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Splitting the t-class group

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Abstract

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Let D be an integral domain and S a saturated multiplicatively closed subset of D. We say that S is a splitting set if for each $0 \neq d \in D$, we can write d as the product $d = sa$, where $s \in S$ and $a \in D$, with $s'D \cap aD = s'aD$ for all $s' \in S$. An important example of a splitting set is the multiplicatively closed set generated by a set of principal primes having the property that for each $0 \neq d \in D$, there is a bound on the length of a product of these primes dividing d. If S is a splitting set, then $T = \{0 \neq t \in D \mid tD \cap sD = tsD$ for all $s \in S\}$ is a saturated multiplicatively closed subset of D. We show that the map from the monoid $T(D)$ of t-ideals of D to the cardinal product $T(D_S) \times_c T(D_T)$, given by $A \to (AD_S, AD_T)$, is an order-preserving monoid isomorphism. Moreover, the induced map $\mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D_S) \times \mathrm{Cl}_t(D_T)$, given by $[A] \to ([AD_S], [AD_T])$, is an isomorphism which splits the t-class group of D. Applications and examples of this splitting are given.

1. Introduction

Let D be a Krull domain. If S is a multiplicatively closed subset of D generated by principal primes, then it is well known that the natural map $\mathrm{Cl}(D) \to \mathrm{Cl}(D_S)$ is an isomorphism, where $\mathrm{Cl}(D)$ is the divisor class group of D. The converse is also true. Thus, for example, D is a UFD if and only if D_S is a UFD. One of the purposes of this paper is to extend this result to arbitrary integral domains as follows. Let D be an integral domain and let S be the multiplicatively closed set generated by a family $\{p_\alpha\}$ of principal primes having the property that for any

$0 \neq x \in D$, there is a bound on the length of a product of the p_α's dividing x. (This is the case, if for example, D has ACC on principal ideals.) Then the natural homomorphism $\mathrm{Cl}_t(D) \rightarrow \mathrm{Cl}_t(D_S)$ is an isomorphism, where $\mathrm{Cl}_t(D)$ is the t-class group of D, that is, the group of t-invertible t-ideals of D modulo the subgroup of nonzero principal fractional ideals of D. (See Section 2 for the definition of a t-ideal.) This has also been shown by Gabelli and Roitman [14] using entirely different techniques.

Actually, we prove a much more general result. Let D be an integral domain and let S be a saturated multiplicatively closed subset of D having the property that for each $0 \neq d \in D$, we have $d = sa$ for some $s \in S$ and $a \in D$ with $s'D \cap aD = s'aD$ for all $s' \in S$. We call such an S a splitting multiplicatively closed set. Then the set $T = \{0 \neq t \in D \mid sD \cap tD = stD \text{ for all } s \in S\}$ is also a saturated multiplicatively closed subset of D. We show that the map $T(D) \rightarrow T(D_S) \times_c T(D_T)$, given by $A \rightarrow (AD_S, AD_T)$, is an order-isomorphism from the monoid of fractional t-ideals of D to the cardinal product of the monoids of fractional t-ideals of D_S and D_T. (See Section 2 for the definition of a cardinal product.) We also show that $A \in T(D)$ is t-invertible (respectively, principal) if and only if both AD_S and AD_T are t-invertible (respectively, principal). Thus we get an isomorphism $\mathrm{Cl}_t(D) \rightarrow \mathrm{Cl}_t(D_S) \times \mathrm{Cl}_t(D_T)$ given by $[A] \rightarrow ([AD_S], [AD_T])$. In particular, the natural map $\mathrm{Cl}_t(D) \rightarrow \mathrm{Cl}_t(D_S)$ is surjective. If S is generated by principal primes as in the preceding paragraph, then S is a splitting multiplicatively closed set and D_T is a UFD, so $\mathrm{Cl}(D_T) = 0$. This gives the isomorphism $\mathrm{Cl}_t(D) \cong \mathrm{Cl}_t(D_S)$ mentioned in the preceding paragraph.

This splitting of the t-class group can also be used to prove certain 'Nagata-type' theorems: if D_S has property X, then D has property X.

Section 2 first reviews some of the necessary facts about t-ideals and the t-class group. Splitting multiplicatively closed sets are then defined and are related to the group of divisibility. Section 3 shows that not only does a splitting multiplicatively closed set split the group of divisibility, but it also splits the monoid of t-ideals and the t-class group as well. Examples and Nagata-type theorems are given in the final Section 4.

2. Splitting sets and the monoid of t-ideals

Let D be an integral domain with quotient field K and group of units $U(D)$. Recall that for a nonzero fractional ideal A of D,

$$A_v = (A^{-1})^{-1} = [D : [D : A]] = \bigcap \{xD \mid xD \supseteq A \text{ where } x \in K\} .$$

An ideal A is said to be a *v-ideal*, *divisorial*, or *reflexive* if $A = A_v$. For properties of the v-operation, the reader is referred to [15, Section 34]. However, we will be mostly interested in the *t-operation* $A \rightarrow A_t$, where

$$A_t = \bigcup \{J_v \mid 0 \neq J \subseteq A \text{ with } J \text{ finitely generated}\} .$$

An ideal A is called a *t-ideal* if $A = A_t$. A t-ideal (respectively, v-ideal) A has *finite type* if $A = (a_1, \ldots, a_n)_t$ (respectively, $A = (a_1, \ldots, a_n)_v$) for some finite subset $\{a_1, \ldots, a_n\} \subseteq A$. While the set of v-ideals may be a proper subset of the set of t-ideals, evidently the set of finite-type v-ideals coincides with the set of finite-type t-ideals. An ideal A is said to be *t-invertible* if there exists an ideal B with $(AB)_t = D$. In this case, we may take $B = A^{-1}$. A t-invertible t-ideal has finite type. For properties of the t-operation, the reader is referred to [18] and [19] and for t-invertibility [17] and [20] may be consulted.

The set $T(D)$ of fractional t-ideals is a monoid with identity D under the t-product $A * B = (AB)_t$. Now $T(D)$ is partially ordered by reverse inclusion: $A \leq B$ if and only if $B \subseteq A$. It is easily seen that this partial order is actually a lattice order. The positive cone $T_+(D)$ of this partial order is the submonoid of integral t-ideals. The subgroup of invertible elements of $T(D)$ is the group $TI(D)$ of t-invertible t-ideals. As previously remarked, a t-invertible t-ideal has finite type, so $TI(D)$ is actually a subgroup of $T^*(D)$, the monoid of finite-type t-ideals. In fact, $TI(D)$ is a directed partially ordered group under $\leq$. Further, let $I(D)$ be the group of invertible fractional ideals and $P(D)$ its subgroup of principal fractional ideals. Thus we have

$$P(D) \subseteq I(D) \subseteq TI(D) \subseteq T^*(D) \subseteq T(D) .$$

Each of these monoids inherits the partial order from $T(D)$ and in each case the positive cone corresponding to this partial order is the set of integral members of that monoid. For example, $I_+(D)$ is the monoid of integral invertible ideals.

The group $P(D)$ is order-isomorphic to $G(D) = K^*/U(D)$, the *group of divisibility* of D, partially ordered by $xU(D) \leq yU(D) \Leftrightarrow x \mid y$ in D, via the correspondence $xD \leftrightarrow xU(D)$. Also, we have the two abelian groups

$$\mathrm{Pic}(D) = I(D)/P(D) \subseteq TI(D)/P(D) = \mathrm{Cl}_t(D) ,$$

where $\mathrm{Pic}(D)$ is the *Picard group* of D and $\mathrm{Cl}_t(D)$ is the *t-class group* of D. When D is a Krull domain, $\mathrm{Cl}_t(D)$ is just the divisor class group $\mathrm{Cl}(D)$. For properties of the t-class group, the reader is referred to [7–10].

Given two partially ordered monoids $(M, \leq)$ and $(N, \leq)$, the *cardinal product* $M \times_c N$ of M and N is the monoid direct product $M \times N$ with the partial order $(a, b) \leq (c, d) \Leftrightarrow a \leq c$ and $b \leq d$. Sometimes we use the notation $M \oplus_c N$ and say *cardinal sum* when we are dealing with an internal direct product.

Mott [21, Theorem 2.1] showed that there is a one-to-one correspondence between the set of convex directed subgroups of $P(D) \cong G(D)$ and the set of saturated multiplicatively closed subsets of D. This correspondence is given as follows. If S is a saturated multiplicatively closed subset of D, then $\langle S \rangle =$

$\{s_1s_2^{-1}D \mid s_1,s_2 \in S\}$ is a convex directed subgroup of $P(D)$ with positive cone $\langle S \rangle_+ = \{sD \mid s \in S\}$. (In $G(D)$, we may identify $\langle S \rangle$ with $U(D_S)/U(D)$; so $G(D)/\langle S \rangle$ is order-isomorphic to $G(D_S)$.) Conversely, if H is a convex directed subgroup of $P(D)$, then $S = \{s \in D \mid sD \in H_+\}$ is a saturated multiplicatively closed subset of D.

