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ON SUPER v-DOMAINS

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Abstract. An integral domain D, with quotient field K, is a v-domain if for each nonzero finitely generated ideal A of D we have $(AA^{-1})^{-1} = D$. It is well known that if D is a v-domain, then some quotient ring D_S of D may not be a v-domain. Calling D a super v-domain if every quotient ring of D is a v-domain we characterize super v-domains as locally v-domains. Using techniques from factorization theory we show that D is a super v-domain if and only if $D[X]$ is a super v-domain if and only if $D + XK[X]$ is a super v-domain and give new examples of super v-domains that are strictly between v-domains and P-domains that were studied in [Manuscripta Math. 35(1981)1-26]

An integral domain D, with quotient field K, is called a v-domain if for every finitely generated nonzero ideal A of D, A is v-invertible, i.e., we have $(AA^{-1})^{-1} = D$ or equivalently $(AA^{-1})_v = D$. Now v-domains, the oldest known notion in, multiplicative ideal theory, according to [9], come defined in various ways. They are called $*$-Prufer if for some star operation, every finitely generated nonzero ideal A is $*$-invertible, i.e., we have $(AA^{-1})^* = D$ (see, e.g., [2]). As is apparent from the above definitions, v-domains are modeled after Prufer domains. Mimicking the proof, for Prufer domain, by Prufer himself, it was shown in [16] that D is a v-domain if and only if every two generated nonzero ideal of D is v-invertible (see also an earlier paper by Gabelli [11] that hints at the possibility.) Call D essential if D has a family $\mathcal{F}$ of prime ideals such that D_P is a valuation domain for each $P \in \mathcal{F}$ and $D = \cap_{P\in\mathcal{F}} D_P$. As indicated in [9] an essential domain is a v-domain and so is the so-called "P-domain". A P-domain here is an essential domain, each of whose quotient rings is essential. The P-domains were initially studied in [17]. It turns out, however, that if D is a v-domain and A a nonzero finitely generated ideal of D, A^{-1} may not even be close to being finitely generated, completely unlike Prufer domains. Also, every quotient ring of a Prufer domain is Prufer. Yet using an example of Heinzer's, [15], of an essential domain with a non-essential quotient ring that cannot be a v-domain, one can show that if D is a v-domain and S a multiplicative set of D, then D_S need not be a v-domain, see section 3 of [9] for a discussion on this example. This raises the questions: (a) if D is a v-domain, under what conditions on a multiplicative set S, or on D, can we be sure that D_S is a v-domain? (b) what are the v-domains whose quotient rings are also v-domains and that are not any of the known examples of v-domains all of whose quotient rings are v-domains and (c) if every proper quotient ring of D is a v-domain, must D be a v-domain? The purpose of this note is to start a discussion on these questions.

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For our part we characterize super v-domains i.e. domains whose quotient rings are all v-domains and discuss some conditions that will ensure that a quotient ring of a v-domain is a v-domain. We show for instance that if D is a v-domain and S is a splitting or a t-splitting set of D then D_S is a v-domain. Using some of these results we give an example schema for super v-domains, showing that these super v-domains are strictly between v-domains and P-domains, we show that D is a super v-domain if and only if $D+XK[X]$ is a super v-domain. We also show that if X is an indeterminate over D, then D is a super v-domain if and only if $D[X]$ is. (The answer to question (c) is that for a one dimensional quasi local domain D a proper quotient ring is the field of fractions of D and hence a v-domain. But a one dimensional quasi local domain need not be a v-domain.)

It seems pertinent to let the reader in on the terminology that we have used above and that we are going to use when we prove our results. Let D be an integral domain with quotient field K and let $F(D)$ be the set of nonzero fractional ideals of D. A star operation is a function $A \mapsto A^*$ on $F(D)$ with the following properties:

If $A, B \in F(D)$ and $a \in K\backslash\{0\}$, then

(i) $(a)^* = (a)$ and $(aA)^* = aA^*$.

