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RINGS BETWEEN D[X] AND K[X]

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Abstract. **Let K be a field, D a subring of K, and X an indeterminate over K. The purpose of this paper is two-fold: to study the rings between D[X] and K[X] and to use these rings to give some interesting examples. Special attention is given to the rings $\mathbf{A + XB[X]}$ and $\mathbf{I(B, A) = \{f(X) \in B[X] \mid f(A) \subseteq A\}}$ where $\mathbf{A \subseteq B}$ is a pair of subrings of K containing D.**

I. Introduction. Let K be a field, D a subring of K, and X an indeterminate over K. The purpose of this paper is two-fold: to study the rings between $D[X]$ and $K[X]$ and to use these rings to give some interesting examples. There are two classes of rings between $D[X]$ and $K[X]$ of particular importance. Let $A \subseteq B$ be a pair of subrings of K with $D \subseteq A$. The ring $A + XB[X] = \{a_0 + a_1X + \cdots + a_nX^n \in B[X] \mid a_0 \in A\}$ is called the *composite of A and B* and the ring $I(B, A) = \{f(X) \in B[X] \mid f(A) \subseteq A\}$ is called the ring of *A-valued B-polynomials*. We have $A[X] \subseteq I(B, A) \subseteq A + XB[X] \subseteq B[X]$.

We show that the intersection of a family of composites is again a composite. This leads to the result that every ring R between $D[X]$ and $K[X]$ has a composite cover, the unique minimal overring of R that is a composite. We use the composite cover of R to study R itself. For example, we show that R is integral over $D[X]$ if and only if its composite cover is integral over $D[X]$. Composite covers are studied in Section II along with other basic properties of rings between $D[X]$ and $K[X]$.

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Sections III-VI use composites to give examples of certain types of rings. Two important special cases of composites are $D + XD_S[X]$, where S is a multiplicatively closed subset of D, and $K_1 + XK_2[X]$, where $K_1 \subseteq K_2$ are fields. In Section III, we show that for any multiplicatively closed subset S, $D + XD_S[X]$ is an S-domain. (Recall that R is an *S-domain* if for each height-one prime ideal P of R, $ht\, P[X] = 1$ in $R[X]$.) In particular, $D[X]$ is always an S-domain. In Section IV, we construct some non-Noetherian Hilbert domains. For example, if D is a Dedekind domain with infinitely many primes and $S = \{d^n\}_{n=0}^{\infty}$ where d is a nonzero nonunit of D, then $D + XD_S[X]$ is a two-dimensional coherent non-Noetherian Hilbert PVMD that is also a ring of Krull type.

Sections V and VI are concerned with divisibility properties. We show that if $K_1 \subsetneq K_2$ are fields, then $K_1 + XK_2[X]$ is half-factorial but not factorial. This example also yields perhaps the simplest example of an atomic ring without unique factorization. We give an example of a ring with the ascending chain condition on principal ideals (ACCP) whose integral closure does not satisfy ACCP.

The final section, Section VII, considers the ring $I(B, A) = \{f(X) \in B[X] \mid f(A) \subseteq A\}$ where $A \subseteq B$ is a pair of rings. This ring generalizes the ring of integer-valued polynomials $I(\mathbb{Q}, \mathbb{Z})$ which has been widely studied. We show that in certain cases, $A + XB[X]$ is the composite cover of $I(B, A)$. We show that for D an integral domain with quotient field K, $I(K, D)$ has ACCP if and only if D has ACCP. Thus $I(\mathbb{Q}, \mathbb{Z})$ is an example of a non-Noetherian Prüfer domain with ACCP.

Rather than give an exhaustive treatment of composites and A-valued B-polynomials in any one context, we show how they arise in many different contexts. The novelty is in fact in the number of different contexts and the simplicity of the examples.

All rings are commutative with identity, usually integral domains. Our terminology and notation will follow that of [**15**] or [**21**].

II. Basic properties. Let K be a field, D a subring of K and let X be an indeterminate over K. The main feature of a ring R between $D[X]$ and $K[X]$ is that the values of elements of R at $X = 0$ split the ring into two parts: $M_R = \{f(X) \in R \mid f(0) = 0\}$ and $S_R = \{f(X) \in R \mid f(0) \neq 0\}$. Of these, M_R is a prime ideal of R, S_R is a multiplicatively closed set and $R_0 = \{f(0) \mid f \in R\}$ is a ring. The map $\pi : R \to R_0$ given by $\pi(f) = f(0)$ is a ring epimorphism with $\ker \pi = M_R$. Hence $R_0 \cong R/M_R$.

If $R_0 \subseteq R$, then R_0 is a subring of R and $R = R_0 + M_R$. This form resembles the celebrated $D+M$-form of Gilmer (cf. [4]). Further, if $R_0 \subseteq R$ and $M_R = XR_1[X]$ for some R_0-algebra R_1, then $R = R_0 + XR_1[X]$ and this is the now familiar composite of R_0 and R_1.

However, we may have $R_0 \not\subseteq R$. For example, consider $\mathbb{Q}[X] \subsetneq R = \mathbb{Q}[X][\sqrt{2}X+\pi] \subsetneq \mathbb{R}[X]$ where $\mathbb{Q}$ and $\mathbb{R}$ denote the rational numbers and real numbers, respectively. Here $\pi \notin R$, but $\pi \in R_0 = \mathbb{Q}[\pi]$. Let $S = R[\pi] = \mathbb{Q}[\pi][X, \sqrt{2}X]$; then $R \subsetneq S = S_0 + M_S$ and $S_0/M_S \cong Q[\pi] \cong R_0/M_R$. This procedure may be applied to any ring R between $D[X]$ and $K[X]$.

THEOREM 2.1. *Let K be a field, D a subring of K and let X be an indeterminate over K. Let R be a ring with $D[X] \subseteq R \subseteq K[X]$. Then there is a unique smallest ring S, $R \subseteq S \subseteq K[X]$, of the form $S = S_0 + M_S$ where $S/M_S \cong R/M_R$.*

PROOF: Let $T = R_0 - R$ and set $S = R[T]$. Then $S_0 = R_0 \subset S$, so $S = S_0 + M_S$. Also, $R/M_R \cong R_0 = S_0 \cong S/M_S$. Clearly S is the smallest such ring.

We now turn our attention to the composites and ask, 'When is a $D + M$-form a composite?' The following proposition provides the answer.

PROPOSITION 2.2. *Let R be a ring satisfying $D[X] \subseteq R \subseteq K[X]$ and suppose that $R = R_0+M_R$. Then R is a composite if and only if $\sum_{i=1}^{n} a_iX^i \in R$ implies that $a_iX \in R$ for each $1 \leq i \leq n$.*

PROOF: Since the implication $\Rightarrow$ is obvious we only prove the implication $\Leftarrow$.

Let $R_1 = \{a \in K \mid aX \in R\}$. Then $D \subseteq R_1 \subseteq K$ and R_1 is closed under addition and subtraction. Now for a, $b \in R_1$, aX, $bX \in R$ and hence $abX^2 = (aX)(bX) \in R$. But then, by hypothesis, $abX \in R$, so $ab \in R_1$. Thus R_1 is a ring and $R = R_0 + XR_1[X]$.