In [22], Mott and Schexnayder considered the question of when $\langle S \rangle \cong U(D_S)/U(D)$ is a cardinal summand of $P(D) \cong G(D)$, that is, when there is a subgroup H of $P(D)$ with $\langle S \rangle \oplus_c H = P(D)$. (Here, of course, H is order-isomorphic to $P(D)/\langle S \rangle \cong G(D_S)$.) They gave a condition ([22, Proposition 4.1], essentially (4) of Theorem 2.2) in terms of multiplicatively closed subsets of D for $\langle S \rangle$ to be a cardinal summand of $P(D)$, which essentially amounts to saying that $\langle S \rangle_+$ is a cardinal summand of $P_+(D)$. They showed that if S is generated by principal primes satisfying certain conditions (which will be given later in this section), then $\langle S \rangle$ is a cardinal summand of $P(D)$. They then observed that this approach could be used to prove known results such as if D is a GCD domain (respectively, UFD), then $D[X]$ is a GCD domain (respectively, UFD); and certain 'Nagata-type' theorems: if S is generated by principal primes (with $\langle S \rangle$ being a cardinal summand) and if D_S is a GCD domain (respectively, UFD), then D is a GCD domain (respectively, UFD). Additional Nagata-type theorems were given in [3], where an alternative characterization for $\langle S \rangle$ being a cardinal summand was used. It is that definition that we now give.

Definition 2.1. A saturated multiplicatively closed subset S of D is said to be a *splitting set* if for each $0 \neq d \in D$, we can write d as the product $d = sa$ for some $s \in S$ and $a \in D$ with $s'D \cap aD = s'aD$ for all $s' \in S$.

For S any multiplicatively closed subset of D, let

$$T = \{0 \neq t \in D \mid sD \cap tD = stD \text{ for all } s \in S\}$$

be the set of all nonzero elements of D that are lcm-prime to each element of S. (Observe that s and t are *lcm-prime* (that is, $sD \cap tD = stD$) if and only if s and t are *v-coprime* (that is, $(s, t)_v = D$).) It is easily proved that T is a saturated multiplicatively closed subset of D. Thus S is a splitting set if and only if $ST = D - \{0\}$. Hence if S is a splitting set, each nonzero element d of D may be written in the form $d = st$ for some $s \in S$ and $t \in T$, and this factorization is unique up to unit factors. We will call T the *complementary multiplicatively closed set for* S or just the *m-complement for* S. Note that T is also a splitting multiplicatively closed set with S for its m-complement. Several conditions equivalent to S being a splitting multiplicatively closed set are given in the next theorem.

Theorem 2.2. *The following conditions are equivalent for a saturated multiplicatively closed set* S *in an integral domain* D.

(1) *S is a splitting multiplicatively closed set.*

(2) *$\langle S \rangle$ is a cardinal summand of $P(D)$, that is, there is a subgroup H of $P(D)$ with $P(D) = \langle S \rangle \oplus_c H$.*

(3) *If A is an integral principal ideal of D_S, then $A \cap D$ is a principal ideal of D.*

(4) *There exists a multiplicatively closed set T such that*

(a) *each element $0 \neq d \in D$ may be written as $d = st$, where $s \in S$ and $t \in T$, and*

(b) *one of the following equivalent four conditions holds:*

(i) *If $st = s't'$, where $s,s' \in S$ and $t,t' \in T$, then $s' = su$ and $t' = tu^{-1}$ where $u \in U(D)$.*

(ii) *If $d = st$ where $s \in S$ and $t \in T$, then $dD_S \cap D = tD$.*

(iii) *For each $s \in S$ and each $t \in T$, $sD \cap tD = stD$.*

(iv) *For each $t \in T$, $tD_S \cap D = tD$.*

Proof. The equivalence of (2) and (4) is essentially given in [22, Proposition 4.1], while the equivalence of (1) and (3) is given in [3, Corollary 1.3]. Certainly (4)$\Rightarrow$(3) and the remarks given in the paragraph preceding Theorem 2.2 show that (1)$\Rightarrow$(4). □

Let S be a splitting multiplicatively closed subset of D. It is easily seen that the saturation $\bar{T}$ of the multiplicatively closed T from (4) of Theorem 2.2 is $TU(D)$ and that $\bar{T} = \{0 \neq t \in D \mid sD \cap tD = stD$ for all $s \in S\}$, that is, $\bar{T}$ is the m-complement for S. It is also easily seen that $D = D_S \cap D_T$. Moreover, $G(D_S)$ (respectively, $G(D_T)$) is order-isomorphic to $\langle \bar{T} \rangle$ (respectively, $\langle S \rangle$) and $P(D) = \langle S \rangle \oplus_c \langle \bar{T} \rangle \cong G(D_S) \oplus_c G(D_T)$. Note that condition (4) of Theorem 2.2 states that $P_+(D) = \langle S \rangle_+ \oplus_c \langle T \rangle_+$. We next define an important special type of splitting set.

Definition 2.3. A splitting multiplicatively closed subset S of D is said to be an *lcm splitting set* if for each $s \in S$ and $d \in D$, $sD \cap dD$ is principal.

Our next proposition gives several characterizations of lcm splitting sets. Of special interest will be lcm splitting sets generated by principal primes, see Definition 2.5.

Proposition 2.4. *The following conditions are equivalent for a splitting multiplicatively closed subset S of D.*

(1) *S is lcm splitting.*

(2) *For $s_1,s_2 \in S$, $s_1D \cap s_2D$ is principal.*

(3) *For $s_1,s_2 \in S$, $s_1D \cap s_2D = sD$ for some $s \in S$.*

(4) *D_T is a GCD domain, where T is the m-complement for S.*

Proof. (1)$\Rightarrow$(2) Clear.

(2)$\Rightarrow$(3) Suppose that $s_1D \cap s_2D = xD$. Write $x = s't'$, where $s' \in S$ and $t' \in T$. Then $t'D = s't'D_S \cap D = (s_1D \cap s_2D)D_S \cap D = D$, so $t' \in U(D)$. Hence $s_1D \cap s_2D = s'D$.

(3)⇒(4) Every principal ideal in D_T has the form sD_T for some $s \in S$. Since $s_1D_T \cap s_2D_T = (s_1D \cap s_2D)D_T$ is principal, the intersection of two principal ideals of D_T is principal. Thus D_T is a GCD domain.

(4)⇒(1) Write $d = s_1t_1$, where $s_1 \in S$ and $t_1 \in T$. Then $sD \cap dD = sD \cap s_1t_1D = sD \cap s_1D \cap t_1D$. By the proof of (2)⇒(3), it is enough to show that $sD \cap s_1D$ is principal. For then $sD \cap s_1D = s_2D$ for some $s_2 \in S$, and hence $sD \cap dD = s_2D \cap t_1D = s_2t_1D$. Since D_T is a GCD domain, $sD_T \cap s_1D_T$ is principal. Since T is also a splitting set, $(sD_T \cap s_1D_T) \cap D$ is principal. Thus $sD \cap s_1D = (sD_T \cap D) \cap (s_1D_T \cap D) = (sD_T \cap s_1D_T) \cap D$ is principal. □

Definition 2.5. A set $\{p_\alpha\}$ of principal prime elements is said to be a *splitting set of principal primes* if the saturated multiplicatively closed set $\{up_{\alpha_1} \cdots p_{\alpha_n} \mid u \in U(D),\ p_{\alpha_i} \in \{p_\alpha\},\ n \geq 0\}$ generated by the p_α's is a splitting set.

It is easily seen [3, Proposition 1.5] that a set $\{p_\alpha\}$ of principal primes is splitting if and only if (1) for each α, $\bigcap_{n=1}^{\infty} p_\alpha^n D = 0$ (equivalently, ht $p_\alpha D = 1$) and (2) for any sequence $\{p_{\alpha_n}\}$ of nonassociate members of $\{p_\alpha\}$, $\bigcap_{n=1}^{\infty} p_{\alpha_n} D = 0$. In the terminology of Mott and Schexnayder [22], $\{p_\alpha\}$ has the UF-property. Also, if D satisfies the ascending chain condition on principal ideals, or more generally, if D is atomic (that is, every nonzero nonunit of D is a finite product of irreducible elements), then any saturated multiplicatively closed set generated by principal primes is a splitting multiplicatively closed set [3, Corollary 1.6]. However, in general a saturated multiplicatively closed set generated by principal primes need not be a splitting set. For a valuation domain (V, M) with principal maximal ideal $M = pV$, $\{p\}$ is a splitting set of primes if and only if V has rank one. Let E be the ring of entire functions and let S be the saturated multiplicatively closed subset of E generated by the principal primes of E. Then S is not a splitting set. Here while each principal prime has height one, so (1) is satisfied, condition (2) is not satisfied.