(ii) $A \subseteq A^*$ and if $A \subseteq B$, then $A^* \subseteq B^*$.

(iii) $(A^*)^* = A^*$.

We may call A^* the $*$-image (or $*$-envelope) of A. An ideal A is said to be a $*$-*ideal* if $A^* = A$. Thus A^* is a $*$-ideal (by (iii)). Moreover (by (i)) every principal fractional ideal, including $D = (1)$, is a $*$- ideal for any star operation $*$.

For all $A, B \in F(D)$ and for each star operation $*$, $(AB)^* = (A^*B)^* = (A^*B^*)^*$. These equations define what is called $*$-multiplication *(or* $*$-product).

Define $A_v = (A^{-1})^{-1}$ and $A_t = \bigcup\{J_v | \ 0 \neq J$ is a finitely generated subideal of $A\}$. The functions $A \mapsto A_v$ and $A \mapsto A_t$ on $F(D)$ are more familiar examples of star operations defined on an integral domain. A v-ideal is better known as a divisorial ideal. The identity function d on $F(D)$, defined by $A \mapsto A$ is another example of a star operation. There are of course many more star operations that can be defined on an integral domain D. But for any star operation $*$ and for any $A \in F(D)$, $A^* \subseteq A_v$. Some other useful relations are: For any $A \in F(D)$, $(A^{-1})^* = A^{-1} = (A^*)^{-1}$ and so, $(A_v)^* = A_v = (A^*)_v$. Using the definition of the t-operation one can show that an ideal that is maximal w.r.t. being a proper integral t-ideal is a prime ideal of D, each ideal A of D with $A_t \neq D$ is contained in a maximal t-ideal of D and $D = \cap D_M$, where M ranges over maximal t-ideals of D. For more on v- and t-operations the reader may consult sections 32 and 34 of Gilmer [12]. Our terminology essentially comes from [12].

Call a multiplicative set S of D a splitting set if S is saturated and for each $d \in D\backslash\{0\}$ we can write $d = d's$ where $s \in S$ and $d' \in D$ such that $(d', t)_v = D$ for all $t \in S$. For more on splitting sets look up [3]. On the other hand a multiplicative set S of D is a t-splitting set if for all $d \in D\backslash\{0\}$ we can write $dD = (AB)_t$ where $B_t \cap S \neq \phi$ and $(A, s)_v = D$ for all $s \in S$. The t-splitting sets were introduced and applied in [4].

Let's call D a super v-domain if every quotient ring of D is a v-domain. Let us be clear about what we are looking for, when we study "super v-domains" as there do exist super v-domains in the form of the P-domains and Prufer domains and the so-called Prufer v-Multiplication domains or PVMDs. PVMDs, by the way, are v-domains such that $aD \cap bD = A_v$ for some finitely generated ideal A, for all

$a, b \in D\backslash\{0\}$, [17]. According to [17] a PVMD is a P-domain. In our study of super v-domains we are looking for v-domains D that are not P-domains yet have the property that D_S is a v-domain for each multiplicative set S of D. In other words we are looking for v-domains D that lie strictly between v-domains and P-domains, with the property that every quotient ring of D is a v-domain.

The first thing that seems to prevent a v-domain from having a quotient ring that is a v-domain seems to be that while for a nonzero finitely generated ideal I we have $(ID_S)^{-1} = I^{-1}D_S$ we have no such general formula for a nonzero ideal I. One way of dealing with a situation like this is to bring in a new definition. Call a quotient ring D_S of D super extending if for each nonzero ideal I of D we have $(ID_S)^{-1} = I^{-1}D_S$. An immediate consequence is that if D_S is super extending, then $(ID_S)_v = I_vD$.

