]$. Then $R = (R, \mathcal{L}_t(R)\backslash\{0\}, \Pi_R)$ for Π_R the set of proper nonzero principal ideals (resp., t-invertible t-ideals, t-ideals of finite type, divisorial ideals) and conversely.*

Proof. (1) Let $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi_D)$ for Π_D the set of proper nonzero principal ideals (resp., t-invertible t-ideals, t-ideals, t-ideals of finite type, divisorial ideals). Then every maximal ideal P of D is a t-ideal. Now consider the prime ideal $P[X]$ in $R[X]$ and note that $P[X]$ can never be a maximal ideal because $R[X]/P[X] \cong (R/P)[X]$ is a polynomial ring over a field and so must have an infinite number of maximal ideals. This forces $P[X]$ to be properly contained in an infinite number of maximal ideals M_α of $R[X]$. Let M be one of them. Then $M = (f, P[X])$. Now, if it were the case that $R = (R, \mathcal{L}_d(R)\backslash\{0\}, \Pi_R)$ for Π_R the set of proper t-ideals, then every maximal ideal of R would be a t-ideal. This would make M a t-ideal with $M \cap D = P \neq (0)$. But then, according to Proposition 1.1 of [24], $M = (M \cap D)[X] = P[X]$, a contradiction to the fact that $P[X] \subsetneq M$. For (2) note that if $D = (D, \mathcal{L}_t(D), \Pi_D)$ for Π_D the set of proper nonzero principal ideals (resp., t-invertible t-ideals, t-ideals of finite type, divisorial ideals), then in each case every maximal t-ideal P of D is divisorial. Now let M be a maximal t-ideal of R. If $M \cap D = (0)$, then M is a t-invertible t-ideal and hence divisorial by Theorem 1.4 of [24]. Next if M is such that $M \cap D \neq (0)$, then $M = (M \cap D)[X]$ where $M \cap D$ is a maximal t-ideal of D and hence divisorial. Now it is easy to show that M is divisorial. Conversely suppose that $R = (R, \mathcal{L}_t(R)\backslash\{0\}, \Pi_R)$ for Π_R the set of proper divisorial ideals. Then every maximal t-ideal M of R is divisorial. Now let P be a maximal t-ideal of D. Then $P[X]$ is a maximal t-ideal of R by Proposition 1.1 of [24] and hence divisorial. But this leads to $P[X] = (P[X])_v = P_v[X]$ and hence to $P = P_v$. □

Before checking how fare the $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi_D)$ for Π_D the set of proper nonzero principal ideals (resp., t-invertible t-ideals, t-ideals, t-ideals of finite type divisorial ideal) under extension to rings of fractions, we need to consider another type of extension, the $D + XL[X]$ construction. Yet to be able to fully appreciate how it works, one needs to learn a little about it, the construction $D + XL[X]$. Let

D be an integral domain with quotient field K, let L be an extension of K and let X be an indeterminate over L. Then $R = D + XL[X] = \{f \in L[X] | f(0) \in D\}$ is an integral domain. Indeed R has two kinds of prime ideals P , ones that intersect D trivially and ones that don't. If $P \cap D \neq (0)$ then $P = P \cap D + XL[X]$ [9, Lemma 1.1] and obviously P is maximal if and only if $P \cap D$ is. It can be shown, as was indicated prior to the proof of Corollary 16 in [2], that if $P = P \cap D + XL[X]$, then P is a maximal t-ideal of R if and only if $P \cap D$ is a maximal t-ideal of D and indeed as $P_v = (P \cap D)_v + XL[X]$, P is divisorial if and only if $(P \cap D)$ is. Moreover, prime ideals of R that are not comparable with $XL[X]$ are of the form $(1 + Xg(X))R$ where $1 + Xg(X)$ is an irreducible element of $L[X]$, [9, Lemmas 1.2, 1.5]. Also as $XL[X]$ is of height one $XL[X]$ is a t-ideal.

Proposition 4. *(1) Let L be an extension of the field of quotients K of an integral domain D. let X be an indeterminate over L and let $R = D + XL[X]$. Then $D = (D, \mathcal{L}_d(D) \backslash \{0\}, \Pi_D)$ for Π_D the set of proper nonzero principal ideals (resp., t-invertible t-ideals, t-ideals, t-ideals of finite type divisorial ideals) if and only if $R = (R, \mathcal{L}_d(R) \backslash \{0\}, \Pi_R)$ with Π_R the set of proper nonzero princpal ideals (resp., t-invertible t-ideals, t-ideals, t-ideals of finite type divisorial ideals) of R and (2) Let L be an extension of the field of quotients K of an integral domain D. let X be an indeterminate over L and let $R = D + XL[X]$. Then $D = (D, \mathcal{L}_t(D), \Pi_D)$ for Π_D the set of proper nonzero principal ideals (resp., t-invertible t-ideals, t-ideals of finite type, divisorial ideals) if and only if $R = (R, \mathcal{L}_t(R), \Pi_R)$ with Π_R the set of proper nonzero princpal ideals (resp., t-invertible t-ideals, t-ideals of finite type divisorial ideals) of R.*

Proof. (1) We only carry the proof through for one case of Π_D leaving the rest to the reader. Let $D = (D, \mathcal{L}_d(D) \backslash \{0\}, \Pi_D)$ for Π_D the set of proper t-ideals (resp. t-ideals of finite type). Then by Proposition 5, every maximal ideal of D is a t-ideal (resp. t-ideal of finite type) and to show that $R = (R, \mathcal{L}_d(R) \backslash \{0\}, \Pi_R)$ with Π_R the set of proper t-ideals (resp. t-ideals of finite type) all we need do is check if every maximal ideal M of R is a t-ideal (resp. t-ideal of finite type). Now, according to the discussion prior to this proposition, any maximal ideal incomparable to $XL[X]$ is principal and hence a t-ideal (of finite type) and any maximal ideal M comparable to $XL[X]$ is of the form $M = P + XL[X]$ where P is a maximal ideal if and only if M is and P is a t-ideal if and only if M is. Since every maximal ideal of D is a t-ideal (resp. t-ideal of finite type) P is a t-ideal (resp. t-ideal of finite type) and it is easy to see that so is M a t-ideal (resp. t-ideal of finite type). For the converse note that for every maximal ideal P of D, $M = P + XL[X]$ is a maximal ideal of R and so is a t-ideal (resp. t-ideal of finite type), but this forces P to be a t-ideal (resp. t-ideal of finite type). We shall do one part of the proof of (2) and leave the rest to the reader. Suppose that $D = (D, \mathcal{L}_t(D), \Pi_D)$ for Π_D the set of proper divisorial ideals of D. Then every maximal t-ideal P is a divisorial ideal of D. Now every maximal t-ideal of R that is incomparable with $XL[X]$ is a prime ideal, hence a principal maximal ideal of the form $(1 + Xg(X))R$ and hence a divisorial ideal. This leaves maximal t-ideal of R that are comparable with $XL[X]$. These are of the form $M = P + XL[X]$ where P is a maximal t-ideal of D. But maximal t-ideals of D are divisorial and $M_v = (P + XL[X])_v = P_v + XL[X] = P + XL[X]$ (by discussion prior to this proposition). Thus $R = (R, \mathcal{L}_t(R) \backslash \{0\}, \Pi_R)$ with Π_R the set of proper nonzero divisorial ideals of R. For the converse let P be a maximal t-ideal of D. Then $M = P + XL[X]$ is a maximal t-ideal. But, by the condition,

M is divisorial which forces P to be divisorial. Whence $D = (D, \mathcal{L}_t(D), \Pi_D)$ for Π_D the set of proper divisorial ideals of D. □

Now we are ready to show that if $R = D_S$, for a multiplicative set S of $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi_D)$ for Π_D the set of proper nonzero principal ideals (resp., t-invertible t-ideals, t-ideals, t-ideals of finite type, divisorial ideals), then it may not generally be the case that $R = (R, \mathcal{L}_d(R)\backslash\{0\}, \Pi_R)$ for Π_R the set of proper nonzero principal ideals (resp., t-invertible t-ideals, t-ideals, t-ideals of finite type, divisorial ideals) of R. Let's first recall from Lemma 3.2 that if every maximal ideal is a t-invertible t-ideal then every maximal ideal is actually invertible. Now let's start constructing examples.