PROPOSITION 2.3. *Let $\{R_j = R_{j0} + XR_{j1}[X]\}_{j \in I}$ be family of composites lying between $D[X]$ and $K[X]$. Then $R = \cap R_j$ is again a composite given by $R = R_0 + XR_1[X]$, where $R_0 = \cap R_{j0}$ and $R_1 = \cap R_{j1}$.*

PROOF: This proposition may be easily proved directly or may be proved using Proposition 2.2.

COROLLARY 2.4. *Let R be a ring satisfying $D[X] \subseteq R \subseteq K[X]$. Then associated with R is a unique composite ring S which is the intersection of all the composites between $D[X]$ and $K[X]$ containing R. Moreover, $S = S_0 + XS_1[X]$ where $S_0 = R_0$, and S_1 is the subring of K generated by $\cup\{A_f \mid f \in R\}$ where A_f is the D-module generated by the coefficients of f.*

COROLLARY 2.5. *Let $\{D_j\}_{j\in I}$ be a family of overrings of D with $D = \cap D_j$. Then $D[X] = \cap D_j[X] = \cap(D + XD_j[X])$.*

The results we have proved give us an idea of how to find a decent ring nearest to the given ring. Now we shall see how the knowledge of a property of a composite cover provides us with information on the ring itself.

Let R be a ring between $D[X]$ and $K[X]$ and suppose that K is the quotient field of D. If the composite cover of R is integral over $D[X]$ then obviously R is integral over $D[X]$. If on the other hand R is integral over $D[X]$ we find that for each $f \in R$, $f \in D_0[X]$ where D_0 is the integral closure of D ([**15**, Theorem 10.7]). Thus $R \subseteq D_0[X]$ and $D_0[X]$ is a composite, so the composite cover of R lies between R and $D_0[X]$ and hence is integral over $D[X]$.

PROPOSITION 2.6. *Let D be an integral domain with quotient field K and let X be an indeterminate over K. Then a ring R between $D[X]$ and $K[X]$ is integral over $D[X]$ if and only if its composite cover is integral over $D[X]$.*

On the other hand, it is easily determined when a composite is integrally closed, completely integrally closed, root closed, or seminormal. We leave the proof of the next theorem to the reader.

THEOREM 2.7. *Let $A \subseteq B$ be a pair of integral domains and let $R = A + XB[X]$.*

(1) *R is integrally closed if and only if B is integrally closed and A is integrally closed in B.*
(2) *R is completely integrally closed if and only if $A = B$ and B is completely integrally closed.*
(3) *R is n-root closed if and only if B is n-root closed and A is n-root closed in B.*
(4) *R is seminormal if and only if A and B are seminormal.*

Also, by applying the Mayer-Vietoris sequence (U, Pic), one can show that $Pic(R) = Pic(A) \oplus NPic(B)$, where $NPic(B) = \ker(Pic(B[X]) \longrightarrow$

$Pic(B)$). Thus $Pic(R) = Pic(A)$ if and only if B is seminormal. (This fact also follows from [**2**, Theorem 1] since R is almost seminormal if and only if B is seminormal.)

By Theorem 2.7, if D is an integral domain with quotient field K and $D \subseteq D_1 \subseteq K$, then $D + XD_1[X]$ is integrally closed if and only if D and D_1 are both integrally closed. Since Prüfer domains are characterized by the fact that every overring is integrally closed, it easily follows that D is Prüfer if and only if each composite $D + XD_1[X]$ is integrally closed where $D \subseteq D_1 \subseteq K$.

In a similar vein, let D be a Schreier domain ([**7**]). Then the following statements can easily be shown to be equivalent: (1) D is a Bézout domain, (2) for every overring D_1 of D, $D + XD_1[X]$ is integrally closed, (3) for every overring D_1 of D, $D + XD_1[X]$ is a Schreier domain, (4) for every pair of overrings $D_1 \subseteq D_2$ of D, D_2 is a quotient ring of D_1. The proof is based on the fact that a Prüfer domain which is also a Schreier domain is Bézout ([**11**, Theorem 2.8]).

A less trivial result is the following:

THEOREM 2.8. *An integral domain D, which is not a field, is a one-dimensional Bézout domain such that every nonzero nonunit belongs to only a finite number of prime ideals if and only if for every two overrings $D_1 \subseteq D_2$ of D, $D_1 + XD_2[X]$ is a GCD-domain.*

PROOF: ($\Rightarrow$) A one-dimensional Bézout domain D with the property that every nonzero nonunit belongs to only a finite number of prime ideals is easily seen to be a generalized UFD ([**31**]). Hence by [**31**, Theorem 3.1], for each multiplicatively closed subset S of D, $D + XD_S[X]$ is a GCD-domain. Since every overring of a Bézout domain is a quotient ring, $D + XD_2[X]$ is a GCD-domain for every overring D_2 of D. But since every overring D_1 of D is itself a one-dimensional Bézout domain in which every nonzero nonunit is contained in only finitely many prime ideals, the result follows.

($\Leftarrow$) By the paragraph preceding Theorem 2.8, D must be a Bézout domain. Since every $D + XD_S[X]$ is a GCD-domain, again by [**31**, Theorem 3.1], D must be a generalized UFD. But is easily seen that a Bézout generalized UFD must have dimension one and have the property that every nonzero nonunit is contained in only finitely many maximal ideals.

If R is a Prüfer domain, then certainly the composite cover of R is

Prüfer. However, the converse is false. For example, $R = \mathbb{Z}+\mathbb{Z}X+X^2\mathbb{Q}[X]$ is not Prüfer, but its composite cover $\mathbb{Z} + X\mathbb{Q}[X]$ is Prüfer.

Call a ring R, $D[X] \subseteq R \subseteq K[X]$, *homogeneous* if whenever $a_0 + a_1X + \cdots + a_nX^n \in R$, each $a_iX^i \in R$. A composite is of course homogeneous (Proposition 2.2). For R homogeneous, let $R_i = \{a \in K \mid aX^i \in R\}$. Certainly R_i is closed under addition and subtraction and $R_iR_j \subseteq R_{i+j}$. Hence each R_i is an R_0-module. Moreover, $D[X] \subseteq R$ implies that $D \subseteq R_0 \subseteq R_1 \subseteq \cdots$. Conversely, such an ascending sequence $D \subseteq R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$ of D-submodules of K satisfying $R_iR_j \subseteq R_{i+j}$ gives rise to the homogeneous ring $R = R_0+R_1X+\cdots$. A simple example of a homogeneous ring that is not a composite is given by $D\left[\frac{X}{d}\right] = D + \frac{1}{d}DX + \frac{1}{d^2}DX^2 + \cdots$ where d is a nonzero nonunit of D.

There are two examples of composites that merit special attention. First, suppose that D is an integral domain with quotient field K. Let S be a multiplicatively closed subset of D. Then $R = D+XD_S[X]$ is a composite between $D[X]$ and $K[X]$. Taking $S = D-\{0\}$, gives $R = D+XK[X]$. The $D + XD_S[X]$ construction is investigated in [**12**] and [**31**]. Also, Sections III and IV use the $D+XD_S[X]$ construction to give examples of S-domains and Hilbert domains.