Lemma 0.1. *If D_S is super extending and D is a v-domain, then D_S is a v-domain.*

Proof. Let $\alpha, \beta \in D_S$. Then $\alpha = \frac{a}{s}, \beta = \frac{b}{t}$ for some $a, b \in D$ and $s, t \in S$ and $(\alpha, \beta)D_S((\alpha, \beta)D_S)^{-1} = (a, b)D_S((a, b)D_S)^{-1} = ((a, b)(a, b)^{-1})D_S$. Now as D_S is super extending we conclude that $((\alpha, \beta)D_S((\alpha, \beta)D_S)^{-1})^{-1} = (((a, b)(a, b)^{-1})D_S)^{-1} = (((a, b)(a, b)^{-1}))^{-1}D_S = D_S$ because in D we have $(((a, b)(a, b)^{-1}))^{-1} = D$. □

But the drawback of Lemma 0.1 is that if D_S happens to be such that $(a, b)^{-1}D_S$ is a finitely generated ideal of D_S for each pair a, b of D, then Lemma 0.1 would be an overkill. Though D_S would have to be a stronger form of a PVMD. All this beside, super extending is too much even for our needs. So let's call D_S simple extending if $(((a, b)(a, b)^{-1})D_S)^{-1} = (((a, b)(a, b)^{-1}))^{-1}D_S$. We do seem to have disadvantages of super extending when working with simple extending and simple extending is sort of too obvious a ploy, but it may work in some interesting ways neatly.

Proposition 1. *Let D be an integral domain and let $\{S_\alpha\}$ be a family of multiplicative sets of D such that $D = \cap D_{S_\alpha}$. If, for each $\alpha \in I$, D_{S_α} is a simple extending quotient ring of D and a v-domain, then D is a v-domain.*

Proof. Note that, as the inverse of an ideal is divisorial, we have $(((a, b)(a, b)^{-1}))^{-1} = \cap(((a, b)(a, b)^{-1}))^{-1}D_{S_\alpha} = \cap((((a, b)(a, b)^{-1}))D_{S_\alpha})^{-1} = \cap D_{S_\alpha} = D$. □

But there is a better result available on the market in the form of Proposition 3.1 of [9]. This result says.

Proposition 2. *Let $\{D_\lambda | \lambda \in \Lambda\}$ be a family of flat overrings of D such that $D = \cap_{\lambda\in\Lambda} D_\lambda$. If each of D_λ is a v-domain, then so is D.*

Let us recall that a prime ideal P is called an associated prime of a principal ideal (a) if P is minimal over an ideal of the form $0 \neq (a) : (b) = \{r \in D | rb \in (a)\} \neq D$. Associated primes of principal ideals, or simply associated primes, of D have been studied by quite a few authors, but our reference in this regard is [5]. According to Proposition 4 of [5], if S is a multiplicative set of D and $\{P_\alpha\}$ is the family of associated primes of principal ideals of D disjoint from S, then $D_S = \cap_\alpha D_{P_\alpha}$.

With Proposition 2 at hand, we can state and prove the following characterization of super v-domains.

Theorem 0.2. ([9, Proposition 3.4])*The following are equivalent for an integral domain D. (1) D_S is a v-domain for every multiplicative set S of D, (2) D_P is* *a v-domain for every prime ideal P of D and (3) D_P is a v-domain for every associated prime P of D.*

Proof. That (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is obvious. For (3) $\Rightarrow$ (1), let S be a multiplicative set of D and let $\mathcal{F} = \{P_\alpha\}$ be the family of associated primes disjoint from S. Then by (3) each of D_{P_α} is a v-domain and by [5, Proposition 4] $D_S = \cap_{P_\alpha \in \mathcal{F}} D_{P_\alpha}$. Thus by Proposition 2, D_S is a v-domain. □

There is, however, a situation in which D_S is a v-domain, whenever D is. That is when the multiplicative set S in D is a splitting set. If S is a splitting set, the set $T = \{t \in D|\ (t,s)_v = D$ for all $s \in S\}$ often denoted as $S^\perp$ is called the m-complement of S. Indeed if S is a splitting set and $T = S^\perp$, then $D = D_S \cap D_T$ and $dD_S \cap D = tD$ where $t \in T$ such that $d = ts$ for some $s \in S$. A splitting set S of D is an lcm splitting set if $sD \cap xD$ is principal for all $s \in S$ and for all $x \in D \backslash \{0\}$.