Example 3.3. Let L be field extension of K with $[L : K] = \infty$, let X be an indeterminate over L and consider $R = D + XL[X]$. Set $S = D\backslash\{0\}$. If every maximal ideal of D is principal (invertible, t-ideal of finite type then so is every maximal ideal of R. But that is not the case for every maximal ideal of R_S. For $R_S = K + XL[X]$ has a maximal ideal that is a t-ideal but not of finite type and hence not invertible, nor principal.

The following example has been taken, almost verbatum, from [25, Example 3.3].

Example 3.4. . There does exist at least one example of a domain D such that each maximal ideal of D is a t-ideal but for some maximal M we have MD_M not a t-ideal. One such example is that of an essential domain that is not a PVMD. (Recall that an integral domain D is essential if D has a set F of primes such that D_p is a valuation domain for each $P \in F$ and $D = \cap_{P\in F} D_P$.). Now the example in question was constructed by Heinzer and Ohm in [22] and further analyzed in [32] and [16]. As it stands the example has all except one maximal ideals of height one primes and hence t-ideals and the other maximal ideal M is a height 2 prime t-ideal. Indeed this is the maximal ideal M such that D_M is a 2-dimensional regular local ring and so with a maximal ideal that is not a t-ideal. Showing that while $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi_D)$ for Π_D the set of t-ideals of D, $D_M \neq (D_M, \mathcal{L}_d(D_M)\backslash\{0\}, \Pi_{D_M})$ for Π_{D_M} the set of t-ideals of D_M.

Now the fact that $D = (D, \mathcal{L}_t(D), \Pi_D)$ can go through the $D + XL[X]$ construction with the various definitions of Π_D can be used to construct, for example a domain of any dimention with t-maximal ideals principal. If that reminds an attentive reader of comments (3) and (4) of Remarks 8 of [33], then so be it. The point however is that the domains $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi_D)$ and $D = (D, \mathcal{L}_t(D), \Pi_D)$, with suitable Π_Ds, do not have the usual Ascending Chain Conditions on ideals (principal or t-)ideals. One may wonder if there are any simple restrictions that will get the beast under control. Yet to prepare to see that, here is another simple set of results that can come in handy when we are dealing with completely integrally closed integral domains. Of course before we bring in those results some introduction is in order. Recall that an integral domain D with quotient field K is completely integrally closed if whenever $rx^n \in D$ for $x \in K$, $0 \neq r \in D$, and every integer $n \geq 1$, we have $x \in D$. It can be shown that an intersection of completely integrally closed domains is completely integrally closed. The go to reference for Krull domains is Fossum's book [14] where you can find that D is a Krull domain if D is a locally finite intersection of localizations at height one primes such that D_P is a discrete valuation domain at each height one prime. Thus a Krull domain is completely integrally closed. Glaz and Vasconcelos [19] called an integral domain D an H-domain if there is an ideal A with $A^{-1} = D$, (or equivalently $A_v = D$) then A contains a finitely generated subideal F such that $A^{-1} = F^{-1}$. They showed that a completely integrally closed H-domain is a Krull domain. In [23, Proposition 2.4] it was shown that D is an H-domain if and only if every maximal t-ideal of D is divisorial. We have in the following a basic result and some of its derivatives.

Proposition 5. *Let D be a completely integrally closed domain. Then (1) $D = (D, \mathcal{L}_t(D), \Pi)$ is a Krull domain if and only if Π is the set of proper divisorial ideals of D, (2) $D = (D, \mathcal{L}_t(D), \Pi)$ is a locally factorial Krull domain if and only if Π is the set of proper invertible integral ideals of D, (3) $D = (D, \mathcal{L}_t(D), \Pi)$ is a Krull domain if and only if Π is the set of proper t-invertible t-ideals of D, (4) Let D be such that D_M is a Krull domain for each maximal ideal M of D. Then $D = (D, \mathcal{L}_t(D), \Pi)$ is a Krull domain if and only if Π is the set of proper divisorial ideals of D* [13] *(5) Let D be an intersection of rank one valuation domains. Then $D = (D, \mathcal{L}_t(D), \Pi)$ is a Krull domain if and only if Π is the set of proper divisorial ideals of D, (6) Let D be an almost Dedekind domain. Then $D = (D, \mathcal{L}_d(D), \Pi)$ is a Dedekind domain if and only if Π is the set of proper divisorial ideals of D.*

Proof. The idea of proof, in each case, is that every maximal t-ideal (maximal ideal) being contained in a proper divisorial ideal must be equal to it and combining this with the fact that D is completely integrally closed we get the Krull domain conclusion. For the locally factorial domain conclusion in (2) we note that every maximal t-ideal of D is invertible and so divisorial. This gives the Krull conclusion and a Krull domain is locally factorial if and only if every height one prime of D is invertible [1, Theorem 1]. For the Dedekind domain conclusion in (6), we note that every maximal ideal is of height one and divisorial, being invertible. So every maximal ideal is a t-ideal and so the domain is Krull and one dimensional. The converse in each case is obvious, in that if D is a Krull domain then D is completely integrally closed and every maximal t-ideal of D is, a t-invertible t-ideal and hence, divisorial. (If D is locally factorial, as in (2), every maximal t-ideal of D is invertible and hence divisorial.) And if D is Dedekind, then D is completely integrally closed and every maximal ideal is invertible and hence divisorial. □

It is well known that D is a Krull domain if and only if every t-ideal of D is a t-product of prime t-ideals of D [35]. As we have seen, the prime t-ideals in a Krull domain happen to be all t-invertible t-ideals, and hence maximal t-ideals and divisorial [24, Proposition 1.3]. Also, according to [37, Theorem 1.10], D is a locally factorial Krull domain if, and only if, every t-ideal of D is invertible.

Call D a t-ACC domain if D satisfies ACC on its t-invertible t-ideals.

Lemma 3.5. *Let D be a t-ACC domain and let I be a proper t-invertible t-ideal of D. Then $\cap(I^n)_t = (0)$. Consequently, in a domain satisfying t-ACC, if A is a proper divisorial ideal of D and I a t-invertible t-ideal then $(AI)_v = A$ implies $I = D$.*

Proof. Because a t-invertible t-ideal is a v-ideal of finite type with I^{-1} of finite type there is no harm in using v for t. Now let $\cap(I^n)_v \neq e$ and let x be a nonzero element in $\cap(I^n)_v$. Then there is a chain of t-invertible t-ideals $xI^{-1} \subseteq (xI^{-2})_v \subseteq ... \subseteq x(I^{-r})_v$ ... which must stop after a finite number of steps, because of the t-ACC restriction. Say $x(I^{-n})_v = x(I^{-n-1})_v$. Cancelling x from both sides we get

$(I^{-n})_v = (I^{-n-1})_v$. Multiplying both sides by I^{n+1} and applying the v-operation we get $I = D$, a contradiction that arises from assuming that there is a nonzero element in $\cap(I^n)_v$. For the consequently part note that $(AI)_v = A$ implies that $A \subseteq (I^n)_v$ for all positive integers n. □

Proposition 6. *Let D be a t-ACC domain. Then (1) $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi_D)$ is a PID if and only if Π_D is the set of proper nonzero principal ideals of D and (2) $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi_D)$ is a Dedekind domain if and only if Π_D is the set of proper invertible ideals of D and (3) $D = (D, \mathcal{L}_t(D), \Pi_D)$ is a Krull domain if and only if Π_D is the set of proper t-invertible t-ideals of D.*