The second important special case is $R = K_1 + XK_2[X]$, where $K_1 \subseteq K_2$ are fields. Note that R has quotient field $K_2(X)$ and that the integral closure of R is $\overline{K_1} + XK_2[X]$ where $\overline{K_1}$ is the algebraic closure of K_1 in K_2. Also, R is Noetherian if and only if $[K_2 : K_1] < \infty$ ([**4**, Theorem 4]). The next theorem shows that the prime ideal structure of $K_1 + XK_2[X]$ is very simple. While parts of Theorem 2.9 follow from Theorem 2.10, we give a proof of Theorem 2.9 to make this paper self-contained.

THEOREM 2.9. *Let K_1 be a subfield of K_2, let X be an indeterminate over K_2 and let $R = K_1 + XK_2[X]$. Then*

(i) *every nonzero prime ideal of R is maximal;*
(ii) *every prime ideal P different from $XK_2[X]$ is principal; and*
(iii) *R is atomic, i. e., every nonzero nonunit of R is a finite product of irreducible elements.*

PROOF: (i) First note that $XK_2[X]$ is maximal since $R/XK_2[X] \cong K_1$. Let P be a nonzero prime ideal of R. Now $X \in P$ implies $(XK_2[X])^2 \subseteq P$ and hence $XK_2[X] \subseteq P$ so $P = XK_2[X]$. So suppose that $X \notin P$. Then for

$N = \{1, X, X^2, \cdots\}$, P_N is a prime ideal in the PID $K_2[X, X^{-1}] = R_N$. (In fact, $R_P \supseteq K_2[X, X^{-1}]$ is a DVR.) So P is minimal and is also maximal unless $P \subsetneq XK_2[X]$. But let $\ell_n X^n + \cdots + \ell_s X^s \in P$ where $\ell_n \neq 0$. Then $X^{n+1} + \ell_n^{-1}\ell_{n+1}X^{n+2} + \cdots + \ell_n^{-1}\ell_s X^s \in P$, so $X \notin P$ implies that $1 + \ell_n^{-1}\ell_{n+1}X + \cdots + \ell_n^{-1}\ell_s X^{s-n} \in P$, a contradiction. So every nonzero prime ideal is maximal.

(ii) If P is different from $XK_2[X]$, then it contains an element of the form $1 + Xf[X]$ where $f(X) \in K_2[X]$. Now if $1 + Xf(X)$ can be factored in $K_2[X]$ it can be written as $(1 + Xg(X)(1 + Xh(X))$. Hence $1 + Xf(X)$ is irreducible in R if and only if it is irreducible in $K_2[X]$.

Now let $1 + Xf(X)$ be irreducible in R and suppose that $1 + Xf(X) \mid h(X)k(X)$ in R. Then $1 + Xf(X) \mid h(X)k(X)$ in $K_2[X]$, and so in $K_2[X]$ we have, say, $1 + Xf(X) \mid h(X)$. Then, in $K_2[X]$, $h(X) = (1 + Xf(X))d(X)$. Now $d(X)$ can be written as $d(X) = aX^r(1 + Xp(X))$. If $r > 0$, $d(X) \in R$, while if $r = 0$, $h(X) = (1 + Xf(X))\,(a(1 + Xp(X))$ and $a \in K_1$ because $h(X) \in R$. In either case, $d(X) \in R$ and so $1 + Xf(X) \mid h(X)$ in R. Consequently, in R every irreducible element of the type $1 + Xf(X)$ is prime.

Now since every element of the form $1 + Xf(X)$ is a product of irreducible elements of the same form and hence is a product of prime elements, it follows that every prime ideal of P different from $XK_2[X]$ contains a principal prime and hence is actually principal.

(iii) Thus a general element of $R = K_1 + XK_2[X]$ can be written as $aX^r(1 + Xf(X))$ where $a \in K_2$ (with $a \in K_1$ if $r = 0$) and $1 + Xf(X)$ is a product of primes.

A variation of the two previous examples is $D + XL[X]$ where L is a field and D is a subring of L. For a detailed study of this construction the reader may consult [**13**]. We only mention the following result from [**13**].

THEOREM 2.10. *Let L be a field, D a subring of L and let X be an indeterminate over L. Then the following statements hold for $R = D + XL[X]$.*

(1) *If P is a nonzero prime ideal of R disjoint from $D^* = D - \{0\}$, then $P = XL[X]$ or P is principal.*
(2) *If A is an ideal of R with $A \cap D^* \neq \phi$, then $A = A \cap D + XL[X] = (A \cap D)R$.*
(3) *Every maximal ideal of R is either principal or of the form $P + XL[X]$ where P is a maximal ideal of D.*

(4) *If S is an overring of R then S is a quotient ring of $S \cap L + XL[X]$.*

There are many other interesting rings between $D[X]$ and $K[X]$. For example, Eakin and Heinzer [**14**] have shown that for any finitely generated abelian group G, there is a Dedekind domain R such that $\mathbb{Z}[X] \subset R \subsetneq \mathbb{Q}[X]$ and R has class group G. Another important class of domains between $D[X]$ and $K[X]$, the A-valued B-polynomials $I(B, A) = \{f \in B[X] \mid f(A) \subseteq A\}$, where $A \subseteq B$ is a pair of rings with $D \subseteq A \subseteq B \subseteq K$, is discussed in Section VII.

III. S-domains. An integral domain D is called an S-*domain* (the S stands for Seidenberg) if for each prime ideal P of D with $ht\,P = 1$, $ht\,P[X] = 1$. D is called a *strong* S-*domain* if D/P is an S-domain for each prime ideal P. The terms S-domain and strong S-domain were coined by Kaplansky in his treatment of the Krull dimension of polynomial rings given in [**21**]. Certainly Noetherian domains and Prüfer domains are S-domains. The purpose of this section is to show that S-domains exist in abundance.

LEMMA 3.1. *For an integral domain D, the following statements are equivalent.*

(1) *D is an S-domain.*
(2) *For each prime ideal P of D with $ht\,P = 1$, D_P is an S-domain.*
(3) *For each prime ideal P of D with $ht\,P = 1, \overline{D_P}$ is a Prüfer domain.*

PROOF: (1)$\Rightarrow$ (2) Now $ht\,P = 1$ implies that $ht\,P[X] = 1$ since D is an S-domain. Hence $ht\,P_P[X] = 1$. So D_P is an S-domain. (2)$\Rightarrow$(3) Now D_P is a one-dimensional domain with $ht\,P_P[X] = 1$. Hence dim $\overline{D_P}[X] = 2$. By a result of Seidenberg [**22**, Theorem 7.23], $\overline{D_P}$ is Prüfer. (3)$\Rightarrow$(1) Let P be a height-one prime ideal of D. Then $\overline{D_P}$ is a one-dimensional Prüfer domain, so dim $\overline{D_P}[X] = 2$. Hence dim $D_P[X] = 2$ since $D_P[X] \subseteq \overline{D_P}[X]$ is integral. But dim $D_P[X] = 2$ implies that $ht\ P_P[X] = 1$ and hence $ht\,P[X] = 1$. So D is an S-domain.