Theorem 0.3. *Let S be a splitting multiplicative set of D and let $T = S^\perp$. If D is a v-domain, then so is D_S. Moreover if S is an lcm splitting set then D_S is a v-domain if and only if D is a v-domain.*

Proof. Suppose that D_S is not a v-domain. That is, there is a pair a, b of D_S such that $(((a,b)(a,b)^{-1})\ D_S)_v \neq D_S$. Since $(r,s)^{-1}D_S = ((r,s)D_S)^{-1}$, for $r, s \in D\backslash\{0\}$, we can take $a, b \in D$ and regard $(a,b)(a,b)^{-1}$ as an ideal of D. Since $(((a,b)(a,b)^{-1})\ D_S)_v \neq D_S$, $(a,b)(a,b)^{-1} \cap S = \phi$. Again since $(((a,b)(a,b)^{-1})\ D_S)_v \neq D_S$ there exist $x, y \in D_S$ such that $((a,b)(a,b)^{-1})\ D_S \subseteq \frac{x}{y} D_S$ where $x \nmid y$ in D_S. As S is a splitting set, we can take $x, y \in T$. But then $y((a,b)(a,b)^{-1})\ D_S \subseteq xD_S$ and $y((a,b)(a,b)^{-1}) \subseteq y((a,b)(a,b)^{-1})\ D_S \cap D \subseteq xD_S \cap D$. As $x \in T$, we have $xD_S \cap D = xD$ ([3], Theorem 2.2). Thus we have $y((a,b)(a,b)^{-1}) \subseteq xD$. Applying the v-operation throughout and noting that D is a v-domain we conclude that $yD \subseteq xD$. But then $yD_S \subseteq xD_S$, a contradiction. Whence D_S is a v-domain. For the moreover part note that $D = D_S \cap D_T$ where D_T is a GCD domain, by Theorem 2.4 of [3]. Thus if S is lcm splitting D_S is a v-domain and so is D_T, being a GCD domain, forcing $D = D_S \cap D_T$ to be a v-domain, by Proposition 2. □

Theorem 0.4. *Let D be an integral domain with quotient field K and let X be an indeterminate over D. Then D is a super v-domain if and only if $D + XK[X]$ is a super v-domain.*

Proof. Let D be a super v-domain. Then by Theorem 4.42 of [7] $T = D + XK[X]$ is a v-domain. Also by Proposition 2.2 of [8], every overring S, and hence every quotient ring S, of T is a quotient ring of $S \cap K + XK[X]$. According to the proof of Proposition 2.2 of [8] the elements of S are of the form $\frac{\alpha + Xf(X)}{1 + Xg(X)}$ where $\alpha \in S \cap L$. Let $U = \{u \in D | u$ is a unit in $S\}$. Then $D_U \subseteq S \cap K$. Let $h \in S$. Then $h = \frac{a + Xf(X)}{b + Xg(X)}$ where, $a, b \in D$ and, $b + Xg(X)$ is a unit in S. This gives $b = b(1 + \frac{X}{b}g(X)(1 + \frac{X}{b}g(X)^{-1}$ and so b is a unit in $S \cap K$, whence $b \in U$. But then $a/b = h(0) \in D_U$. Noting that $h(0) \in S \cap K$ we conclude that $D_U = S \cap K$. This leads to the conclusion that S is a quotient ring of $D_U + XK[X]$. Since D is a super v-domain D_U is a v-domain and so is $D_U + XK[X]$. Next, by the proof of Proposition 2.2 of [8], denoting by $U(S)$ the set of units of S we have $U(S) = \{f \in D_U + XK[X] | f = u + Xg(X)$, where u is a unit in $D_U\}$ and as elements of the form $1 + Xg(X)$ are finite products of height one primes of $D_U + XK[X]$ ([7], Theorem 4.21) we conclude that $U(S)$ is a splitting set generated by primes.