Proof. We shall prove (3) and explain why it should work for the other two cases. For (3) note that $D = (D, \mathcal{L}_t(D), \Pi_D) \Leftrightarrow \forall A \in \mathcal{L}_t(D)$ $(A \neq D \Rightarrow \exists \pi \in \Pi_D(A \subseteq \pi))$ where Π is the set of proper t-invertible t- (resp., nonzero principal, invertible) ideals. Then, by the condition, every maximal t-ideal (maximal ideal) of D is t-invertible (resp., principal, invertible). By Lemma 3.5 we have for each maximal t-ideal M (maximal ideal M) $\cap(M^n)_v = (0)$ (resp., $\cap M^n = (0)$, since powers of principal (invertible) ideals are v-ideals). Thus each maximal t-ideal (maximal ideal) is of height one. Thus D is of t-dimension one (resp., of dimension one). Now, in each case, MD_M is of height one and principal, forcing D_M to be a rank one valuation domain for each maximal t-ideal (maximal ideal) M. This makes D completely integrally closed, for $D = \cap D_M$ where M ranges over maximal t-ideals (maximal ideals). Now apply Proposition 5, using the fact that each maximal t-ideal (maximal ideal) is divisorial, being a t-invertible t-ideal (principal (invertible) ideal). The converse is obvious in each case. □

Proposition 7. *Let D be a t-ACC domain. Then (1) $D = (D, \mathcal{L}_t(D), \Pi)$ is a UFD if and only if Π is the set of proper nonzero principal ideals of D and (2) $D = (D, \mathcal{L}_t(D), \Pi)$ is a locally factorial Krull domain if and only if Π is the set of proper invertible ideals of D.*

Proof. We shall prove (1) and explain why it should work for the other case. For (1) note that $D = (D, \mathcal{L}_t(D), \Pi) \Leftrightarrow \forall A \in \mathcal{L}_t(D)$ $(A \neq D \Rightarrow \exists \pi \in \Pi(A \subseteq \pi))$ where Π is the set of proper nonzero principal (invertible) ideals. Then, by the condition, every maximal t-ideal of D is principal (invertible). By Lemma 3.5 we have for each maximal t-ideal M, $\cap M^n = (0)$, since powers of principal (invertible) ideals are v-ideals. Thus each maximal t-ideal is of height one. Thus D is of t-dimension one. Now, in each case, MD_M is of height one and principal, forcing D_M to be a rank one valuation domain for each maximal t-ideal. This makes $D = \cap D_M$, where M ranges over maximal t-ideals, a completely integrally closed domain. Now apply Proposition 5, using the fact that each maximal t-ideal is divisorial, being principal or invertible. This gets us the Krull conclusion. Now recall that in a Krull domain D, $A_t = (P_1^{n_1}...P_r^{n_r})_t$. Then D is locally factorial by [37, Theorem 1.10] and D is factorial because every principal ideal is a product of prime powers. The converse, in each case, is obvious in that a UFD (locally factorial Krull domain) is Krull every maximal t-ideal of whose is principal (resp., invertible). □

An integral domain D that satisfies ACC on integral divisorial ideals is called a Mori domain. Obviously a Noetherian domain is a Mori domain. It is easy to check that for every nonzero integral ideal A of a Mori domain D there are elements $a_1, .., a_r \in A$ such that $A_v = (a_1, .., a_r)_v$. So the inverse of a nonzero ideal of a Mori domain is a v-ideal of finite type. Hence a v-invertible ideal in a Mori domain is t-invertible. It is well known that a domain D is a Krull domain if, and only if, every nonzero ideal of D is t-invertible (see e.g. [34, Theorem 2.5]) and thus a Krull domain is Mori too. Noting that a Mori domain is a t-ACC domain and that Noetherian is Mori too, we have the following direct corollaries. Yet when their environments afford independent proofs we include them too.

Corollary 2. *Let D be a Mori domain. Then (1) $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi)$ is a PID if and only if Π is the set of proper nonzero principal ideals of D, (2) $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi)$ is a Dedekind domain if and only if Π is the set of proper invertible ideals of D, (3) $D = (D, \mathcal{L}_t(D), \Pi)$ is a Krull domain if and only if Π is the set of proper t-invertible t-ideals of D, (4) $D = (D, \mathcal{L}_t(D), \Pi)$ is a UFD if and only if Π is the set of proper nonzero principal ideals of D and (5) $D = (D, \mathcal{L}_t(D), \Pi)$ is a locally factorial Krull domain if and only if Π is the set of proper invertible ideals of D.*

Corollary 3. *Let D be a Noetherian domain. Then (1) $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi)$ is a PID if and only if Π is the set of proper nonzero principal ideals of D and (2) $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi)$ is a Dedekind domain if and only if Π is the set of proper invertible ideals of D.*

Proof. $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi) \Leftrightarrow \forall A \in \mathcal{L}_d(D)\backslash\{0\}$ $(A \neq D \Rightarrow \exists \pi \in \Pi(A \subseteq \pi))$ where Π is the set of proper nonzero principal (invertible) ideals. Start with a proper nonzero ideal A of D. Then by the condition $A \subseteq \pi_1$ for some $\pi_1 \in \Pi$. Let $A_1 = A\pi_1^{-1}$. Then $A \subsetneq A_1$. If $A_1 = D$ we have $A = \pi_1$ a principal (invertible) ideal and we are done. If $A_1 \neq D$ then by the condition $A_1 \subseteq \pi_2$ where π_2 is a proper principal (invertible) ideal. Let $A_2 = A_1\pi_2^{-1} = A(\pi_1\pi_2)^{-1}$. Then $A \subsetneq A_1 \subsetneq A_2$ and at a general stage $A \subsetneq A_1 \subsetneq A_2 \subsetneq ... \subsetneq A_r \subsetneq ...$ where $A_i = A(\Pi_{j=1}^{i}\pi_j)^{-1}$. (If there is any doubt about the chain being strictly ascending, take a cue from Lemma 3.5.) Now, being a Noetherian domain, D cannot afford an infinite strictly ascending chain of proper integral ideals. Whence at some stage n, $A_n = A(\Pi_{j=1}^{n}\pi_j)^{-1} = D$, forcing $A = \Pi_{j=1}^{n}\pi_j$. This makes a typical ideal A of D principal (invertible). The converse in both cases is obvious. □

Corollary 4. *Let D be a Mori domain. Then (1) $D = (D, \mathcal{L}_v(D), \Pi)$ is a UFD if and only if Π is the set of proper nonzero principal ideals of D, (2) $D = (D, \mathcal{L}_v(D), \Pi)$ is a locally factorial Krull domain if and only if Π is the set of proper invertible integral ideals of D, (3) $D = (D, \mathcal{L}_v(D), \Pi)$ is a Krull domain if and only if Π is the set of proper t-invertible t-ideals of D.*

Proof. According to [37], given that I is an invertible fractional ideal we have $(IB)_v = IB_v$ for all $B \in F(D)$. Thus proving (3) may be sufficient. For (3) let A be a proper integral v-ideal of the Mori domain D. By the condition, A must be contained in a proper integral t-invertible t-ideal I_1 of D. That is $A \subseteq I_1$. Let $A_1 = (AI_1^{-1})_v$. If $A_1 = D$ we have nothing more to prove because then $A = I_1$, a proper t-invertible t-ideal of D. So let's assume that $A_1 \subsetneq D$. Then by Lemma 3.5, $A \subsetneq A_1$. Since A_1 is proper there is a proper t-invertible t-ideal $I_2 \in \Pi$ such that $A_1 \subseteq I_2$. Let $A_2 = (A_1I_2^{-1})_v. = (AI_1^{-1}I_2^{-1})_v$. If $A_2 = D$ we get $A = (I_1I_2)_v$. So let us assume A_2 is proper. This gives $A \subsetneq A_1 \subsetneq A_2 \subsetneq D$. Continuing in this manner we get a strictly ascending chain $A_1 \subsetneq A_2 \subsetneq ... \subsetneq A_j \subsetneq ...$ where $A_j = (A(\Pi_{i=1}^{j}I_i^{-1}))_v$. Now because D is Mori this strictly ascending chain of divisorial ideals cannot go on for ever and so $A_j = D$ for some j, forcing $A = (\Pi_{i=1}^{j} I_i)_v$ and making A a t-invertible t-ideal. The converse is obvious because in a Krull domain every nonzero ideal is t-invertible and so are its v-ideals. (All proper v-ideals are in Π.). □

Finally, consider the following scheme of results.