Kabbaj [**20**] has shown that if D is an S-domain, then $D[X]$ is an S-domain. Our next theorem shows that for any integral domain D, $D[X]$ is an S-domain. Hence $D[\{X_\alpha\}]$ is an S-domain for any nonempty set $\{X_\alpha\}$

of indeterminates. However, $D[X]$ need not be a strong S-domain. For, if $D[X]$ is a strong S-domain, so must be its homomorphic image D. However, even if D is a strong S-domain, $D[X]$ need not be a strong S-domain (cf. [**3**, Proposition 2.1]).

THEOREM 3.2. *For any integral domain D, $D[X]$ is an S-domain.*

PROOF: Let Q be a height-one prime ideal of $D[X]$. By Lemma 3.1, it suffices to show that $\overline{D[X]_Q}$ is Prüfer. If $Q \cap D = 0$, then $D[X]_Q$ is a DVR. So we may suppose that $P = Q \cap D \neq 0$. Then $ht\, P = 1$ and $Q = P[X]$. But then D_P is an S-domain. So by Lemma 3.1, $\overline{D_P}$ is Prüfer. Thus $\overline{D_P}(X)$ is also Prüfer [**15**, Theorem 33.4]. But then $\overline{D[X]_Q} = \overline{D[X]_{P[X]}} = \overline{D_P(X)} = \overline{D_P}(X)$ is Prüfer. (Here the last equality follows from [**16**, Theorem 3].)

COROLLARY 3.3. *Let D be an integral domain and S a multiplicatively closed subset of D. Then $D + XD_S[X]$ is an S-domain.*

PROOF: Let $R = D + XD_S[X]$ and let P be a height-one prime ideal of R. First suppose that $P \cap S \neq \phi$. Then $P \supseteq XD_S[X]P = XD_S[X]$. But since $ht\, P = 1$, $P = XD_S[X]$. But then $P \cap S = \phi$, a contradiction. Thus we must have $P \cap S = \phi$. Then P_S is a height-one prime ideal in $R_S = D_S[X]$. By Theorem 3.2, R_S is an S-domain. Hence $R_P = {R_S}_{P_S}$ is also an S-domain by (1)⇒(2) of Lemma 3.1. Thus R is an S-domain by (2)⇒ (1) of Lemma 3.1.

It is easily seen that for any nonempty set $\{X_\alpha\}$ of indeterminates over D, and any multiplicatively closed subset S of D, $D + \{X_\alpha\}D_S[\{X_\alpha\}]$ is an S-domain. However, if K is the quotient field of D, then $D + XK[X]$ is a strong S-domain if and only if D is a strong S-domain [**23**, Theorem 5.2].

IV. Hilbert domains. A commutative ring R is called a *Hilbert ring* if every prime ideal of R is an intersection of maximal ideals of R. In [**26**] it was shown that if $D \subseteq K$ where K is a field, then $D + XK[X]$ is a Hilbert domain if and only if D is a Hilbert domain. Thus if D is a PID that is not a field and K is the quotient field of D, then $D + XK[X]$ is a two-dimensional, non-Noetherian, Bézout-Hilbert domain in which every maximal ideal is principal. In this section we give further examples of non-Noetherian Hilbert domains with special properties.

THEOREM 4.1. *Let D be an integral domain and S a multiplicatively closed subset of D with the property that for a prime P of D with $P \cap S \neq \phi$, then $Q \cap S \neq \phi$ for each prime $0 \neq Q \subseteq P$. Then $R = D + XD_S[X]$ is a Hilbert domain if and only if D and D_S are Hilbert domains.*

PROOF: ($\Leftarrow$) Let Q be a prime ideal of R. Suppose that $Q \cap S \neq \phi$. Then $XD_S[X] = XD_S[X]Q \subseteq Q$, so $Q = Q \cap D + XD_S[X]$. Since D is a Hilbert domain, $Q \cap D$ is an intersection of maximal ideals, hence so is Q. So we may suppose that $Q \cap S = \phi$. Then since $D_S[X]$ is a Hilbert domain, $Q_S = \cap_\alpha \mathcal{M}_\alpha$ where $\{\mathcal{M}_\alpha\}$ is the set of maximal ideals of $D_S[X]$ containing Q_S. Then $Q = \cap_\alpha(\mathcal{M}_\alpha \cap R)$. So it suffices to show that each $\mathcal{M}_\alpha \cap R$ is a maximal ideal of R. So let M be a maximal ideal of $D_S[X]$. Then $M = N_S$ where N is a prime ideal of $D[X]$. Now M maximal implies $M \cap D_S$ is maximal since D_S is Hilbert. If $M \cap D_S = 0$, then D_S is a field and hence R is a Hilbert domain [**26**, Theorem 5]. So we may assume that $M \cap D_S \neq 0$. Then by the hypothesis on S, $(M \cap D_S) \cap D = N \cap D$ must also be maximal. Since $N \supsetneq (N \cap D)[X]$, N must be a maximal ideal of $D[X]$. Hence $D[X]/N \subseteq R/M \cap R \subseteq D_S[X]/M = D_S[X]/N_S = D[X]/N$ since $D[X]/N$ is a field. Therefore $M \cap R$ is a maximal ideal.

($\Rightarrow$) Suppose that R is a Hilbert domain. Then $D \cong R/XD_S[X]$ is also a Hilbert domain. Suppose that D_S is not a Hilbert domain. Let Q be a nonzero prime ideal of D with $Q \cap S = \phi$. Since D is a Hilbert domain, $Q = \cap_\alpha \mathcal{M}_\alpha$ where $\{\mathcal{M}_\alpha\}$ is the set of maximal ideals of D containing Q. Since $Q \cap S = \phi$ by the hypothesis on S, each $\mathcal{M}_\alpha \cap S = \phi$. Hence $Q_S = \cap \mathcal{M}_{\alpha S}$ is an intersection of maximal ideals of D_S. So every nonzero prime ideal of D_S is an intersection of maximal ideals. Since D_S is not a Hilbert domain, 0_S is not an intersection of maximal ideals. Hence there is a nonzero element $u \in D$ such that u is in every nonzero prime ideal of D_S. Consider $u + X \in R$. Let P be prime ideal of R minimal over $(u + X)$ with $P \cap D = 0$. (Such a prime P exists since $(u + X) \cap (D - \{0\}) = \phi$.) If Q is a prime ideal of R with $Q \supsetneq P$, then $Q \cap D \neq 0$. For otherwise in $D_S[X]$, $0 \neq P_S \subsetneq Q_S$ would both contract to 0. Now if $Q \cap S \neq \phi$, then $X \in XD_S[X] \subseteq Q$; while if $Q \cap S = \phi$, then $u \in (Q_S \cap D_S) \cap D \subseteq Q$. So every prime ideal of R properly containing P contains both u and X. Hence P is not the intersection of the maximal ideals containing it, contradicting the fact that R is a Hilbert domain. So D_S must also be a Hilbert domain.