If S is generated by a splitting set $\{p_\alpha\}$ of primes, then every principal ideal of D_T (where T is the m-complement for S) is a product of principal prime ideals of the form $p_\alpha D_T$. Hence D_T is a UFD. Since a UFD is a GCD domain, S is an lcm splitting set by Proposition 2.4. Conversely, suppose that S is a splitting multiplicatively closed set and that D_T is a UFD, where T is the m-complement for S. For $s \in S$ a nonunit, we may write $sD_T = p_1D_T \cdots p_nD_T$, where $p_i \in S$ and p_iD_T is a principal prime ideal of D_T. Since T is also a splitting set, $p_iD = p_iD_T \cap D$ is a principal prime ideal. Thus S is generated by a set of principal prime elements, necessarily an lcm set of principal primes, since S is an lcm splitting set. We summarize these equivalencies in the next proposition.

Proposition 2.6. *The following conditions are equivalent for a saturated multiplicatively closed subset S of D.*

(1) *S is generated by a set of prime elements* $\{p_\alpha\}$ *satisfying* (a) *for each* α,

$\bigcap_{n=1}^{\infty} p_\alpha^n D = 0$, *and* (b) *for any sequence* $\{p_{\alpha_n}\}$ *of nonassociate members of* $\{p_\alpha\}$, $\bigcap_{n=1}^{\infty} p_{\alpha_n} D = 0$.

(2) *S is generated by a splitting set of principal primes.*

(3) *S is generated by a set of principal prime elements and S is a splitting set.*

(4) *S is a splitting set and* D_T *is a UFD, where T is the m-complement for S.* □

Thus a splitting set generated by principal primes is an lcm splitting set. For atomic domains (for example, domains with ACC on principal ideals, Noetherian domains, or Krull domains), the converse is also true.

Corollary 2.7. *Let D be an atomic integral domain. Then a saturated multiplicatively closed subset S of D is an lcm splitting set if and only if S is generated by principal primes.*

Proof. ($\Leftarrow$) Suppose that S is generated by principal primes. Since the conditions of Proposition 2.6(1) are obviously satisfied, S is an lcm splitting set.

($\Rightarrow$) Suppose that S is an lcm splitting set for D and let T be the m-complement for S. Since $P_+(D)$ is order isomorphic to $P_+(D_S) \times_c P_+(D_T)$ by Theorem 2.2 and the remarks following it, D_T is atomic. (The fact that D_T is atomic also follows from [3, Corollary 2.2].) By Proposition 2.4, D_T is a GCD domain. Thus D_T is an atomic GCD domain and hence is a UFD. By Proposition 2.6, S is generated by principal primes. □

Examples of lcm splitting sets not generated by principal primes will be given in Section 4.

3. Splitting the t-class group

Let S be a splitting multiplicatively closed subset of D with T the m-complement for S. We have seen in the previous section that each nonzero principal ideal dD of D has a unique representation of the form $dD = (sD)(tD)$ $(= sD \cap tD)$, where $s \in S$ and $t \in T$. Moreover, $sD = dD_T \cap D$ and $tD = dD_S \cap D$. Stated in terms of the monoid $P_+(D)$ of nonzero principal ideals of D, we have that the map $P_+(D) \to P_+(D_S) \times_c P_+(D_T)$, given by $A \to (AD_S, AD_T)$, is a monoid order-isomorphism. Or, in terms of the group of all nonzero principal fractional ideals of D (respectively, the group of divisibility of D), we have that the map $P(D) \to P(D_S) \times_c P(D_T)$, given by $A \to (AD_S, AD_T)$ (respectively, the map $G(D) \to G(D_S) \times_c G(D_T)$, given by $xU(D) \to (xU(D_S), xU(D_T)))$, is an order-isomorphism.

The purpose of this section is to show that similar results hold for the monoid of t-ideals and the t-class group. We show that if S is a splitting multiplicatively closed set with m-complement T, then the map $T(D) \to T(D_S) \times_c T(D_T)$, given by $A \to (AD_S, AD_T)$, is an order-isomorphism. Moreover, the image of $P(D)$ (respectively; $TI(D)$, $T^*(D)$) under this map is $P(D_S) \times_c P(D_T)$ (respectively; $TI(D_S) \times_c TI(D_T)$, $T^*(D_S) \times_c T^*(D_T)$). Hence the map $\mathrm{Cl}_t(D) \to Cl_t(D_S) \times \mathrm{Cl}_t(D_T)$, given by $[A] \to ([AD_S], [AD_T])$, is a group isomorphism. In particular, $\mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D_S)$ is surjective. Here as usual, $[A]$ denotes the equivalence class of $A \in TI(D)$ in $\mathrm{Cl}_t(D)$.

The next lemma is the key observation needed to prove Theorem 3.2 which states that if S is a splitting multiplicatively closed set, then a t-ideal may be written as a t-product of an ideal generated by elements from S and an ideal generated by elements from T, the m-complement for S.

Lemma 3.1. *Let S be a splitting multiplicatively closed subset of D with T the m-complement for S. Let $s_1, \dots, s_n \in S$ and $t_1, \dots, t_n \in T$. Then $(s_1t_1, \dots, s_nt_n)_v = ((s_1, \dots, s_n)(t_1, \dots, t_n))_v$.*

Proof. Put $s = s_1 \dots s_n$, $\hat{s}_i = s/s_i$, $t = t_1 \dots t_n$, and $\hat{t}_i = t/t_i$. Note that for $1 \le i, j \le n$, $\hat{s}_i\hat{t}_jD = \hat{s}_iD \cap \hat{t}_jD$ since $\hat{s}_i \in S$ and $\hat{t}_j \in T$. Then

$$(s_1t_1, \dots, s_nt_n)^{-1} = s_1^{-1}t_1^{-1}D \cap \cdots \cap s_n^{-1}t_n^{-1}D = s^{-1}t^{-1}\Big(\bigcap_{i=1}^{n} \hat{s}_i\hat{t}_iD\Big)$$

while

$$((s_1, \dots, s_n)(t_1, \dots, t_n))^{-1} = \bigcap_{1 \le i,j \le n} s_i^{-1}t_j^{-1}D = s^{-1}t^{-1}\Big(\bigcap_{1 \le i,j \le n} \hat{s}_i\hat{t}_jD\Big).$$

Since

$$\bigcap_{1 \le i,j \le n} \hat{s}_i\hat{t}_jD = \bigcap_{1 \le i,j \le n} (\hat{s}_iD \cap \hat{t}_jD) = \bigcap_{i=1}^{n} (\hat{s}_iD \cap \hat{t}_iD) = \bigcap_{i=1}^{n} \hat{s}_i\hat{t}_iD,$$

we have that $(s_1t_1, \dots, s_nt_n)^{-1} = ((s_1, \dots, s_n)(t_1, \dots, t_n))^{-1}$, and hence the desired result. □

Theorem 3.2. *Let S be a splitting multiplicatively closed subset of D with T the m-complement for S. Let $A = (\{a_\alpha\})$ (each $a_\alpha \neq 0$) be an integral ideal of D. For each α, write $a_\alpha = s_\alpha t_\alpha$, where $s_\alpha \in S$ and $t_\alpha \in T$. Then $A_t = ((\{s_\alpha\})(\{t_\alpha\}))_t$. In particular, $A_t = ((S_1)(T_1))_t$, where $S_1 = \{s \in S \mid st \in A \text{ for some } t \in T\}$ and $T_1 = \{t \in T \mid st \in A \text{ for some } s \in S\}$.*

Proof. Certainly $A = (\{a_\alpha\}) = (\{s_\alpha t_\alpha\}) \subseteq (\{s_\alpha\})(\{t_\alpha\})$, so $A_t \subseteq ((\{s_\alpha\})(\{t_\alpha\}))_t$. Conversely, let $0 \neq x \in ((\{s_\alpha\})(\{t_\alpha\}))_t$. Then $x \in ((s_{\beta_1}, \dots, s_{\beta_n})(t_{\gamma_1}, \dots, t_{\gamma_m}))_v$, where $s_{\beta_1}, \dots, s_{\beta_n} \in \{s_\alpha\}$ and $t_{\gamma_1}, \dots, t_{\gamma_m} \in \{t_\alpha\}$. Thus

$$x \in ((s_{\beta_1}, \ldots, s_{\beta_n}, s_{\gamma_1}, \ldots, s_{\gamma_m})(t_{\beta_1}, \ldots, t_{\beta_n}, t_{\gamma_1}, \ldots, t_{\gamma_m}))_v$$
$$= (s_{\beta_1}t_{\beta_1}, \ldots, s_{\beta_n}t_{\beta_n}, s_{\gamma_1}t_{\gamma_1}, \ldots, s_{\gamma_m}t_{\gamma_m})_v$$
$$= (a_{\beta_1}, \ldots, a_{\beta_n}, a_{\gamma_1}, \ldots, a_{\gamma_m})_v \subseteq A_t ,$$

where the next to the last equality follows from Lemma 3.1. □

The next theorem concerns the relationship between localization at a splitting multiplicatively closed set and the t-operation.