Proposition 8. *Suppose that D satisfies ACCP. Then (1) $D = (D, \mathcal{L}_d(D)\backslash\{0\}, \Pi)$ is a PID if and only if Π is the set of proper nonzero principal ideals of D and (2) $D = (D, \mathcal{L}_t(D), \Pi)$ is a UFD if and only if Π is the set of proper nonzero principal ideals of D.*

Proof. $D = (D, \mathcal{L}_\star(D)\backslash\{0\}, \Pi) \Leftrightarrow \forall A \in \mathcal{L}_\star(D)\backslash\{0\}$ $(A \neq D \Rightarrow \exists \pi \in \Pi(A \subseteq \pi))$ where Π is the set of proper nonzero principal ideals of D and $\star = d$ or t. Then, by the condition, for any maximal (maximal t-ideal) M of D we have $M \subseteq \pi$ for some $\pi \in \Pi$ and so $M = \pi D$. Claim that, because of the ACCP, M is of height one. (For if not, then there is $Q \subseteq \cap \pi^n D$. So for every nonzero $x \in Q$, x is divisible by every power of π, giving rise to an infinite ascending chain $xD \subsetneq \frac{x}{\pi}D \subsetneq \frac{x}{\pi^2} \subsetneq ... \subsetneq \frac{x}{\pi^n}D \subsetneq ...$ which is impossible in the presence of ACCP on D.) Now MD_M is principal and of height one, making D_M a rank one discrete valuation domain and making $D = \cap D_M$ completely integrally closed with every maximal (t-) ideal principal. This makes D a Krull domain with every height one prime a principal ideal and so a UFD. Finally, a UFD with every height one prime maximal is a PID. The converse, in each case is straightforward. □

The above Proposition may revive an old question touched on in [34]: If D has ACCP and M a maximal t-ideal, must M be of height one? We couldn't answer it then and we had to resort to using the strong ACCP: D has ACCP and D_M has ACCP for every maximal t-ideal M. Now I have taken the route of using the t-ACC and this gives rise to: If D has t-ACC, must D_M have ACCP for each maximal t-ideal M?

The ideas touched on above got developed in a different direction, in [41], using predicates; a demonstration of the fact that the change of language is at times the change of degrees of freedom. The point of this exercise here is to indicate how pre-Riesz monoids can help create examples. My response to any, possibly, raised eyebrows is that there are examples of direct translations from ring theoretic results to results on monoids in Halter-Koch's book [20] and in most of his work close to the end of his career. (See for instance his paper on mixed invertibility [21].) There are other examples, even in ideal theory, almost all results on PVMDs, originally proved using the t-operation are now being stated and proved for the so-called w-operation. So, hopefully, there's no problem with some applications being duplicated. The other reason for the "duplication", if you see it that way, is that the other paper is not published yet and mere reference to unpublished work may not help the reader. There is also the thought that as "contains means divides" some of these ideas may be extended to non-commutative systems or to nearrings. Finally, if the notation irks some readers, I adopted the notation $D = (D, \mathcal{L}_t(D), \Pi)$ etc., because it enables me to state and prove a number of theorems in one go. It is also a shorthand way of recording one's thoughts about domains of this type.

4. Virtual factoriality

Now let me end the article with a mention of something positive, a kind of unique factorization. Recall that two positive elements a, b of a p.o. group G are said to be disjoint if $h \leq a, b$ implies $h \leq e$, the identity of the group. While this definition works admirably in p.o. groups, it can cause confusion in the monoid set up. If, following Birkhoff, (see comment on page 220 of [5]),we liken disjointness with "relative primeness" or coprimenesses, then in an integral domain there are two kinds of coprimenesses: One that comes from the lack of non-unit common factors in a set of elements and one that comes from the elements being disjoint in the group of divisibility of the domain. As the theory of factorization in integral domains developed, the two kinds got distinct names.

Definition 4.1. Let x, y be two elements in $D\backslash\{0\}$, where D is an integral domain. Then x, y are coprime if $z|x, y$ implies that z is a unit in D, i.e., if $GCD(x, y)$ is a unit and x, y are v-coprime if $xD \cap yD = xyD$, i.e., if $(x, y)_v = D$.

As indicated in [40], page 389, the notion of x, y being v-coprime gets translated to xD, yD being disjoint in $G(D) = \{xD|x \in K\backslash\{0\}\}$ ordered by $xD \leq yD \Leftrightarrow xD \supseteq yD$. As is also indicated in [40] there are several shades of coprimality, thanks to the star operations. We may say that x, y are $\star$-coprime if $(x, y)^\star = D$. But the kinds of coprimality outlined in the definition above are the two extremes. Again, it was indicated in [40] that while $x, y \in D\backslash\{0\}$ being v-coprime implies that x, y are coprime, yet it is not the case the other way around. That is the notion of v-coprimality is more reliable. And in a general monoid situation our best bet would be to stick with the most reliable, unless a specific situation requires a deeper digging. It turns out that the best use of v-coprimality comes from the results that if, for $a, b, c \in D\backslash\{0\}$, a and c are v-coprime and if $a|bc$, then $a|b$, and $(a, bc)_v = D$ if and only if $(a, b)_v = (a, c) = D$, see Propositions 2.2 and 2.3 of [40].

We note that if in a Riesz monoid $< M, *, e, \leq >$ we have $e \leq a \leq b * c$ where $a \wedge c = e$, i.e., a and c are disjoint, then $a \leq b$. This can be shown by noting that a is primal. Generally if for $< M, *, e, \leq >$, M^+ is the positive cone of a directed p.o. group G one can prove the same result, because if $a \wedge c$ (resp., $a \vee c$) exists in G then $x * (a \wedge c) = (x * a) \wedge (x * c)$ (resp., $x * a \vee c = (x * a) \vee (x * c)$) for all $x \in G$ and hence for all $x \in G^+$.

Proposition 9. *Let M^+ be the positive cone of a directed p.o. group that is abelian. Then in $M^+\backslash\{e\}$ $a \leq b * c$, where $a \wedge c = e$, implies that $a \leq b$.*

Proof. Note that as $a \wedge c = e$ we have $b = b * (a \wedge c) = (b * a) \wedge (b * c)$, by the remark prior to the statement of this proposition. Now $a \leq a * b, b * c$ implies $a \leq (b * a) \wedge (b * c) = b$. □

A result similar to the above can also be proved in, what we have chosen to call, the reverse multiplicative lattice $\Delta_\star(D) =< I^\star(D), +^\star, \times^\star, \wedge, \vee, D >$.

Proposition 10. *If $A, B, C \in \Delta_\star(D) =< I^\star(D), +^\star, \times^\star, \wedge, \vee, D >$ such that $A \wedge C = D$, then $A \leq B \times^\star C$ implies that $A \leq B$.*

Proof. Recall that reverse containment rules in $\Delta_\star(D)$. Consequently D is the least element of the lattice, $X \leq Y$ in $\Delta_\star(D)$ if and only if $X \supseteq Y$ and $X \wedge Y$ becomes $(X +^\star Y) = (X + Y)^\star$. Thus the statement "$A \leq B \times^\star C$ where $A \wedge C = D$" translates to "$A \supseteq B \times^\star C$ where $(A + C)^\star = D$". Now as we also have $A \supseteq A \times^\star B$ we can say that $A \supseteq A \times^\star B, B \times^\star C$ and so, as A is a $\star$-ideal we have $A \supseteq A \times^\star B +^\star B \times^\star C = (AB * BC)^\star = (B(A + C))^\star = (B(A + C)^\star)^\star = B$, giving us $A \supseteq B$ which translates back to $A \leq B$. □