Some examples of multiplicatively closed sets with the property that

$P \cap S \neq \phi$ implies that $Q \cap S \neq \phi$ for each prime $0 \neq Q \subseteq P$ include $S = D - \{0\}$, $S = D - (P_1 \cup \cdots \cup P_n)$ where $P_1, \cdots, P_n$ are height-one maximal ideals of D and $S = \{d^n\}_{n=0}^{\infty}$ where $0 \neq d \in D$ with D a one-dimensional domain.

COROLLARY 4.3. *Let D be a PID (resp., Dedekind domain) and let S be a multiplicatively closed subset of D containing a nonunit such that D_S has infinitely many prime ideals. Then $R = D + XD_S[X]$ is a non-Noetherian Hilbert domain which is a coherent GCD-domain (resp., PVMD).*

PROOF: Since D_S has infinitely many prime ideals, so does D; hence D and D_S are both Hilbert domains. Since dim $D = 1$, the previous theorem applies to show that R is a Hilbert domain. Since $D_S \supsetneq D$, R is not Noetherian. Suppose that D is a PID, then by [**12**, Corollary 1.2] R is a GCD-domain, while by [**12**, Theorem 4.32] R is coherent. Suppose that D is a Dedekind domain. Again by [**12**, Theorem 4.32], R is coherent. Since R is coherent and integrally closed, by [**29**, Theorem 2], R is a PVMD.

Example 4.4. Let D be Dedekind domain with infinitely many primes and let $S = \{d^n\}_{n=0}^{\infty}$ where d is a nonzero nonunit of D. Then $R = D + XD_S[X]$ is a two-dimensional coherent non-Noetherian Hilbert PVMD that is even a ring of Krull type (i. e., R is a locally finite intersection of essential valuation overrings). For by [**19**, Proposition 16], a PVMD is a ring of Krull type if and only if every nonzero nonunit belongs to only a finite number of maximal t-ideals. However, if P is a prime ideal of R, then either $P \cap S = \phi$ and $ht\, PD_S[X] = 1$ so $ht\, P = 1$, or $P \cap S \neq \phi$ and $P = P \cap D + XD_S[X]$. But since there are only finitely many primes of D intersecting S nontrivially, the set of height-two primes is finite. So for P a prime ideal of R minimal over $0 \neq f \in R$, either $ht\, P = 2$ or P_S is also minimal over $fR_S = fD_S[X]$. So the set of such primes P is finite.

However, $R = D + XD_S[X]$ may be Hilbert without the set S satisfying the hypothesis of Theorem 4.1. For example, take $R = \mathbb{Z} + X\mathbb{Z}_S[X]$ where either $S = \mathbb{Z} - \{0\}$ or $S = \{d^n\}_{n=0}^{\infty}$, d a nonzero nonunit of $\mathbb{Z}$. Then R is a Hilbert domain by Theorem 4.1 and hence so is $R[Y] = \mathbb{Z}[Y] + X\mathbb{Z}[Y]_S[X]$ by [**21**, Theorem 31]. But S is a multiplicatively closed subset of $\mathbb{Z}[Y]$ with $S \cap (d, Y) \neq \phi$ while $S \cap (Y) = \phi$. It seems reasonable to conjecture that $D + XD_S[X]$ is a Hilbert domain if and only if both D and D_S are Hilbert domains.

V. Divisibility properties I. If an integral domain R satisfies ACCP, then R is atomic, that is, every nonzero nonunit of R is a product of irreducible elements. However, an atomic domain need not satisfy ACCP [**18**]. Also in [**18**, Proposition 2.1], it is observed that if $A \subseteq B$ is a pair of domains with $U(B) \cap A = U(A)$ (here $U(B)$ is the set of units of B), then B has ACCP implies that A has ACCP. In particular, if $A \subseteq B$ is integral and B has ACCP, then A has ACCP. We first show that the converse is false.

Example 5.1. An integral domain R which satisfies ACCP, but whose integral closure does not satisfy ACCP.

Let $\overline{\mathbb{Z}}$ be the ring of all algebraic integers and $R = \mathbb{Z} + X\overline{\mathbb{Z}}[X]$. Then $\overline{R} = \overline{\mathbb{Z}}[X]$ is not even atomic since $\overline{\mathbb{Z}}$ is not atomic (for an atomic Bézout domain is a PID). However, R satisfies ACCP. For if not, then there is an infinite properly ascending chain of principal ideals of R. Since the degrees of the polynomials generating these principal ideals are nonincreasing, the degrees eventually stabilize. The principal ideals in $\overline{\mathbb{Z}}$ generated by the leading coefficients of these polynomials gives an infinite ascending chain $a_1\overline{\mathbb{Z}} \subsetneq a_2\overline{\mathbb{Z}} \subsetneq \cdots$ where each $a_n/a_{n+1} \in \overline{\mathbb{Z}}$. Thus all $a_n \in \mathbb{Q}[a_1]$. Let $A = \overline{\mathbb{Z}} \cap \mathbb{Q}[a_1]$. Then $a_1A \subsetneq a_2A \subsetneq \cdots \subsetneq A$, a contradiction since A is Dedekind.

Note that for R a ring between $D[X]$ and $K[X]$, R has ACCP if and only if for every $n \geq 0$, any ascending chain of principal ideals generated by polynomials of degree n terminates. If K is the quotient field of D, the following proposition may be used to show that a ring R satisfies ACCP.

PROPOSITION 5.2. *Let D be an integral domain with quotient field K. Let R be a ring with $D[X] \subseteq R \subseteq K[X]$. Then R has ACCP if and only if $R \cap K$ has ACCP and for each ascending chain of polynomials $f_1R \subseteq f_2R \subseteq f_3R \subseteq \cdots$ where the $f_i \in R$ all have the same degree, then there is a $0 \neq d \in R \cap K$ such that $df_i \in (R \cap K)[X]$.*

PROOF: ($\Rightarrow$) Since $U(R\cap K) = U(R)\cap K$, R has ACCP implies that $R\cap K$ has ACCP. The chain $f_1R \subseteq f_2R \subseteq \cdots$ stops, say $f_nR = f_{n+1}R = \cdots$. So $f_{n+1} = u_if_i$, where u_i is necessarily a unit of $R \cap K$. Since $f_n \in K[X]$, there exists a $0 \neq d \in D \subseteq R \cap K$ with $df_n \in D[X] \subseteq R$. But then the coefficients of $df_{n+i} = u_idf_n$ all lie in $R \cap K$.

($\Leftarrow$) Let $f_1R \subseteq f_2R \subseteq \cdots$ be an ascending chain in R. Since deg $f_{i+1} \leq$ deg f_i, eventually all the f_i have the same degree, so without loss of generality, we can assume that deg $f_1 =$ deg $f_2 = \cdots$. By hypothesis there exists a $0 \neq d \in R \cap K$ with each $df_i \in (R \cap K)[X]$. Now $f_i R \subseteq f_{i+1} R$ implies $f_i = f_{i+1}\alpha$ where $\alpha \in R$ has degree 0, so $\alpha \in R \cap K$. Hence $df_i(R \cap K)[X] \subseteq df_{i+1}(R \cap K)[X]$. But $R \cap K$ has ACCP and hence so does $(R \cap K)[X]$. So for large n, $f_n(R \cap K)[X] = f_{n+1}(R \cap K)[X] = \cdots$, and hence $f_n R = f_{n+1} R = \cdots$.