Theorem 3.3. *Let S be a splitting multiplicatively closed subset of D and let T be the m-complement for S. Let $A = (\{a_\alpha\})$ $(a_\alpha \neq 0)$ be an integral ideal of D. For each α, let $a_\alpha = s_\alpha t_\alpha$, where $s_\alpha \in S$ and $t_\alpha \in T$. Then $(AD_S)_t \cap D = (\{t_\alpha\})_t$. In particular, if A is generated by elements of T, then $(AD_S)_t \cap D = A_t$, and hence $(AD_S)_t = A_t D_S$.*

Proof. Since $AD_S = (\{t_\alpha\})D_S$, it suffices to prove the result where each $a_\alpha = t_\alpha \in T$. Now $A_t \subseteq (A_t D_S)_t \cap D = (AD_S)_t \cap D$ since $(AD_S)_t = (A_t D_S)_t$ ([19, Lemma 3.4] or [25, Lemma 4]). Let $0 \neq x \in (AD_S)_t \cap D$, so $x \in ((t_1, \ldots, t_n)D_S)_v \cap D$ for some finite subset $\{t_1, \ldots, t_n\} \subseteq \{t_\alpha\}$. Write $x = st$, where $s \in S$ and $t \in T$. Then $t \in ((t_1, \ldots, t_n)D_S)_v$, so $t((t_1, \ldots, t_n)^{-1}D_S) = t((t_1, \ldots, t_n)D_S)^{-1} \subseteq D_S$. Hence $t(t_1^{-1}D_S \cap \cdots \cap t_n^{-1}D_S) \subseteq D_S$. Multiplying both sides by $t_1 \ldots t_n$ yields that

$$tt_2 \ldots t_n D_S \cap \cdots \cap tt_1 \ldots t_{n-1} D_S \subseteq t_1 \ldots t_n D_S .$$

Contracting back to D, Theorem 2.2 gives that

$$tt_2 \ldots t_n D \cap \cdots \cap tt_1 \ldots t_{n-1} D \subseteq t_1 \ldots t_n D .$$

Then dividing by $t_1 \ldots t_n$ gives that

$$t(t_1, \ldots, t_n)^{-1} = tt_1^{-1}D \cap \cdots \cap tt_n^{-1}D \subseteq D .$$

Hence $t \in (t_1, \ldots, t_n)_v$. Thus $x \in A_t$. The remaining statements are now immediate. □

Corollary 3.4. *With the notation of Theorem* 3.3 *let* $\emptyset \neq S_1 \subseteq S$ *and* $\emptyset \neq T_1 \subseteq T$. *Then* $(S_1)_t \cap (T_1)_t = ((S_1)(T_1))_t$.

Proof. Now $(S_1)_t \cap (T_1)_t$ is a t-ideal, so by Theorem 3.2 $(S_1)_t \cap (T_1)_t = ((S_2)(T_2))_t$, where $S_2 \subseteq S$ and $T_2 \subseteq T$. Now $(T_1)_t \supseteq ((S_2)(T_2))_t \supseteq (S_2)(T_2)$, so $(T_1)_t D_S \supseteq (S_2)(T_2)D_S = (T_2)D_S$. Hence

$$(T_1)_t = ((T_1)D_S)_t \cap D = (T_1)_t D_S \cap D \supseteq (T_2)D_S \cap D \supseteq (T_2) .$$

By similar reasoning, we also have $(S_1)_t \supseteq (S_2)$. Hence

$$\begin{aligned}(S_1)_t \cap (T_1)_t &= ((S_2)(T_2))_t \subseteq ((S_1)_t(T_1)_t)_t \\ &= ((S_1)(T_1))_t \subseteq (S_1)_t \cap (T_1)_t . \qquad \square\end{aligned}$$

For an arbitrary multiplicatively closed subset S of an integral domain D, if A is a t-ideal, AD_S need not be a t-ideal [26]. The next corollary shows that this cannot happen if S is a splitting set.

Corollary 3.5. *With the notation of Theorem* 3.3, *if B is an* (*integral*) *t-ideal of D, then BD_S is an* (*integral*) *t-ideal of D_S. In fact, for a nonzero ideal A of D, $A_tD_S = (AD_S)_t$. If E is a t-ideal of D_S, then $E \cap D$ is a t-ideal of D.*

Proof. Let A be a nonzero integral ideal of D. By Theorem 3.2 and Corollary 3.4, we have $A_t = ((S_1)(T_1))_t = (S_1)_t \cap (T_1)_t$, for some $S_1 \subseteq S$ and $T_1 \subseteq T$. Then

$$\begin{aligned}A_tD_S &= ((S_1)_t \cap (T_1)_t)D_S = (S_1)_tD_S \cap (T_1)_tD_S \\ &= (T_1)_tD_S = ((AD_S)_t \cap D)D_S = (AD_S)_t .\end{aligned}$$

Dividing through by an appropriate element of D shows that the equality $A_tD_S = (AD_S)_t$ holds for nonzero fractional ideals as well. Hence if B is a t-ideal, then so is BD_S. It is well known and easily proved that if E is a t-ideal of D_S, where S is any multiplicatively closed set, then $E \cap D$ is a t-ideal of D. However, we offer the following alternative proof for the case where S is a splitting multiplicatively closed set. If E is an integral t-ideal of D_S, then $E = (AD_S)_t$ for some integral ideal A of D generated by elements of T, and hence $E \cap D = (AD_S)_t \cap D = A_t$ is a t-ideal by Theorem 3.3. $\square$

Our next theorem summarizes our observations that a splitting multiplicatively closed set S, with m-complement T, gives both a product and intersection decomposition for integral t-ideals.

Theorem 3.6. *Let S be a splitting multiplicatively closed subset of D and let T be the m-complement for S. Let A be a nonzero integral ideal of D.*

(1) *There exist subsets $S_1 \subseteq S$ and $T_1 \subseteq T$ so that $A_t = ((S_1)(T_1))_t$. Moreover, this product representation is unique in that $(S_1)_t = A_tD_T \cap D$ and $(T_1)_t = A_tD_S \cap D$.*

(2) *There exist subsets $S_1 \subseteq S$ and $T_1 \subseteq T$ so that $A_t = (S_1)_t \cap (T_1)_t$. Moreover, this intersection representation is unique in that $(S_1)_t = A_tD_T \cap D$ and $(T_1)_t = A_tD_S \cap D$. Thus $A_t = (AD_S)_t \cap (AD_T)_t$, so the t-operation on D is induced by the t-operations on D_S and D_T.*

Proof. By Theorem 3.2 and Corollary 3.4, the representations given in (1) and (2) exist. If $A_t = ((S_1)(T_1))_t$, then $A_t = ((S_1)(T_1))_t = (S_1)_t \cap (T_1)_t$; so

$$A_t D_S = ((S_1)_t \cap (T_1)_t) D_S = (S_1)_t D_S \cap (T_1)_t D_S = (T_1)_t D_S ,$$

and hence

$$A_t D_S \cap D = (T_1)_t D_S \cap D = ((T_1) D_S)_t \cap D = (T_1)_t .$$

But T is also a splitting multiplicatively closed set with S for its m-complement. Hence $A_t D_T \cap D = (S_1)_t$. This proves (1) and the first part of (2). Since $D = D_S \cap D_T$, $(AD_S)_t \cap (AD_T)_t \subseteq D$, and hence

$$\begin{aligned}(AD_S)_t \cap (AD_T)_t &= ((AD_S)_t \cap D) \cap ((AD_T)_t \cap D)\\ &= (A_t D_S \cap D) \cap (A_t D_T \cap D) = (T_1)_t \cap (S_1)_t = A_t .\end{aligned}$$

Dividing through by an appropriate element of D shows that for any nonzero fractional ideal A of D, $A_t = (AD_S)_t \cap (AD_T)_t$, that is, in the terminology of [1], the t-operation on D is induced by the t-operations on D_S and D_T. □

Suppose that P is a prime t-ideal. Then from $P = (PD_S \cap D) \cap (PD_T \cap D)$ we see that either $P = PD_S \cap D$ or $P = PD_T \cap D$. Thus either $P \cap S = \emptyset$ or $P \cap T = \emptyset$, but not both; and there is a bijection between the prime t-ideals P with $P \cap S = \emptyset$ (respectively, $P \cap T = \emptyset$) and the prime t-ideals of D_S (respectively, D_T). Moreover, this correspondence preserves maximal t-ideals.