Proposition 10 serves several purposes. First of all, because there is no concept of invertibility or cancellation in $\Delta_*(D)$, nor in $\mathcal{L}_*(D)$, it shows that results such as "$a \leq b * c$ and $a \wedge c = e$, implies that $a \leq b$" may hold in monoids that are neither divisibility nor cancellative. It also shows that while in some monoids $a \wedge b = e$ may make sense as the elements a, b being disjoint, i.e. $h \leq a, b$ implies $h \leq e$, in others such as $\mathcal{L}_*(D)$ a, b being disjoint may well be equivalent to $a \vee b = 1$ where 1 is the largest element of the lattice. Of course in $\mathcal{L}_*(D)$ $a \wedge b = 0$, if and only if $a = 0$ or $b = 0$. Then, as pointed out, after definition 12, in [27], there are monoids in which $a \wedge b_i = e$ for $i = 1, .., n$ but $a \wedge (b_1 * ... * b_n) > e$. One purpose of studying monoids is to see if some kind of unique factorization exists in them and it helps to separate or collect factors of an element of a monoid by disjointness or lack of it, in the monoid envioronment. So if in a monoid M, elements $a, b_1, ..., b_n$ such that $a \wedge b_i = e$, for $i = 1, ..., n$, implies $a \wedge (b_1 * ... * b_n) = e$ we know that we are in a safe environment. This can happen in cases indicated in the following proposition. Yet before that let's call M a multiplicative $\wedge$-semilattice if $a \wedge b \in M$ for all $a, b \in M$ and for all $x \in M$ we have $x * (a \wedge b) = x * a \wedge x * b$, $\wedge$ is an associative and commutative binary operation and $x = x \wedge x$ for all $x \in M$. Also call a monoid M $\wedge$-smooth if whenever $a \wedge b$ exists for some $a, b \in M$ we have $x = x * a \wedge x * b$ for all $x \in M$ and if $a \wedge (x \wedge y)$ or $(a \wedge x) \wedge y)$ exists we have $a \wedge (x \wedge y) = (a \wedge x) \wedge y)$.

Proposition 11. *(1) Let M be a $\wedge$-smooth monoid. If, for $a, b_1, ..., b_n \in M$, we have $a \wedge b_i = e$, for $i = 1, ..., n$ in M, then $a \wedge (b_1 * ... * b_n) = e$. Conversely if, for $a, b_1, ..., b_n \in M$, we have $a \wedge (b_1 * ... * b_n) = e$, then $a \wedge b_i = e$ for all $i = 1, 2, ..., n$. (2). If M is a multiplicative $\wedge$-semi-lattice and if, for $a, b_1, ..., b_n \in M$, we have $a \wedge b_i = e$, for $i = 1, ..., n$, in M, then $a \wedge (b_1 * ... * b_n) = e$.*

Proof. (1). The case of $n \leq 2$ is provided by the definition. Suppose that we have established the proposition for $n \leq r$ for $r \geq 2$. That is if $a \wedge b_i = e$, for $i = 1, ..., r$ in M, then $a \wedge (b_1 * ... * b_r) = e$. Let $a \wedge b_{r+1} = e$ and consider $a \wedge (b_1 * ... * b_{r+1})$. Since $a \leq a * b_1 * ... * b_r$ we have $a = a \wedge a * b_1 * ... * b_r$ and so $a \wedge (b_1 * ... * b_{r+1}) = (a \wedge (a * b_1 * ... * b_r)) \wedge (b_1 * ... * b_{r+1}) = (a \wedge ((a * b_1 * ... * b_r)) \wedge (b_1 * ... * b_{r+1}) = a \wedge ((b_1 * ... * b_r) * (a \wedge b_{r+1}))$ (and because $(a \wedge b_{r+1}) = e$ we have)

$a \wedge (b_1 * ... * b_{r+1}) = a \wedge ((b_1 * ... * b_r)$. But by the induction hypothesis, we have $a \wedge ((b_1 * ... * b_r) = e$, So $a \wedge (b_1 * ... * b_{r+1}) = e$. Thus (1) holds for all integers > 1. For the converse note that as $b_i \leq b_1 * b_2 * ... * b_n$ we have $b_i = b_i \wedge (b_1 * b_2 * ... * b_n)$. Thus $a \wedge b_i = a \wedge (b_i \wedge (b_1 * b_2 * ... * b_n) = a \wedge (b_1 * b_2 * ... * b_n \wedge b_i) = (a \wedge b_1 * b_2 * ... * b_n) \wedge b_i = e \wedge b_i$.

(2). Follows from (1), because a multiplicative $\wedge$-semilattice is $\wedge$-smooth. □

Dually we can talk about multiplicative $\vee$-semilattices, and $\vee$-smooth monoids, and noting that the dual of $a \wedge b = e$ is $a \vee b = 1$, the largest element, we can state and prove the dual of Proposition 11, by replacing $\wedge$ by $\vee$, $\leq$ by $\geq$ and e by 1.

Next, there are monoids such as the monoid of nonzero finitely generated integral ideals of D, partially ordered by inclusion (or by reverse inclusion) denoted by $f(D)$ (respectively by $\varphi(D)$) under multiplication of ideals. We can also talk about $f_*(D)$ and $\varphi_*(D)$ of integral $*$-ideals of finite type closed under $*$-multiplication.

Now in a Pre-Riesz monoid, as we have defined it, there seems to be no indication if a pre-Riesz monoid is $\wedge$-smooth (or $\vee$-smooth). One way of dealing with the situation is to require that we deal only with smooth Pre-Riesz monoids. Thus we have the obvious corollary.

Corollary 5. *Given that M is a $\wedge$-smooth pre-Riesz monoid. If, for $a, b_1, ..., b_n \in M^+$, we have $a \wedge b_i = e$, for $i = 1, ..., n$ in M, then $a \wedge (b_1 * ... * b_n) = e$. Conversely if $a \wedge (b_1 * ... * b_n) = e$, $b_1, ..., b_n \in M^+$, then $a \wedge b_i = e$.*

My interest in $\wedge$ (or $\vee$)-smooth pre-Riesz monoids arose from the fact that they are as amenable to factorization as Riesz monoids. That was the reason why I looked into the bases of pre-Riesz groups via Conrad's F-condition, with Y.C. Yang [36]. I was hoping to find the ultimate building blocks of factorization, in the positive cones of pre-Riesz groups, as Conrad did in the form of a basic element in the case of lattice ordered groups. Let's call an element h of a monoid M a homogeneous element if $h > e$ (i.e. h is strictly positive) and for all $u, v \in (e, h]$ ($=$ for all $e < u, v \leq h$) we yhave $e < l \leq u, v$). It is easy to see that if x is a homogeneous element of M, then so is each $e < t \leq x$. Let's first establish that homogeneous elements are not too hard to find.

Lemma 4.2. *Let M be a pre-Riesz monoid such that $M^+ = M$. Then the following hold. (1). For $x, y, h \in M$, $e < h \leq x$ and $x \wedge y = e$ implies $h \wedge y = e$. (2). For $x, y, h_1, h_2 \in M$, $e < h_1, h_2 \leq x$ and $x \wedge y = e$ implies $h_i \wedge y = e$ and if in addition $h_1 \wedge h_2 = e$, then h_1, h_2, y are mutually disjoint. (3) If $x_1, ..., x_n \in M$ such that x_i are mutually disjoint and $e < h_{i1}, h_{i2} \leq x_i$ for some $i \in \{1, ..., n\}$, such that $h_{i1} \wedge h_{i2} = e$, then $\{x_1, ..., x_{i-1}, h_{i1}, h_{i2}, x_{i+1}, ..., x_n\}$ are mutually disjoint. (4). If $e < h, k \in M$, then h and k are nondisjoint if and only if there is $0 < t \leq h, k$.*

Proof. (1) If h, y are non-disjoint then by the pre-Riesz property, there is $e < r \leq h, y$. But since $h \leq x$ we have $e < r \leq x, y$ a contradiction. Now (2) follows from (1) by noting that each of h_i is disjoint with y and if we add to it the fact that $h_1 \wedge h_2 = e$ we have the conclusion that $\{h_1, h_2, y\}$ are mutually disjoint. For (3) note that $\{h_{i1,} h_{i2}, x_j | i \neq j\}$ are mutually disjoint by (2). (4) follows from the definition of a pre-Riesz monoid. □