In **[32]**, Zaks introduced the notion of a half-factorial domain and gave a detailed study of half-factorial Krull domains in **[33]**. Recall that a domain R is called a *half-factorial domain* (HFD) if (1) R is atomic and (2) for each nonzero nonunit $x \in R$, $x = x_1 \cdots x_m = y_1 \cdots y_n$ where the x_i's and y_j's are all irreducible, implies that $m = n$. Certainly a UFD is half-factorial and a half-factorial domain satisfies ACCP. We next show that if $K_1 \subseteq K_2$ are fields, then $K_1 + XK_2[X]$ is half-factorial. However, $K_1 + XK_2[X]$ is factorial (equivalently, Krull) if and only if $K_1 = K_2$.

THEOREM 5.3. *Let A be a subring of a field K. Then $R = A + XK[X]$ is a HFD if and only if A is a field.*

PROOF: ($\Rightarrow$) Clearly, R a HFD implies that A is a HFD. Suppose that A is not a field, so there is an irreducible element $a \in A$. Then $X = a^n(X/a^n)$ for all $n \geq 1$. Thus A must be a field.

($\Leftarrow$) Suppose that A is a field. By Theorem 2.9, $R = A + XK[X]$ is atomic. The proof of Theorem 2.9 shows that an irreducible element of R is of the form aX where $a \in K$ or $a(1 + Xf[X])$ where $a \in A$, $f(X) \in K[X]$, and $1 + Xf(X)$ is irreducible in $K[X]$. Thus for any $g(X) \in R$, the number of irreducible factors in a representation of $g(X)$ as a product of irreducible factors from R is the same as the number of irreducible factors in a representation of $g(X)$ as a product of irreducible factors from the PID $K[X]$. Hence R is a HFD.

A careful choice of fields $K_1 \subseteq K_2$ can yield HFD's with some interesting properties. If $K_1 \subsetneq K_2$ is algebraic, then $R = K_1 + XK_2[X]$ has integral closure $K_2[X]$, a Euclidean domain. If actually $[K_2 : K_1] < \infty$, then $K_1 + XK_2[X]$ is a Noetherian HFD that is not integrally closed. If K_1 is algebraically closed in K_2, then $K_1 + XK_2[X]$ is an integrally closed non-Noetherian HFD. Of course, $K_1 + XK_2[X]$ always satisfies ACCP.

Example 5.4 A HFD R with the property that $R[Y]$ is not a HFD. Let $R = \mathbb{R} + X\mathbb{C}[X]$, so R is a Noetherian HFD. But in $R[Y] = \mathbb{R}[Y] +$

$X\mathbb{C}[Y][X]$ we have $(X(1+iY))(X(1-iY)) = X \cdot X(1+Y^2)$, two factorizations into irreducibles of different lengths. So $R[Y]$ is not a HFD. (Note that for R a Krull domain, $R[Y]$ is a HFD if and only if $Cl(R)$ is either 0 or $\mathbb{Z}/2\mathbb{Z}$ [**33**, Theorem 2.4].)

The proof of ($\Rightarrow$) in Theorem 5.3 shows that if $A + XB[X]$ is a HFD, then A is a HFD and $U(A) = A \cap U(B)$. However, Example 5.4 shows that the converse if false. Here is another example.

Example 5.5 Let $n > 1$ and $R = \mathbb{Q}[t^n] + X\mathbb{Q}[t][X]$. Then for m with $1 \leq m < n$, $(t^m X)^n = (t^n)^m X^n$ are two factorizations of $t^{nm}X^n$ into irreducibles of length n and $m+n$, respectively. Hence R is not a HFD.

The usual example of an integral domain without unique factorization is $\mathbb{Z}[\sqrt{-5}]$ with $3 \cdot 7 = (1+2\sqrt{-5})(1-2\sqrt{-5})$. This example entails a nontrivial amount of explanation. A much simpler example is $R = \mathbb{R} + X\mathbb{C}[X]$. We have already remarked that R is a Noetherian HFD. Here clearly $X \cdot X = (iX)(-iX)$ are two factorizations of X^2 into irreducibles, but X and iX (or $-iX$) are not associates, so R is not a UFD. Of course, R is not integrally closed, while $\mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain.

If we wish to avoid complex numbers, we may take $R = \mathbb{Q} + X\mathbb{R}[X]$. Here R is a non-Noetherian HFD. For each $r \in \mathbb{R} - \{0\}$, rX is irreducible and two irreducible elements $r_1 X$, $r_2 X$ are associates if and only if $r_1/r_2 \in \mathbb{Q}$. Thus for $a \in \mathbb{R} - \mathbb{Q}$, $X^2 = (aX)(a^{-1}X)$ are distinct factorizations. In particular, X^2 has an uncountable number of distinct nonassociate irreducible factors and factorizations into irreducible elements. This example also conveys well the difference between an element being irreducible and being prime. For X is irreducible, but is not prime since $X \mid (\sqrt{2}X)(\sqrt{2}X)$, but $X \nmid \sqrt{2}X$.

VI. Divisibility properties II. In this section, we use composites to give some examples of almost GCD-domains, almost Bézout domains, and almost factorial domains.

An integral domain D is said to be *an almost* GCD-*domain* (resp., *almost Bézout domain*) if for $a, b \in D - \{0\}$, there exists an $n = n(a, b) \geq 1$ with $a^n D \cap b^n D$ (*resp.*, (a^n, b^n)) principal. It is easily seen that an almost Bézout domain is an almost GCD-domain. Almost GCD-domains were introduced in [**30**], while almost Bézout domains were introduced in [**1**].

THEOREM 6.1. *Let $K \subseteq L$ be a pair of fields with L purely inseparable* *over K (that is, char $K = p > 0$ and for each $\ell \in L$, there exists a natural number $n = n(\ell)$ with $\ell^{p^n} \in K$). Then every ring R between $K[X]$ and $L[X]$ is a one-dimensional almost Bézout domain.*

PROOF: Since $K[X] \subseteq L[X]$ is an integral extension, dim $R =$ dim $K[X] = 1$. For each $f \in L[X]$, $f^{p^n} \in K[X]$ for n large enough. Hence for $f, g \in R$, $f^{p^n}, g^{p^n} \in K[X]$ for some $n \geq 1$. But $(f^{p^n}, g^{p^n})K[X]$ is principal. Hence $(f^{p^n}, g^{p^n})R$ is principal.

Thus if $K \subseteq L$ is a pair of fields with L purely inseparable over K, then $R = K + XL[X]$ is an almost Bézout domain. It can be shown ([**1**, Example 4.14]) that R satisfies the stronger property that for any subset $\{a_\alpha\}$ of R, there exists an $n = n(\{a_\alpha\})$ with $(\{a_\alpha^n\})$ principal if and only if there is a bound on the degree of inseparability.