Let $T_+(D)$ be the monoid of integral t-ideals under the t-product. If A is an integral t-ideal of D, then AD_S is an integral t-ideal of D_S and AD_T is an integral t-ideal of D_T. This gives a map $\theta : T_+(D) \to T_+(D_S) \times T_+(D_T)$, where $\theta(A) = (AD_S, AD_T)$. Now

$$\begin{aligned}\theta(A * B) = \theta((AB)_t) &= ((AB)_t D_S, (AB)_t D_T) = ((ABD_S)_t, (ABD_T)_t)\\ &= ((AD_S BD_S)_t, (AD_T BD_T)_t) = \theta(A) * \theta(B) .\end{aligned}$$

If $\theta(A) = \theta(B)$, then $A = AD_S \cap AD_T = BD_S \cap BD_T = B$. Finally, if X is an integral t-ideal of D_S and Y is an integral t-ideal of D_T, then $X \cap D$ and $Y \cap D$ are integral t-ideals of D, and hence so is $(X \cap D) \cap (Y \cap D)$. But

$$\begin{aligned}&\theta((X \cap D) \cap (Y \cap D))\\ &\quad = (((X \cap D) \cap (Y \cap D))D_S, ((X \cap D) \cap (Y \cap D))D_T) = (X, Y)\end{aligned}$$

since

$$((X \cap D) \cap (Y \cap D))D_S = (X \cap D)D_S \cap (Y \cap D)D_S$$
$$= (X \cap D)D_S \cap D_S = X \cap D_S = X$$

and similarly for the second variable.

The map θ actually extends to an isomorphism $\theta : T(D) \to T(D_S) \times T(D_T)$, given by $\theta(A) = (AD_S, AD_T)$, where A is now a fractional ideal of D. Certainly AD_S and AD_T are still t-ideals and θ is still a monoid homomorphism. Suppose that $\theta(A) = \theta(B)$. Write $A = \frac{1}{a}A'$ and $B = \frac{1}{b}B'$, where A' and B' are integral t-ideals of D. Then $\frac{1}{a}A'D_S = \frac{1}{b}B'D_S$ implies that $bA'D_S = aB'D_S$ and $\frac{1}{a}A'D_T = \frac{1}{b}B'D_T$ implies that $bA'D_T = aB'D_T$, i.e., $\theta(bA') = \theta(aB')$. Hence $bA' = aB'$, so $A = \frac{1}{ab}(bA') = \frac{1}{ab}(aB') = B$. Thus θ is still injective. Let $(E, F) \in T(D_S) \times T(D_T)$. Then $E = \frac{1}{e}E'$ and $F = \frac{1}{f}F'$, where $e \in D_S$, E' is an integral t-ideal of D_S, $f \in D_T$, and F' is an integral t-ideal of D_T. Moreover, we can take $e \in T$ and $f \in S$. Let A be an integral t-ideal of D with $AD_S = E'$ and $AD_T = F'$. Then

$$\tfrac{1}{ef}AD_S = \tfrac{1}{e}AD_S = \tfrac{1}{e}E' = E \quad \text{and} \quad \tfrac{1}{ef}AD_T = \tfrac{1}{f}AD_T = \tfrac{1}{f}F' = F\,.$$

So $\theta(\frac{1}{ef}A) = (E, F)$ and hence θ is surjective. This proves part of our next theorem.

Theorem 3.7. *Let D be an integral domain, S a splitting multiplicatively closed subset for D, and T the m-complement for S. Then the map $\theta : T(D) \to T(D_S) \times_c T(D_T)$, given by $\theta(A) = (AD_S, AD_T)$, is a monoid order-isomorphism. Moreover, for $A \in T(D)$, A is integral (respectively; principal, of finite type, t-invertible) if and only if both AD_S and AD_T are integral (respectively; principal, of finite type, t-invertible).*

Proof. We have already seen that θ is a monoid isomorphism. Moreover, θ maps the positive cone $T_+(D)$ of $T(D)$ to the positive cone $(T(D_S) \times_c T(D_T))_+ = T_+(D_S) \times_c T_+(D_T)$ of $T(D_S) \times_c T(D_T)$. Thus A is integral if and only if AD_S and AD_T are both integral. Also, A is a unit of $T(D)$ if and only if (AD_S, AD_T) is a unit of $T(D_S) \times T(D_T)$. Thus A is t-invertible if and only if AD_S and AD_T are both t-invertible. If A is principal, certainly AD_S and AD_T are both principal. Conversely, suppose that AD_S and AD_T are both principal. Choose $0 \neq a \in D$ so that aA is an integral t-ideal. Then aAD_S and aAD_T are both principal, so $aAD_S = tD_S$ for some $t \in T$ and $aAD_T = sD_T$ for some $s \in S$. Thus $aA = (aAD_S \cap D) \cap (aAD_T \cap D) = tD \cap sD = stD$ is also principal. Hence A itself is principal. To show that A has finite type if and only if both AD_S and AD_T have finite type, we may restrict ourselves to integral t-ideals. If $A = (a_1, \ldots, a_n)_t$, then $AD_S = (a_1, \ldots, a_n)_t D_S = ((a_1, \ldots, a_n)D_S)_t$ has finite type, as does AD_T. Conversely, suppose that AD_S and AD_T both have finite type.

Then $AD_S = ((t_1, \ldots, t_n)D_S)_t$ and $AD_T = ((s_1, \ldots, s_m)D_T)_t$. For $A' = (s_1, \ldots, s_m)(t_1, \ldots, t_n)$, we have $A'_t D_S = AD_S$ and $A'_t D_T = AD_T$. Hence $A'_t = A$, so A has finite type. □

Corollary 3.8. *With the notation of Theorem* 3.7, *the map* $\bar{\theta} : \mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D_S) \times \mathrm{Cl}_t(D_T)$, *given by* $\bar{\theta}([A]) = ([AD_S], [AD_T])$, *is a group isomorphism. In particular, the natural map* $Cl_t(D) \to \mathrm{Cl}_t(D_S)$ *is a group epimorphism and is an isomorphism if and only if* $\mathrm{Cl}_t(D_T) = 0$.

Proof. By Theorem 3.7, the monoid isomorphism $\theta : T(D) \to T(D_S) \times T(D_T)$ restricts to a group isomorphism $\theta : TI(D) \to TI(D_S) \times TI(D_T)$, where $TI(D)$ is the subgroup of $T(D)$ consisting of t-invertible t-ideals. Moreover, the image of the subgroup $P(D)$ of principal fractional ideals is $P(D_S) \times P(D_T)$. Thus the induced map

$$\begin{aligned}\bar{\theta} : \mathrm{Cl}_t(D) = TI(D)/P(D) &\to (TI(D_S) \times TI(D_T))/(P(D_S) \times P(D_T)) \\ &\cong TI(D_S)/P(D_S) \times TI(D_T)/P(D_T) \\ &= \mathrm{Cl}_t(D_S) \times \mathrm{Cl}_t(D_T)\,,\end{aligned}$$

given by $[A] \to ([AD_S], [AD_T])$, is a group isomorphism. Hence the map $\mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D_S)$ is surjective and is an isomorphism if and only if $\mathrm{Cl}_t(D_T) = 0$. □

We remark that similar results do *not* carry over for $\mathrm{Pic}(D)$. Suppose that we are in the set-up of Theorem 3.7. If A is an invertible ideal of D, then certainly AD_S and AD_T are both invertible. Thus we get an order-preserving monomorphism $I(D) \to I(D_S) \times_c I(D_T)$ by restricting θ to $I(D)$. However, this map need *not* be surjective. For example, let (D, M) be a two-dimensional local Krull domain that is not factorial but which has a nonzero principal prime p. Take $S = \{up^n \mid u \in U(D),\ n \geq 0\}$. Then $D_S = D[1/p]$ is a one-dimensional Krull domain, that is, a Dedekind domain. Now D_S can *not* be a PID, for then D_S would be a UFD, and hence by Nagata's Theorem, D would also be a UFD. Here $D_T = D_{(p)}$ is a DVR. The map $I(D) \to I(D_S) \times I(D_T)$ cannot be surjective, for $I(D) = P(D)$ and hence we would have $I(D_S) = P(D_S)$, that is, D_S is a PID, a contradiction. Moreover, the group monomorphism $\mathrm{Pic}(D) \to \mathrm{Pic}(D_S) \times \mathrm{Pic}(D_T)$ need not be surjective either. For in our example, $\mathrm{Pic}(D) = 0$ while $\mathrm{Pic}(D_S) \neq 0$. Also see Example 4.5.