Proposition 12. *Let M be a pre-Riesz monoid such that $M^+ = M$, i.e. M is conic. If M satisfies "CFC: every strictly positive element exceeds at most a finite number of mutually disjoint strictly positive elements", then every strictly positive element of M exceeds at least one homogeneous element and at most a finite number of mutually disjoint homogeneous elements.*

Proof. Let $e < x \in M$ and let there be $e < u, v \leq x$. The pre-Riesz condition provides that if u, v are non-disjoint there is a strictly positive l such that $e < l \leq u, v \leq x$. Now if x is such that for all $u, v \in (e, h]$ we have $e < l \leq u, v$, then x is a homogeneous element. If x is not homogeneous then there must be at least two disjoint elements u_1, v_1 preceding x. Let n be the largest number of mutually disjoint strictly positive elements of M preceding x and let $\{x_1, ..., x_n\}$ be a set of n elements such that $0 < x_i \leq x$ where x_i are mutually disjoint. Then each of x_i is a homogeneous element. For if not, and say x_i is not homogeneous, then there exist at least two elements $e < h_{i1}, h_{i2} \leq x_i$ such that $h_{i1} \wedge h_{i2} = e$. By Lemma 4.2 the set $\{x_1, ...x_{i-1}, h_{i1}, h_{i2}, x_{i+1}, ..., x_n\}$ of $n+1$ elements, consists of mutually disjoint strictly positive elements preceding x, a contradiction. Whence each of x_i is homogeneous. Now to establish beyond doubt that x exceeds at least one homogeneous element we proceed as follows. Let $e < x \in M$ and let there be $e < u, v \leq x$. The pre-Riesz condition provides that if u, v are non-disjoint there is a strictly positive l such that $e < l \leq u, v \leq x$. Now, as noted above, if x is such that for all $u, v \in (e, h]$ we have $e < l \leq u, v$, then x is a homogeneous element. If x is not homogeneous then there is a pair of strictly positive disjoint elements h, k preceding x. If either of h, k is homogeneous we can stop. If not, we can find a pair h_1, h_2 of strictly positive disjoint elements preceding h, noting that h_1, h_2, k are mutually disjoint by Lemma 4.2, in particular h_2, k are disjoint. If neither of h_1, h_2 is homogeneous, find $e < h_{11}, h_{12} \leq h_1$ to get $e < h_{11}, h_{12}, h_2, k$ mutually disjoint preceding x, noting that in particular $\{h_{12}, h_2, k\}$ are mutually disjoint. Assuming that neither of h_{11}, h_{12} is homogeneous and repeating the previous step with h_{11} to get $e < h_{111}, h_{112} \leq h_{11}$ such that $h_{111} \wedge h_{12} = e$; noting that by Lemma 4.2 $\{h_{111}, h_{112}, h_{12}, h_2, k\}$ are mutually disjoint and in particular $\{h_{112}, h_{12}, h_2, k\}$ are mutually disjoint. Now this cannot go on indefinitely as each new step increases the number of mutually disjoint strictly positive elements preceding x by one and makes the number of mutually disjoint elements preceding x tend to infinity, contradicting CFC. □

Remark 4.3. "CFC" in the statement of Proposition 12 stands for Conrad's F-Condition, as used in [7]. The plan in the proof of the latter part of Proposition 12 is taken from Theorem 5.2 of [7].

The notion of a homogeneous element was developed in [31] for the study of factoriality in general Riesz groups, without any reference to commutativity. So it should work well in a commutative Riesz monoid. We now proceed to show that the notion of factoriality based on homogeneous elements in Riesz groups works well in some pre-Riesz monoids. As a first step, let's call two homogeneous elements $h, k \in M$ similar, denoted as $h \sim k$, if there is $e < t \leq h, k$, that is if h and k are non-disjoint ($h \wedge k \neq e$). Indeed if we use $(e, h]$ to mean the set of all $e < u \leq h$ and if h is a homogeneous element of M, then each $u \in (e, h]$ is homogeneous.

Proposition 13. *Let h and k be two homogeneous elements in a pre-Riesz monoid M. Then the following are equivalent. (1) $h \wedge k = e$, (2) for every pair $(a, b) \in (e, h] \times (e, k]$ we have $a \wedge b = e$ (3) for some pair $(a, b) \in (e, h] \times (e, k]$ we have $a \wedge b = e$.*

Proof. (1) $\Rightarrow$ (2) Suppose (2) does not hold then for some pair $(a, b) \in (e, h] \times (e, k]$ we have $a \wedge b \neq e$ which, in a pre-Riesz monoid, means that there is $e < t \leq a, b$. But then $e < t \leq h, k$ which forces $h \wedge k \neq e$.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). Suppose that $h \wedge k \neq e$. Then there is $e < t \leq h, k$. This makes t a homogeneous element. Now let $(a, b) \in (e, h] \times (e, k]$. Since $e < t, a \leq h$ and h is homogeneous we have $t \wedge a \neq e$. Also since we are in a pre-Riesz monoid, there is a $e < t_1 \leq t, a$. Similarly there is a $e < t_2 \leq t, b$. Now since t is homogeneous, $e < t_1, t_2 \leq t$ and we are in a pre-Riesz monoid, there must be a $e < t_3 \leq t_1, t_2$. But then $t_3 \leq a, b$, forcing $a \wedge b \neq e$. In other words, negation of (1) implies the negation of (3). □

By negating the constituent statements in Proposition 13 we get the following statement.

Proposition 14. *Let h and k be two homogeneous elements in a pre-Riesz monoid M. Then the following are equivalent. (1) $h \wedge k \neq e$, (2) for every pair $(a, b) \in (e, h] \times (e, k]$ we have $a \wedge b \neq e$, (3) for some pair $(a, b) \in (e, h] \times (e, k]$ we have $a \wedge b \neq e$.*

Proposition 15. *Let $h(M)$ be the set of all homogeneous elements of a pre-Riesz monoid M. Then similarity is an equivalence relation on H.*

Proof. The proof follows exactly the same lines as the proof of (6) of Proposition 1.1 of [31]. That is, reflexivity and symmetry being clear, all we need is check if transitivity works. For this let $u, v, w \in h(M)$ such that $u \sim v$ and $v \sim w$. Now as $u \sim v$ we have $u \wedge v \neq e$. By the pre-Riesz property there is $e < t_1 \leq u, v$. Next as $v \sim w$ we have $v \wedge w \neq e$ and by the pre-Riesz property there is $e < t_2 \leq v, w$. Now as $e < t_1, t_2 \leq v$ and as v is homogeneous, and t_1, t_2 strictly positive, we have $t_1 \wedge t_2 \neq e$. But the pre-Ries property again gives $e < t_3 \leq t_1, t_2$ which implies that $e < t_3 \leq t_1 \leq u$ and $e < t_3 \leq t_2 \leq w$. Implying $e < t_3 \leq u, w$ which is the same as saying that $u \wedge w \neq e$ or $u \sim w$. □

Proposition 16. *Let M be a pre-Riesz monoid. Let $e < x, y \in M$ such that $x \wedge y = e$. If h is a homogeneous element, then h must be disjoint with at least one of x, y. Generally if $x_1, ..., x_n$ are mutually disjoint strictly positive elements of a pre-Riesz monoid M and h is a homogeneous element of M, then h must be disjoint with at least $n-1$ of x_i. Consequently if $x_1, ..., x_n$ are mutually disjoint strictly positive elements of a $\wedge$-smooth pre-Riesz monoid M and h is a homogeneous element of M with $h \leq x_1 * ... * x_n$, then $h \leq x_i$ for exactly one $1 \leq i \leq n$.*