The domain $R = K + XL[X]$ is also an example of a general almost factorial domain. Let us recall the definition of a general almost factorial domain as given in [**30**]. Let D be an integral domain. Two elements $x, y \in D$ are called v-*coprime* if $xD \cap yD = xyD$. A nonzero nonunit $b \in D$ is called a *prime block* if for all x, y non-v-coprime with b there exist a natural number $n(x, y)$ and $d \in D$ such that $x^n, y^n \in dD$ and at least one of x^n/d, y^n/d is v-coprime to b. For example, if q is a principal prime such that $\cap(q^n) = 0$, then every power of q is a prime block. It can be shown that if x is a product of finitely many prime blocks, then some power of x is uniquely expressible as a product of mutually v-coprime prime blocks. Now if D is an integral domain with the property that for every nonzero nonunit x, x^n is expressible as a product of prime blocks for some $n = n(x)$, then D is called a *general almost factorial domain*. It is easily seen that an almost factorial domain of Storch (i. e., Krull domain with torsion divisor class group) is a general almost factorial domain.

Our next example is a slight generalization of an example given in [**30**].

Example 6.2. Let $K \subseteq L$ be a pair of fields with L purely inseparable over K. Then $R = K + XL[X]$ is a general almost factorial domain.

Because R is atomic and because every element of the form $1+Xf(X)$ is a product of prime powers p_i with the property that $\cap_{j=1}^{\infty}(p_i^j) = 0$, it is sufficient to show that aX is a prime block for every $a \in L$. For this we note that if $f(X)$ and $g(X)$ are both non-v-coprime with aX, then $f(X) = bX^r f_1(X)$ and $g(X) = cX^s g_1(X)$ where $r, s > 0$ and f_1, g_1 are of the form $1+Xh(X)$. Now for some n, b^{p^n}, $c^{p^n} \in K$, and $f^{p^n} = b^{p^n} X^{rp^n} f_1^{p^n}$ and $g^{p^n} = c^{p^n} X^{sp^n} g_1^{p^n}$. So if $d = X^t$ where $t = \min(rp^n, sp^n)$, then at least one of f^{p^n}/d, g^{p^n}/d is v-coprime with aX. Finally, all the prime powers being prime blocks and aX^n being a prime block for all $a \in L$ and $n \in \mathbb{N}$, we conclude that $R = K + XL[X]$ is indeed a general almost factorial domain.

In the previous example, R is always a HFD, but R is not integrally closed unless $K = L$.

VII. Integer valued functions. Let $A \subseteq B$ be a pair of rings, not necessarily integral domains. The set $I(B, A) = \{f(X) \in B[X] \mid f(A) \subseteq A\}$ is easily seen to be a subring of $B[X]$. Since for $f(X) \in I(B, A)$, $f(0) \in A$, we even have $A[X] \subseteq I(B, A) \subseteq A + XB[X]$. The second containment is strict unless $A = B$ (for $f(X) = a + bX \in I(B, A) \iff a, b \in A$). Also, $I(B, A)$ is homogeneous if and only if $I(B, A) = A[X]$. The case where $B = \mathbb{Q}$ and $A = \mathbb{Z}$, the integer valued polynomials, has received wide attention. Most of the papers concerning $I(B, A)$ have been restricted to the case where A is a Dedekind domain with finite residue fields and B is its quotient field. Here is a sampling of some recent papers concerning rings of integer valued polynomials: [**5**]-[**8**], [**17**], [**24**], and [**27**].

It is easily seen that the composite cover of $I(\mathbb{Q}, \mathbb{Z})$ is $\mathbb{Z} + X\mathbb{Q}[X]$. We next show that this holds for any domain all of whose proper homomorphic images are finite.

PROPOSITION 7.1. *(a) Let R be an integral domain with quotient field K. Suppose that for each $0 \neq r \in R$, $R/(r)$ is finite. Then the composite cover of $I(K, R)$ is $R + XK[X]$.*
(b) Let $A \subsetneq B$ be rings where A is finite. Then the composite cover of $I(B, A)$ is $A + XB[X]$.

PROOF: (a) Let r be a nonzero nonunit of R and let $R/(r) = \{r_1 + (r), \cdots, r_n + (r)\}$. Set $f(X) = \frac{1}{r}(X - r_1)\cdots(X - r_n) \in K[X]$. Now for $a \in R$, $a + (r) = r_i + (r)$ for some i, so $a - r_i = sr$ for some $s \in R$. Hence $f(a) = \frac{1}{r}(sr)\Pi_{j\neq 1}(a - r_j) \in R$. So $f(X) = \frac{1}{r}X^n + \cdots \in I(K, R)$ and hence $I(K, R)$ has composite cover $R + XK[X]$.
(b) For each $b \in B$, $f(X) = b(\Pi_{a\in A}(X - a)) \in I(B, A)$.

Our next result is the analogue of Theorem 2.7 for $I(K, R)$. The statement of (1) without proof is given in [**10**, Corollaire 1, p. 304] while (4) is reported to be in [**9**].

PROPOSITION 7.2. *Let R be an integral domain with quotient field K.*

(1) *$I(K, R)$ is integrally closed if and only if R is integrally closed.*
(2) *$I(K, R)$ is n-root closed if and only if R is n-rooted class.*
(3) *$I(K, R)$ is seminormal if and only if R is seminormal.*
(4) *$I(K, R)$ is completely integrally closed if and only if R is completely integrally closed.*

PROOF: (1) ($\Rightarrow$) Clear. ($\Leftarrow$) Let $f \in K(X)$ be integral over $I(K, R)$. Then $f \in K[X]$. Now $f(X)^n+g_1(X)f(X)^{n-1}+\cdots+g_{n-1}(X)f(X)+g_n(X) = 0$ for some $g_i(X) \in I(K, R)$. For $r \in R$, $f(r)^n + g_1(r)f(r)^{n-1} + \cdots + g_n(r) = 0$. Since each $g_i(r) \in R$, $f(r)$ is integral over R and hence $f(r) \in R$. So $f \in I(K, R)$. The proofs of (2) and (3) are similar.

(4) ($\Rightarrow$) Clear. ($\Leftarrow$) Suppose that R is completely integrally closed. Let f be almost integral over $I(K, R)$. Then $f \in K[X]$. Let $g(X) \in I(K, R)$ with $g(X)f(X)^k \in I(K, R)$ for all $k \geq 1$. Hence $g(r)f(r)^k \in R$ for all $k \geq 1$. So if $g(r) \neq 0$, $f(r)$ is almost integral over R and hence in R. So $f(r) \in R$ for all $r \in R$ except possibly for the finite set of roots of $g(X)$ which lie in R. So it suffices to observe that if $f(X) \in K[X]$ has $f(r^*) \notin R$ for some $r^* \in R$, then $f(r) \notin R$ for an infinite number of $r \in R$. For let $f(X) = a_0 + a_1X + \cdots + a_nX^n$ and let d be a nonzero nonunit of R (we can assume that $R \neq K$) with $da_i \in R$ for $i = 1, \cdots, n$. Then for $k \geq 1$,

$$\begin{aligned} f(r^* + d^k) &= a_0 + a_1(r^* + d^k) + \cdots + a_n(r^* + d^k)^n \\ &= (a_0 + a_1r^* + \cdots + a_nr^{*n}) + (a_1d^k + 2a_nr^*d^k + \cdots + a_nd^{kn}) \\ &= f(r^*) + r_k \end{aligned}$$

where $r_k \in R$. Hence $f(r^* + d^k) \notin R$.