4. Applications

The purpose of this section is to provide applications of the splitting $\mathrm{Cl}_t(D_S) \times \mathrm{Cl}_t(D_T)$ for $\mathrm{Cl}_t(D)$, where S is a splitting multiplicatively closed subset of D. For the most part, we will be interested in the case where S is an lcm splitting set, usually where S is generated by principal primes. We then prove 'Nagata-type' theorems: if D_S has property X, then D has property X. Other Nagata-type theorems are given in [3].

Theorem 4.1. *Let D be an integral domain, S an lcm splitting set, and T the m-complement for S. Then $D = D_S \cap D_T$, where D_T is a GCD domain. Every finite-type integral t-ideal A of D has the form $A = s(AD_S \cap D) = s(t_1, \ldots, t_n)_v$, where $s \in S$ and $t_1, \ldots, t_n \in T$. Moreover, the map $\mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D_S)$ given by $[A] \to [AD_S]$ is a group isomorphism.*

Proof. By Proposition 2.4, D_T is a GCD domain, so $\mathrm{Cl}_t(D_T) = 0$. Hence every finite-type t-ideal of D_T is principal. In particular, a finite-type integral t-ideal of D_T has the form sD_T for some $s \in S$. Thus $A = ((AD_T \cap D)(AD_S \cap D))_t = s(AD_S \cap D) = s(t_1, \ldots, t_n)_v$ for some $t_1, \ldots, t_n \in T$. By Corollary 3.8, the natural map $\mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D_S) \times \mathrm{Cl}_t(D_T)$ is an isomorphism. Since $\mathrm{Cl}_t(D_T) = 0$, the natural map given by $[A] \to ([AD_S], [AD_T]) \to [AD_S]$ is an isomorphism. □

Our next result is the special case of Theorem 4.1 where S is generated by principal primes.

Theorem 4.2. *Let $\{p_\alpha\}$ be a set of splitting primes, let S be the saturated multiplicatively closed subset of D that they generate, and let T be the m-complement for S. Then $D = D_S \cap D_T$, where D_T is a UFD with principal prime ideals $\{p_\alpha D_T\}$. Every t-ideal A of D has the form $p_{\alpha_1} \cdots p_{\alpha_n} B$, where $B = AD_S \cap D$ is a t-ideal of D generated by elements of T. Moreover, the map $\mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D_S)$, given by $[A] \to [AD_S]$, is an isomorphism.*

Proof. By Proposition 2.6, D_T is a UFD. Since every t-ideal of D_T has the form $p_{\alpha_1} \cdots p_{\alpha_n} D_T$, the result concerning A follows. The last statement follows from Theorem 4.1. □

It is known ([16] or [22]) that if S is an lcm splitting set and D_S is a GCD domain, then D is a GCD domain. We extend this result to Prüfer v-multiplication domains (PVMD's). Recall that an integral domain D is a PVMD if every finite-type t-ideal of D is t-invertible, that is, $T^*(D) = TI(D)$.

Theorem 4.3. *Let S be an lcm splitting set for an integral domain D. Then D is a PVMD (respectively, GCD domain) if and only if D_S is a PVMD (respectively, GCD domain). Moreover, $\mathrm{Cl}_t(D)$ is naturally isomorphic to $\mathrm{Cl}_t(D_S)$ via the map $[A] \to [AD_S]$.*

Proof. It is well known that for any multiplicatively closed set S, if D is a PVMD (respectively, GCD domain), then D_S is a PVMD (respectively, GCD domain).

Conversely, suppose that S is lcm splitting and that T is the m-complement for S. We have an isomorphism $T^*(D) \to T^*(D_S) \times T^*(D_T)$ that takes $TI(D)$ to $TI(D_S) \times TI(D_T)$. Since D_T is a GCD domain, $T^*(D_T) = TI(D_T) = P(D_T)$. Thus D is a PVMD (respectively, GCD domain) if and only if $T^*(D_S) = TI(D_S)$ (respectively, $T^*(D_S) = P(D_S)$), that is, if and only if D_S is a PVMD (respectively, GCD domain). □

When D is a Krull domain, the natural homomorphism $\varphi : \mathrm{Cl}(D) \to \mathrm{Cl}(D_S)$ is surjective for any multiplicatively closed subset S of D; and, as we have already observed, φ is an isomorphism if and only if the saturation of S is generated by principal primes. However, in general, the natural homomorphism $\varphi : \mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D_S)$ need be neither surjective nor injective. In fact, in [8, Theorem 4.8], it was shown that for any two abelian groups G and H, there is an integral domain D and a multiplicatively closed subset S of D with $\mathrm{Cl}_t(D) = G$ and $\mathrm{Cl}_t(D_S) = H$. However, in [2, Theorem 2.3], we showed that $\varphi : \mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D_S)$ is injective when S is generated by principal primes. In [14], Gabelli and Roitman studied conditions under which φ is surjective. In particular, they showed that φ is surjective (and hence an isomorphism) when (in our terminology) S is generated by a splitting set of principal primes. They also gave an example in which S was generated by principal primes, but φ was not surjective. Note, though, that φ may be surjective, and hence an isomorphism, when S is generated by a nonsplitting set of principal primes. For example, this is the case if D is a valuation domain of dimension greater than one with principal maximal ideal $M = fD$ and $S = \{f^n\}$. For, in this case, $\mathrm{Cl}_t(D) = \mathrm{Cl}_t(D_S) = 0$. Another method for constructing such examples is given in the next paragraph. Finally, D. Nour El Abidine [11] has shown that φ is surjective when S is generated by principal primes if $[D : I]$ has finite type for each finitely generated ideal I of D.

Here is an easy way, motivated by [11], to construct examples of integral domains R with nonsplitting sets S generated by principal primes for which the natural map $\mathrm{Cl}_t(R) \to \mathrm{Cl}_t(R_S)$ is an isomorphism. Let D be an integral domain with quotient field K and S a multiplicatively closed subset of D generated by principal primes $\{p_\alpha\}$. Let $R = D + XK[X] \subsetneq K[X]$; so $R = D + M$ with $M = XK[X]$. (Alternatively, one may use $K[[X]]$ in place of $K[X]$.) Now each $p_\alpha \in D$ is also a principal prime in R since $p_\alpha R = p_\alpha D + XK[X]$ is a prime ideal of R. But in R, $\bigcap p_\alpha^n R \supseteq XK[X]$, so $\{p_\alpha\}$ is *not* a splitting set of primes for R. We have the following commutative square:

$$\begin{array}{ccc} \mathrm{Cl}_t(D) & \longrightarrow & \mathrm{Cl}_t(D+M) = \mathrm{Cl}_t(R) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \mathrm{Cl}_t(D_S) & \longrightarrow & \mathrm{Cl}_t(D_S+M) = \mathrm{Cl}_t(R_S)\,. \end{array}$$

Now each horizontal map is an isomorphism by [8, Theorem 3.12]. By [2, Theorem 2.3], α and β are always injective. Clearly α is an isomorphism if and only if β is an isomorphism. Suppose that S is a splitting set for D generated by principal primes, so $\mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D_S)$ is an isomorphism. Thus $\mathrm{Cl}_t(R) \to \mathrm{Cl}_t(R_S)$ is an isomorphism, too. Let $\bar{S}$ be the saturation of S in R. Then $\bar{S}$ is generated by a set of *non*splitting primes and $\mathrm{Cl}_t(\mathrm{R}) \to \mathrm{Cl}_t(R_{\bar{S}})$ is an isomorphism. Note that $\mathrm{Cl}_t(R)$ can be chosen to be any abelian group.