Proof. Suppose on the contrary that h is non-disjoint with both of x and y. Then since we are working inside a pre-Riesz monoid there are $e < t_1 \leq x, h$ and $e < t_2 \leq y, h$. But then $e < t_1, t_2 \leq h$ and h is homogeneous. This leads to the existence of $e < t_3 \leq t_1, t_2$ and to $e < t_3 \leq x, y$ which contradicts the disjointness of x and y. For the general case let $x = x_i$ and $y = y_j, i \neq j$. As we are working in a $\wedge$-smooth environment where $h \leq b * c$ and $h \wedge c = e$ implies $h \leq b$ and as, being homogeneous h cannot be non-disjoint with more than one disjoint elements, we can, by assuming $x_i \wedge h = e$ for $i \neq j$, conclude that $h \leq x_j$ for exactly one j. □

Proposition 17. *Let h_1, h_2 be two similar homogeneous elements in a $\wedge$-smooth pre-Riesz monoid M. Then $h_1 * h_2$ is a homogeneous element similar to both h_i. Generally if $h_1, h_2, ..., h_n$ are mutually similar homogeneous elements of a $\wedge$-smooth pre-Riesz monoid M, then $h_1 * ... * h_n$ is a homogeneous element similar to each of h_i.*

Proof. Let $e < u, v \leq h_1 * h_2$. Claim that $u \wedge h_i \neq e$ and $v \wedge h_i \neq e$. For if, say $u \wedge h_1 = e$, then $u \leq h_1 * h_2$ implies that $u \leq h_2$. But as $e < u \leq h_2$ and h_2 is homogeneous with $h_1 \wedge h_2 \neq e$ we must have $u \wedge h_1 \neq e$, by Proposition 14, a contradiction. Now $u \wedge h_1 \neq e$ implies the existence of $e < t_1 \leq u, h_1$, $v \wedge h_1 \neq e$ implies the existence of $e < t_2 \leq v, h_1$ and as h_1 is homogeneous, $t_1 \wedge t_2 \neq e$. But this, by the pre-Riesz property, means that there is $e < t_3 \leq t_1, t_2$. That $e < t_3 \leq u, v$ is now obvious. Thus for any pair of strictly positive elements u, v preceding $h_1 * h_2$ we have $u \wedge v \neq e$ and so $h_1 * h_2$ is homogeneous. Now suppose that we have established the general statement for $n \leq r$. Then, by the induction hypothesis, $H = h_1 * ... * h_r$ is homogeneous similar to each of h_i. If h_{r+1} is homogeneous similar to any of h_i, h_{r+1} is similar to H, by Proposition 14. By the case of $n = 2$, $H * h_{r+1}$ is homogeneous similar to H and h_{r+1} and hence to all of h_i. □

Having established all the requirements of a theory of factoriality we proceed as follows.

Theorem 4.4. *Let M be a $\wedge$-smooth pre-Riesz monoid and let $x \in M$. If x is expressible as $h_1 * ... * h_r$ a sum/product of finitely many homogeneous elements, then x is expressible, uniquely, as a sum/product of mutually disjoint homogeneous elements, upto permutation of summands/factors.*

Let $x = h_1 * ... * h_r$, where h_i are homogeneous. Pick, say, h_1 and pick all the factors/summands similar to h_1. Suppose that, by a relabeling, the first n_1 factors/summands of x are similar to h_1, the rest are disjoint because similarity is an equivalence relation. Let $H_1 = h_1 * ... * h_{n_1}$. Then $x = H_1 * h_{n_1+1} * ... * h_r$, where $h_{n_1+1}, ..., h_r$ are all disjoint with h_1 and hence with H_1, via Proposition 13. Now repeat the previous step with h_{n_1+1}, collecting all the homogeneous summands/factors similar to h_{n_1+1} and assume relabeling, if necessary, that $h_{n_1+1}, ..., h_{n_2}$ is the set of all homogeneous factors of $h_{n_1+1} * ... * h_r$, (and hence of x) similar to h_{n_1+1} and form $H_2 = h_{n_1+1} * ... * h_{n_2}$, via Proposition 17 Since each of h_i ($n_1 + 1 \leq i \leq n_2$) is disjoint with $h_1$we coclude that $x = H_1 * H_2 * h_{n_2+1} * ... * h_r$ where the rest of the h_i are disjoint with $H_1 * H_2$. Repeat the first step with h_{n_2+1} and so on to get $x = H_1 * H_2 * H_3 * ... * H_n$ where each H_i disjoint with all the previous ones. Because similarity is an equivalence relation H_i are mutually disjoint, being separated on the basis of similarity and disjointness and of course each of H_i is homogeneous by construction. There's an alternative method that may perhaps be easier for some, though harder in practice. Select from $H(x) = \{h_1, ..., h_r\}$ a set $\mathcal{K}_k = \{h_{01}, h_{02}, ..., h_{0k}\}$ of mutually disjoint homogeneous factors/summands of x. If there is a member of $H(x)$ that is disjoint with each member of $\mathcal{K}_k$, then label it as h_{0k+1} and form $\mathcal{K}_{k+1} = \{h_{01}, h_{02}, ..., h_{0k+1}\}$. Repeat untill you get to a stage $\mathcal{K}_n = \{h_{01}, h_{02}, ..., h_{0n}\}$ where h_{0i} are mutually disjoint and there's no member $h_j$of $H(x)$ left such that $h_j \wedge h_{0i} = e$. In this case claim that we have a maximal set of mutually disjoint homogeneous factors/summands of x. Now suppose that there are two sets $\mathcal{K}_m = \{h_{01}, h_{02}, ..., h_{0m}\}$ and $\mathcal{K}_n = \{k_{01}, k_{02}, ..., k_{0n}\}$ where $m < n$. As $\mathcal{K}_m$ is maximal, and as we are dealing with homogeneous elements, each k_{0i} is similar to exactly one of h_{0j} and similars can replace similars. By relabeling we can assume that k_{0i} replaces h_{0i}. But then the extra ones in $\mathcal{K}_n$ would have to be similar to some that they were disjoint with in $\mathcal{K}_n$. Once that is settled take the maximal set $\mathcal{K}_n = \{h_{01}, h_{02}, ..., h_{0n}\}$ and form: $H(h_{0i}) = \{h \in H(x) | h \sim h_{0i}\}$. Then $\{H(h_{0i})\}$ is a partition of $H(x)$. Next write $H_i = \Pi_* h$ where h varies over $H(h_{0i})$. Then by construction and by Propositions 13, 14 and 17$x = H_1 * ... * H_n$ is a product of mutually disjoint homogeneous elements. Now suppose that $x = H_1 * ... * H_n = K_1 * ... * K_m$ where $H_1, ..., H_n$ (resp., $K_1, ..., K_m$) are mutually disjoint homogeneous elements. Since $H_1 \leq x = K_1 * ... * K_m$, $H_1 \leq K_j$ for exactly one j, by Propositions 16. Similarly as $K_j \leq x = H_1 * ... * H_n$ and as, being homogeneous, K_j cannot be nondisjoint with two disjoint elements we conclude that $K_j \leq H_1$. That is $K_j = H_1$. Repeating with H_2 and so on we conclude that $n \leq m$. Similarly repeating the whole process with K_i we get $m \leq n$. Now relabeling K_j we can get the correspondence $H_i = K_i$.

If M is a $\wedge$-smooth pre-Riesz monoid with a non-empty set $h(M)$ of homogeneous elements we can always form a sub-semigroup $H^*(M) = \{h_1 * h_2 * ... * h_n | h_i \in h(M)\}$ of finite sums/products of members of $h(M)$. The sub-semigroup $H^*(M)$ can be made into a monoid $H_h(M) = H^*(M) \cup \{e\}$ by throwing in e as an empty sum/product. Call a directed p.o. monoid M semi homogeneous if M is conic and each strictly positive element of M is a finite sum of homogeneous elements and call M virtually factorial (v-factorial) if M is semi-homogeneous such that every strictly positive element of M is uniquely expressible as a sum/product of finitely many mutually disjoint elements.