For a polynomial $f(X) \in I(\mathbb{Q}, \mathbb{Z})$ of degree n, $n!\,f(X) \in \mathbb{Z}[X]$. A similar result holds for $I(K, R)$ for any integral domain R.

PROPOSITION 7.3. *Let R be an integral domain with quotient field K. For each $n \geq 0$, there exists a $0 \neq t_n \in R$ so that $t_nf(X) \in R[X]$ for all $f(X) \in I(K, R)$ with $\deg f(X) \leq n$.*

PROOF: Now for $n = 0$, we may take $t_0 = 1$. Assume that $0 \neq t_{n-1} \in R$ has been chosen so that $t_{n-1}g(X) \in R[X]$ for all $g(X) \in I(K, R)$ with deg

$g(X) \leq n-1$. If R is equal to K, we may take $t_n = 1$. So suppose that r_0 is a nonzero nonunit of R. Let $f(X) = a_0 + a_1X + \cdots + a_nX^n \in I(K, R)$ have degree n. Now $a_0 + r_0a_1X + \cdots + r_0^na_nX^n = f(r_0X) \in I(K, R)$ as is $r_0^nf(X) = r_0^na_0 + r_0^na_1X + \cdots + r_0^na_nX^n$. Hence $(r_0^n - 1)a_0 + (r_0^n - r_0)a_1X + \cdots + (r_0^n - r_0^{n-1})a_{n-1}X^{n-1} = r_0^nf(X) - f(r_0X) \equiv g(X) \in I(K, R)$. By induction, $t_{n-1}g(X) \in R[X]$, that is, $t_{n-1}(r_0^n - r_0^i)a_i \in R$ for $i = 0, \cdots, n-1$. Put $t_n = t_{n-1}\Pi_{i=0}^{n-1}(r_0^n - r_0^i) \in R$. Since r_0 is a nonzero nonunit, each $r_0^n - r_0^i \neq 0$, so $t_n \neq 0$. Certainly $t_na_i \in R$ for $i = 0, \cdots, n-1$. Now $a_0 + a_1 + \cdots + a_n = f(1) \in R$, so $t_na_0 + \cdots + t_na_{n-1} + t_na_n = t_nf(1) \in R$; hence $t_na_n \in R$. So $t_nf(X) \in R[X]$.

COROLLARY 7.4. *([**27**, Corollary 3]). Suppose that R is an integral domain with quotient field K. Suppose that R contains an infinite field. Then $I(K, R) = R[X]$.*

PROOF: Let K_0 be an infinite subfield of R. Suppose that in the proof of Proposition 7.3, $t_1, \cdots, t_{n-1}$ have been chosen so that they all lie in K_0. It suffices to show that we can find an $0 \neq r_0 \in K$ so that $r_0^n - r_0^i \neq 0$ for $0 \leq i \leq n-1$. For then $0 \neq t_n = t_{n-1}\Pi_{i=0}^{n-1}(r_0^n - r_0^i) \in K_0$ and $t_nf(X) \in R[X]$ implies that $f(X) \in R[X]$. However, since K_0 is infinite and since the equations $X^n - 1 = 0$, $X^n - X = 0, \cdots, X^n - X^{n-1} = 0$ have only finitely many solutions, the desired $r_0 \in K$ must exist.

Let R be an integral domain. Let us call R a *bounded factorization domain* (BFD) if for each nonzero nonunit $a \in R$, there exists a natural number $N(a)$, so that if $a = a_1 \cdots a_s$ where each a_i is nonunit, then $s \leq N(a)$. (Equivalently, any strictly ascending chain of principal integral ideals starting at Ra, has length at most s.) Certainly a BFD has ACCP, but the converse is false. The domain in Example 5.1 is a BFD, but its integral closure is not.

THEOREM 7.5. *Let R be an integral domain with quotient field K. Let T be a domain with $R[X] \subseteq T \subseteq R + XK[X]$. Suppose that for each $n \geq 0$, there exists an $0 \neq r_n \in R$ so that $r_nf \in R[X]$ for all $f \in T$ with $\deg f \leq n$. Then T has ACCP (resp., is a BFD) if and only if R has ACCP (resp., is a BFD).*

PROOF: (1) This is a special case of Proposition 5.2.
(2) Suppose that R is a BFD. Let $0 \neq f \in T$ have degree n and leading coefficient b. Write $f = g_1 \cdots g_sg_{s+1} \cdots g_m$ where $g_1, \cdots, g_m \in T$ are

nonunits with $g_1, \cdots, g_s \in R$ and $g_{s+1}, \cdots, g_m \in T$ have degree ≥ 1. Now $g_{s+1} \cdots g_m$ has degree n, so $m - s \leq n$. Also, $r_n g_{s+1} \cdots g_m \in R[X]$, say it has leading coefficient $c \in R$. Then $r_n b = g_1 \cdots g_s c$. But R is a BFD, so there is a bound on the number of factors for $r_n b$ and hence on s. Thus the m in $f = g_1 \cdots g_s g_{s+1} \cdots g_m$ has an upper bound. Conversely, if T is a BFD, it is easily seen that R is a BFD without any additional hypothesis on T.

COROLLARY 7.6. *Let R be an integral domain with quotient field K. Then $I(K, R)$ satisfies ACCP (is a BFD) if and only if R satisfies ACCP (is a BFD).*

PROOF: Combine Proposition 7.3 and Theorem 7.5.

Proposition 7.2 and Corollary 7.6 together with the well known fact that $I(\mathbb{Q}, \mathbb{Z})$ is a Prüfer domain yields the following interesting example.

Example 7.7. $I(\mathbb{Q}, \mathbb{Z})$ is a two-dimensional completely integrally closed Prüfer domain that is a BFD and hence has ACCP.

Actually, if R is any Dedekind domain satisfying (1) char $D = 0$, (2) D/P is finite for each nonzero prime ideal P, and (3) if $f(X) \in D[X]$ is a nonconstant polynomial then the equation $f(X) \equiv 0$ (P) has a solution for infinitely many primes P, then $I(K, R)$ is a Prüfer domain ([**8**, Theorem 2]) and is a Hilbert domain ([**6**]). It is easily seen that $I(K, R)$ has Krull dimension two. By Proposition 7.2, $I(K, R)$ is completely integrally closed and by Corollary 7.6, $I(K, R)$ is a BFD and hence has ACCP. A. Grams [**18**] has also given examples of non-Noetherian Prüfer domains satisfying ACCP. It is interesting to note that if M is a height-two maximal ideal of $I(K, R)$, then $I(K, R)_M$ is a two-dimensional valuation domain and hence is neither completely integrally closed nor satisfies ACCP, even though $I(K, R)$ is both completely integrally closed and satisfies ACCP.

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