A Krull domain D is of course characterized by the property that $D = \bigcap D_P$, where this intersection (running over the height-one prime ideals) has finite character, and each D_P is a DVR. Let us call a domain D *weakly Krull* if $D = \bigcap \{D_P \mid \mathrm{ht}\, P = 1\}$, where the intersection has finite character. Weakly Krull domains, although not called such, were studied in [5]. An integral domain D is said to be *weakly factorial* [4] if every nonzero nonunit of D is a product of primary elements. It is known [6] that D is weakly factorial if and only if D is weakly Krull and $\mathrm{Cl}_t(D) = 0$. A (weakly) Krull domain is said to be *almost* (*weakly*) *factorial* if some power of each element is a product of primary elements, or equivalently, if $\mathrm{Cl}_t(D)$ is torsion ([5]). Finally, an integral domain is a *Mori domain* if it has ACC on integral v-ideals, or equivalently, on integral t-ideals. The next theorem gives a sampling of some Nagata-type theorems that may be obtained.

Theorem 4.4. *Let S be a saturated multiplicatively closed set generated by a set of splitting principal primes. If D_S is weakly Krull* (*respectively*; *weakly factorial, almost weakly factorial, Krull, almost factorial Krull, factorial, Mori*), *then so is D.*

Proof. Let T be the m-complement for S. Then D_T is a UFD and $D = D_S \cap D_T$. Thus if D_S is weakly Krull, Krull, or Mori, then so is D. Moreover, $\mathrm{Cl}_t(D) \cong \mathrm{Cl}_t(D_S)$. Hence $\mathrm{Cl}_t(D) = 0$ (respectively, is torsion) if and only if $\mathrm{Cl}_t(D_S) = 0$ (respectively, is torsion). Thus if D_S is weakly factorial, then D_S is weakly Krull and $\mathrm{Cl}_t(D_S) = 0$. Thus D is weakly Krull and $\mathrm{Cl}_t(D) = 0$, so D is weakly factorial. The other statements follow in a similar fashion. □

The case in Theorem 4.4 where D_S is a Mori domain has been given by Roitman [24, Theorem 5.1].

Example 4.5. Let D be an integral domain with quotient field K and let X be an indeterminate over D. Then $S = \{uX^n \mid u \in U(D),\ n \geq 0\}$ is the lcm splitting set generated by the principal prime X in $D[X]$ and its m-complement is $T = \{f(X) \in D[X] \mid f(0) \neq 0\}$. Here $D[X]_S = D[X, X^{-1}]$, and $D[X]_T = K[X]_{(X)}$ is a DVR. By Theorem 4.2 the map $\mathrm{Cl}_t(D[X]) \to \mathrm{Cl}_t(D[X, X^{-1}])$ given by $[A] \to [AD[X, X^{-1}]]$ is an isomorphism. The map $\mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D[X])$ given by $[A] \to [AD[X]]$ is an isomorphism if and only if D is integrally closed [13, Theorem 3.6]. Thus the map $\mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D[X, X^{-1}])$ is an isomorphism if and only if D is integrally closed. It is interesting to contrast this to the situation for Pic( ). If D is integrally closed, then the natural map $\psi : \mathrm{Pic}(D) \to \mathrm{Pic}(D[X, X^{-1}])$ is also an isomorphism. Although the natural map $\varphi : \mathrm{Pic}(D[X]) \to \mathrm{Pic}(D[X, X^{-1}])$ is always injective, unlike the case for the t-class group, φ need not be surjective. In fact, φ is surjective if and only if ψ is surjective (and hence an isomorphism). We recall that the natural map $\mathrm{Pic}(D) \to \mathrm{Pic}(D[X])$ is an isomorphism if and only if D is seminormal and ψ is an isomorphism if and only if D is quasinormal, and that a seminormal integral domain need not be quasinormal [23].

Similar results hold for $D[[X]]$, where X is a power series indeterminate over D. Then $D[[X]]_S = D[[X]][X^{-1}]$, where now $S = \{uX^n \mid u \in U(D[[X]]), n \geq 0\}$, and $D[[X]]_T = K[[X]] \cap L$, where L is the quotient field of $D[[X]]$, is a DVR.

Example 4.5 admits a generalization to semigroup rings which we state as our next theorem.

Theorem 4.6. *Let $(G, \leq)$ be a lattice ordered abelian group and let Γ be a l-submonoid of G with $G_+ \subseteq \Gamma$. Then for any integral domain D, $S = \{uX^g \mid u \in U(D), g \in \Gamma\}$ is an lcm splitting set for the monoid domain $D[X; \Gamma]$. Hence the map $\mathrm{Cl}_t(D[X; \Gamma]) \to \mathrm{Cl}_t(D[X; G])$, given by $[A] \to [AD[X; G]]$, is an isomorphism. In particular, the map $\mathrm{Cl}_t(D[X; G_+]) \to \mathrm{Cl}_t(D[X; G])$ is an isomorphism.*

Proof. Let K be the quotient field of D. By the proof of [15, Theorem 18.6], the map $w : K[X; G] \to G \cup \{\infty\}$ given by $w(0) = \infty$ and $w(\sum_{i=1}^{n} a_i X^{g_i}) = \inf\{g_i\}$ (where each $a_i \neq 0$ and the g_i's are distinct) extends to a semi-valuation on the quotient field of $K[X; G]$. Moreover, $w^{-1}(G_+) \cup \{0\}$ is a Bézout domain.

Clearly S is a saturated multiplicatively closed subset of $D[X; \Gamma]$. Let

$$T = \{X^g(r_1 X^{g_1} + \cdots + r_n X^{g_n}) \mid g \in \Gamma \cap -\Gamma,$$
$$g_i \in \Gamma \text{ are distinct with } r_i \neq 0 \text{ and } \inf\{g_1, \ldots, g_n\} = 0\}\,.$$

Alternatively, $T = \{f \in D[X; \Gamma] \mid w(f) \in \Gamma \cap -\Gamma\}$. This shows that T is also a saturated multiplicatively closed subset of $D[X; \Gamma]$.

Let $f = \sum_{i=1}^{n} a_i X^{g_i} \in D[X; \Gamma]$ and let $\alpha = \inf\{g_i\}$. Then $\alpha \leq g_i$, so $g_i - \alpha \geq 0$, and hence $g_i - \alpha \in G_+ \subseteq \Gamma$. Put $h_i = g_i - \alpha$, so $h_i \in \Gamma$ and $\inf\{h_i\} = 0$. Then $f = X^\alpha h$, where $h = \sum_{i=1}^{n} a_i X^{h_i}$. Now since Γ is an l-submonoid, $\alpha \in \Gamma$, so $X^\alpha \in S$ and $h \in T$. Moreover, this representation is unique up to units. For if $uX^g f_1 = u'X^{g'} f'_1$, where $u, u' \in U(D)$, $g, g' \in \Gamma$, and $w(f_1), w(f'_1) \in \Gamma \cap -\Gamma$, then $g + w(f_1) = g' + w(f'_1)$, so $g = g' + (w(f'_1) - w(f_1))$, where $w(f'_1) - w(f_1) \in \Gamma \cap -\Gamma$. Thus uX^g and $u'X^{g'}$ differ by a unit factor from $D[X; \Gamma]$.

Now $D[X; \Gamma]_S = D[X; G]$. It remains to show that $D[X; \Gamma]_T$ is a GCD domain. We claim that $D[X; \Gamma]_T \supseteq w^{-1}(G_+) \cup \{0\}$. This will show that $D[X; \Gamma]_T$ is a Bézout domain since $w^{-1}(G_+) \cup \{0\}$ is a Bézout domain. Now a typical element of $w^{-1}(G_+)$ may be written in the form

$$\frac{f}{g} = \frac{r_1X^{g_1} + \cdots + r_nX^{g_n}}{s_1X^{h_1} + \cdots + s_mX^{h_m}} = \frac{X^{g_0}f_1}{X^{h_0}g_1},$$

where each $r_i, s_i \in D - \{0\}$, each $g_i, h_i \in G_+ \subseteq \Gamma$, and $\inf\{g_i\} = w(f) \geq w(g) = \inf\{h_i\}$. So $f = X^{g_0}f_1$ and $g = X^{h_0}g_1$, where $g_0 = w(f)$ and $h_0 = w(g)$. Now $g_1 \in T$ and $g_0 = w(f) \geq w(g) = h_0$ implies that $g_0 - h_0 \geq 0$ and hence $g_0 - h_0 \in \Gamma$. So

$$\frac{f}{g} = \frac{X^{g_0 - h_0}f_1}{g_1} \in D[X; \Gamma]_T .$$

Hence $w^{-1}(G_+) \cup \{0\} \subseteq D[X; \Gamma]_T$. □

For example, we may take G to be a cardinal product of copies of $\mathbb{Z}$ and Γ to be the positive cone of G. This gives that $\mathrm{Cl}_t(D[\{X_i\}]) \to \mathrm{Cl}_t(D[\{X_i, X_i^{-1}\}])$ is an isomorphism. Or suppose that we take $G = \mathbb{Z} \oplus_L \mathbb{Z}$, the lexicographic direct sum, and $\Gamma = G_+$. Then $D[X; \Gamma] \cong D[X, Y, \{X/Y^n\}_{n=1}^{\infty}]$. So

$$\mathrm{Cl}_t(D[X, Y, \{X/Y^n\}_{n=1}^{\infty}]) \cong \mathrm{Cl}_t(D[X, Y, X^{-1}, Y^{-1}]) \cong \mathrm{Cl}_t(D[X, Y]) .$$

Note that if $\leq$ is a total order on G, then any submonoid $\Gamma \supseteq G_+$ is an l-submonoid of G. In particular, for any submonoid Γ of $\mathbb{Z} \oplus_L \mathbb{Z}$ with $\Gamma \supseteq (\mathbb{Z} \oplus_L \mathbb{Z})_+$, we have $\mathrm{Cl}_t(D[X; \Gamma]) \cong \mathrm{Cl}_t(D[X, Y])$.