Corollary 6. *Let M be a $\wedge$-smooth pre-Riesz monoid with $h(M) \neq \phi$. Then the submonoid $H_h(M) = \{h_1 * h_2 * ... * h_n | h_i \in h(M)\} \cup \{e\}$ is a v-factorial monoid. Moreover $H_h(M)$ is a $\wedge$-smooth pre-Riesz monoid. Consequently if a $\wedge$-smooth pre-Riesz monoid M satisfies CFC, then $H_h(M)$ is a v-factorial monoid.*

Proof. That $H_h(M)$ is v-factorial follows from Theorem 4.4. For the pre-Riesz part let $e < x, y \in H(M)$, under the induced partial order from M, and suppose that $x \wedge y \neq e$. Since $x \wedge y \neq e$ means there is $g \leq x, y$ with $g \nleq e$ and since $H_h(M)$ is conic by construction we conclude that there is $e < t \leq x, y$ in $H_h(M)$. The $\wedge$-smooth part follows because the partial order is induced. The consequently part is obvious. □

This brings us to the examples of virtual factoriality in pre-Riesz monoids. We would be selective, as they are strewn all over. For a start let us recall that a nonzero finitely generated ideal A of a domain D is called primitive if $A \subseteq aD$ implies that a is a unit, for each $a \in D$ and A is called super-primitive if $A_v = D$. Call D a PSP (primitive is super-primitive) domain if every primitive finitely generated ideal of D is super-primitive. It was established in [36] that D is a PSP domain if and only if $G(D)$, the group of divisibility of D, is a pre-Riesz group. Indeed, in light of what we have established in this paper, D is a PSP domain if the monoid of nonzeero principal ideals of D is a pre-Riesz monoid. Also, as a Riesz monoid is a pre-Riesz monoid and a pre-Schreier domain is a domain D whose monoid of nonzero principal ideals is a Riesz monoid and it has been established, often, that a pre-Schreier domain is a PSP domain, (see [28], [38], etc.) Finally let's recall that $r \in D\backslash\{0\}$ is called rigid if for all $x, y | r$ we have $x|y$ or $y|x$ and as in [42] let's call D semirigid if every nonzero nonunit of D is expressible as a finite product of rigid elements.

Example 4.5. (1) Let D be a semirigid PSP domain then $G(D)^+$ or $m(D)$ the monoid of nonzero principal ideals of D is a v-factorial monoid. Of course if D is a GCD domain and semirigid we get a semirigid GCD domain of [43] where $m(D)$ is a factorial monoid. Example 3.7 of [43] serves as an example of a semirigid Schreier domain. Of course a UFD is a PSP domain and can be treated as a semirigid PSP domain. (2) Let D be PSP domain of finite t-character, then the monoid $H_h(m(D))$ generated by homogeneous elements of $m(D) = \{xD | x \in D\backslash\{0\}\}$ is a v-factorial monoid. (D is of finite t-character if every nonzero nonunit of D is contained in at most a finite numver of maximal t-ideals of D.) (3) Let D be a PSP domain such that every maximal t-ideal M of D is associated to a homogeneous element h of D, i.e $M = M(h) = \{x \in D | (x, h)_v \neq D\}$. Then $H_h(m(D))$ is a v-factorial monoid.

Illustration: Let for $xD, yD \in m(D), xD \wedge yD$ stand for $(xD + yD)_v$. Then xD, yD are disjoint if $(xD + yD)_v = D$ and non-disjoint if $(xD + yD)_v \neq D$. Obviously $m(D)$ is pre-Riesz if for every finite set $x_1, ..., x_n$, $(\sum x_i D)_v = D$ or $\sum x_i D \subseteq dD$ for some nonzero non unit d and for $x \in D\backslash\{0\}, xD$ is homogeneous if for each pair of nonzero principal ideals containing xD are non-disjoint (i.e. for all nonunits $u, v | x$, u, v have a non unit common divisor). Of course a rigid element has this property. So if (1) D is a semirigid PSP (pre-Schreier or GCD) domain, all of $m(D)$ is v-factorial. For (2) recall from section 4 of [36] that a PSP domain

D is of finite t-character if and only if every nonzero non unit of D is divisible by at least one homogeneous element and by at most a finite number of homogeneous elements. Once we note that $h(m(D))$, the set of homogeneous elements of $m(D)$, is non-empty, we conclude, via Corollary 6, that $H_h(m(D))$ is a v-factorial monoid. The reader can find several examples of non-pre-Schreier PSP domains of finite t-character in section 3, e.g. Example 3.7, of [42]. For (3) our arguments are the same as the ones for (2) An example of a PSP domain is $D = Z + XR[[X]]$, where Z is the ring of integers and R is the field of real numbers, as explained in the illustration of Example 4.9 of [36].

Let, for a star operation $\star$ of finite character, $\varphi_\star(D)$ denote the set of (nonzero) $\star$-ideals of finite type. Then $\varphi_\star(D)$ is a monoid under $\star$-multiplication $\times^\star : I \times^\star J = (IJ)^\star$, where $I, J \in \varphi_\star(D)$, while $\varphi_\star(D)$ is closed under $\star$-sum $+^\star : I +^\star J = (I + J)^\star$, allowing for $K \times^\star (I +^\star J) = K \times^\star I +^\star K \times^\star J = (KI + KJ)^\star$. Taking $I \leq J \Leftrightarrow I \supseteq J$ in $\varphi_\star(D)$ we have $(I +^\star J) = \inf(I, J) = I \wedge J$ we have a $\wedge$-semilattice, because $+^\star$ is indeed associative. With the order defined as we have D is the least element of $\varphi_\star(D)$. That $\varphi_\star(D)$ is $\wedge$-smooth because generally in $\varphi_\star(D)$ we have $K \times^\star I \wedge K \times^\star J) = K \times^\star (I \wedge J)$. Now requiring that whenever, for $I_1, ..., I_n \in \varphi_\star(D)$, $(I_1, ..., I_n)^\star$ exceeds some member of $\varphi_\star(D)$, which indeed is the case, we make $\varphi_\star(D)$ into a $\wedge$-smooth pre-Riesz monoid, with $e = D$. The idea took root in [11], when pre-Riesz monoids were not in plain sight, with just the notion of factoriality in Riesz groups of [31] and some of the work on factorization to go on. A nonzero $\star$-ideal $\mathbf{h}$ of finite type was called, in [11], a homogeneous ideal if for every pair $\mathbf{J}, \mathbf{k}$ of proper $\star$-ideals of finite type containing $\mathbf{h}$ we had $(\mathbf{j}, \mathbf{k})_v \neq D$ and now, in more general terms, it is: $\mathbf{j} \wedge \mathbf{k} \neq e$ for all $\mathbf{j}, \mathbf{k} \in (e, \mathbf{h}]$. It was shown in [11] that $\mathbf{h} \in \varphi_\star(D)$ is homogeneous if and only if $\mathbf{h}$ is contained in a unique maximal $\star$-ideal of D. Later, in [4], a domain D was called $\star$-semi homogeneous ($\star$-SH) domain if every nonzero principal ideal of D was expressible as a $\star$-product of homogeneous ideals. According to Theorem 6 of [4]: Given that $\star$ is a star operation of finite character. If I is a nonzero $*$-ideal of finite type in a $\star$-SH domain D such that $I \neq D$, then I is uniquely expressible as a $\star$-product of mutually $\star$-comaximal $\star$-homogeneous ideals. With the above description and illustration, the following results and examples can be established. Indeed it was shown in [11] that $\varphi_\star(D)$ satisfied CFC if and only if D was of finite $\star$-character, (see also [42]).

Proposition 18. *Let $\star$ be a finite character star operation defined on an integral domain D. Then the monoid $\varphi_\star(D)$ of nonzero $\star$-ideals of finite type of D, under $\star$-multiplication, is a $\wedge$-smooth pre-Riesz monoid with order defined by $I \leq J$ if and only if $I \supseteq J$, for $I, J \in \varphi_\star(D)$. Moreover the following hold: (1) D is a $\star$-SH domain if and only if $\varphi_\star(D)$ is a v-factorial monoid, (2) If $h(\varphi_\star(D)) \neq \phi$, then $H_h(\varphi_\star(D))$ is a v-factorial monoid.*

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