Example 4.7. Let D be a GCD domain with quotient field K. For $f(X) \in D[X]$, $C(f)$, the content of f, is the ideal of D generated by the coefficients of f. It is easily seen that the set $S = D - \{0\}$ is a lcm splitting set in $D[X]$ with m-complement $T = \{f(X) \in D[X] \mid C(f)_v = D\}$. Here $D[X]_S = K[X]$ is a PID, and $D[X]_T = D^v$, the Kronecker function ring for v, which is a Bézout domain. We have $T(D[X]) \cong T(K[X]) \times_c T(D[X]_T)$. This gives yet another proof that if D is a GCD domain, then so is $D[X]$. Actually, since D is integrally closed, we have

$$\begin{aligned} T(D)/P(D) &\cong T(D[X])/P(D[X]) \\ &\cong T(K[X])/P(K[X]) \times T(D[X]_T)/P(D[X]_T) \\ &\cong T(D[X]_T)/P(D[X]_T) . \end{aligned}$$

If D is a pseudo-principal domain (every v-ideal is principal), then $S = \{df \mid d \in D - \{0\},\ f \in U(D[[X]])\}$ is a splitting multiplicatively closed set in $D[[X]]$, with m-complement $T = \{f(X) = \sum_{i=0}^{\infty} a_iX^i \in D[[X]] \mid (\{a_i\})_v = D\}$. Here D, $D[[X]]$, $D[[X]]_S$, and $D[[X]]_T$ are not so easy to describe or relate. For example, it is well known that D may be a UFD while $D[[X]]$ need not be a UFD [12].

Example 4.5 admits another generalization which may be used to give examples of splitting sets S with neither S nor its m-complement T an lcm splitting set.

Example 4.8. Let $A \subseteq B$ be a pair of integral domains. We say that this pair satisfies (*) if (1) for $0 \neq b \in B$, $b = au$ where $a \in A$ and $u \in U(B)$ and (2) $b = au = a'u'$ $(a, a' \in A,\ u, u' \in U(B))$ implies $u/u' \in U(A)$. Note that the pair $A \subseteq B$ satisfies (*) precisely when the map $P_+(A) \to P_+(B)$ given by $Ax \to Bx$ is an isomorphism or, equivalently, when $P(A) \to P(B)$ is an *order-isomorphism*. The following pairs of domains satisfy (*):

(1) $A \subseteq A$, (2) $K \subseteq L$, where K and L are fields (note that if $A \subseteq B$ satisfies (*) and if A or B is a field, then so is the other), (3) $A \subseteq \hat{A}$, $\hat{A}$ is the completion of A, where A is a quasi-complete local domain (that is, the map $J \to J\hat{A}$ is a lattice isomorphism), and (4) $A \subseteq A(\{Y_\alpha\}) = \{f/g \mid f, g \in A[\{Y_\alpha\}],\ C(g) = A\}$, where A is Bézout.

Let $D = A + XB[X]$. Let $S = \{uX^n \mid u \in U(A)$ for $n = 0$ and $u \in U(B)$ for $n \geq 1\}$. Then S is a saturated multiplicatively closed subset of D. In fact, S is the saturation of $\{X^n\}$. Let $T = \{f(X) \in D \mid f(0) \neq 0\}$. Clearly T is a saturated multiplicatively closed set. Now by (1) of (*), $ST = D - \{0\}$. By (2) of (*) this representation is unique up to unit factors. Hence S is a splitting multiplicatively closed set with m-complement T.

Here $D_S = B[X, X^{-1}]$ and $D_T = (K + XL[X])_{T'}$, where $T' = \{f(X) \in K + XL[X] \mid f(0) \neq 0\}$ (K (respectively, L) is the quotient field of A (respectively, B)). Note that D_T is atomic with irreducible elements uX, where $u \in U(B)$. In fact, D_T has ACC on principal ideals. Thus S is lcm splitting $\Leftrightarrow D_T$ is a GCD domain $\Leftrightarrow D_T$ is a UFD $\Leftrightarrow$ S is generated by principal primes $\Leftrightarrow XD$ is prime $\Leftrightarrow A = B$. In this case, $K = L$, so $D_T = K[X]_{T'} = K[X]_{(X)}$ is a DVR. T is lcm splitting $\Leftrightarrow D_S = B[X, X^{-1}]$ is a GCD domain $\Leftrightarrow B$ is a GCD domain $\Leftrightarrow A$ is a GCD domain; while T is generated by principal primes $\Leftrightarrow D_S = B[X, X^{-1}]$ is a UFD $\Leftrightarrow B$ is a UFD $\Leftrightarrow A$ is a UFD.

Thus neither S nor T is lcm splitting if and only if $A \subsetneq B$ and B is not a GCD domain. Thus if A is a quasi-complete local domain that is not complete and not a UFD, then in $D = A + X\hat{A}[X]$ we get a multiplicatively closed set S such that neither S nor its m-complement is lcm splitting.

If D is weakly factorial, then every saturated multiplicatively closed subset of D is a splitting set, in fact, this property characterizes weakly factorial domains [6, Theorem]. Moreover, if D is weakly factorial, then the finite-character representation $D = \bigcap \{D_P \mid \operatorname{ht} P = 1\}$ gives rise to the order-isomorphism $G(D) \to \bigoplus_c G(D_P)$. We next show that this isomorphism may be extended to $T(D)$.

Theorem 4.9. *Let D be weakly factorial. Then the natural map $T(D) \to \bigoplus_c \{T(D_P) \mid \operatorname{ht} P = 1\}$, given by $A \to (AD_P)_{\operatorname{ht} P=1}$, is an order-isomorphism.*

Proof. Since D is weakly factorial, $S = D - P$ is a splitting multiplicatively closed set. This gives rise to the order-preserving homomorphism $T(D) \to T(D_P)$, given by $A \to AD_P$. Since $D = \bigcap \{D_P \mid \mathrm{ht}\, P = 1\}$ has finite character, we get an order-preserving homomorphism $\theta : T(D) \to \bigoplus_c \{T(D_P) \mid \mathrm{ht}\, P = 1\}$, given by $\theta(A) = (AD_P)_{\mathrm{ht}\, P=1}$. Note that if $\theta(A) \leq \theta(B)$, then each $BD_P \subseteq AD_P$, so $B = \bigcap BD_P \subseteq \bigcap AD_P = A$; hence $A \leq B$. Thus θ is injective with $A \leq B \Leftrightarrow \theta(A) \leq \theta(B)$. It remains to show that θ is surjective. It suffices to show that for each height-one prime ideal P_0 of D and each $E \in T(D_{P_0})$, there is an $A \in T(D)$ with $AD_{P_0} = E$ and $AD_Q = D_Q$ for each height-one prime ideal Q different from P_0. Write $E = e^{-1}E'$, where $0 \neq e \in D$ and E' is an integral D_{P_0}-ideal. Let $A' = E' \cap D$, so A' is an integral t-ideal with $E' = A'D_{P_0}$. Moreover, for $Q \neq P_0$, $A'D_Q = D_Q$. For, if $E' = D_{P_0}$, then $A' = D$; while if $E' \subsetneq D_{P_0}$, then A' is P_0-primary, so $A'D_Q = D_Q$. From the previously mentioned isomorphism $G(D) \cong \bigoplus \{G(D_P) \mid \mathrm{ht}\, P = 1\}$, we can choose $0 \neq f \in D$ with $fD_{P_0} = eD_{P_0}$ and $fD_Q = D_Q$ for $Q \neq P_0$. Then $\frac{1}{f}A'D_{P_0} = E$, while $\frac{1}{f}A'D_Q = D_Q$ for $Q \neq P_0$. □

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