But then, by Theorem 0.3, $S = (D_U + XK[X])_{U(S)}$ is a v-domain. For the converse note that if T is a multiplicative set in D, then $(D + XK[X])_T = D_T + XK[X]$ which is a v-domain if and only if D_T is a v-domain. Thus if $D + XK[X]$ is a super v-domain, then so is D. □

Some super v-domains such as the P-domains have the property that D_P is a valuation domain for every associated prime of a principal ideal of D. Now if P is an associated prime of a principal ideal, one can easily show that D_P is t-local, i.e., PD_P is a t-ideal [9]. This may lead one to ask if a t-local super v-domain is close to a valuation domain. The answer is: Close but not too close, as there does exist a one dimensional completely integrally closed integral domain $\mathcal{N}$, due to Nagata [18] and [19], that is not a valuation domain and a one dimensional quasi local domain is t-local. (Of course a completely integrally closed domain is a v-domain.) Now, trivially, $\mathcal{N}$ has the property that every quotient ring of $\mathcal{N}$ is $\mathcal{N}$ or $qf(\mathcal{N})$. Thus, albeit trivially, $\mathcal{N}$ serves as an example of a super v-domain. This gives us the following example.

Example 0.5. Let F be the quotient field of $\mathcal{N}$ and let X be an indeterminate on F. Then $\mathcal{N}{+}XF[X]$ is a super v-domain.

Illustration: By Theorem 0.4, every quotient ring S of $\mathcal{N}{+}XF[X]$ is a quotient ring $(\mathcal{N}{+}XF[X])_U$ of $\mathcal{N}{+}XF[X]$, by a multiplicative set U generated by elements of the form $1 + Xg(X)$, or a quotient ring of $F[X]$. Since $\mathcal{N}{+}XF[X]$ is a v domain and elements of the form $1 + Xg(X)$ being products of height one primes, U is a splitting set and by Theorem 0.3, $(\mathcal{N}{+}XF[X])_U$ is a v-domain. Also since $F[X]$ is a PID every quotient ring of $F[X]$ is a PID and hence a v-domain. So, every quotient ring of $\mathcal{N}{+}XF[X]$ is indeed a v-domain.

Indeed $\mathcal{N}{+}XF[X]$ provides a "non-trivial" example of a super v-domain and Theorem 0.4 provides a scheme for producing super v-domains of any Krull dimension.

Next call a domain D a v-local domain if D is quasi local such that the maximal ideal M of D is divisorial. Of course, the situation can drastically change if we relax "t-local" to "v-local".

Proposition 3. *An integral domain D is a v-local v-domain if and only if D is a valuation domain with maximal ideal M principal.*

Proof. Let D be a v-local v-domain and let A be a nonzero finitely generated ideal of D. Then $AA^{-1} = D$. For if $AA^{-1} \neq D$ we must have $AA^{-1} \subseteq M$. But as M is a v-ideal and D a v-domain we have $D = (AA^{-1})_v \subseteq M_v = M$ a contradiction. Whence every nonzero finitely generated ideal of D is invertible and hence principal, because D is v-local and hence quasi local. Thus D is a valuation domain. Now the maximal ideal being divisorial means $M_v \neq D$ which means that there is a pair of elements a, b of D such that $M \subseteq (a/b)D$ where $a \nmid b$. Since $a \nmid b$ and D is a valuation domain $M \subseteq (a/d)D$ a principal ideal of D. But then M is principal because M is the maximal ideal. The converse is obvious. □

Let's recall from Griffin [13, Theorem 5] that D is a PVMD if and only if for every finitely generated nonzero ideal I of D we have $(II^{-1})_t = D$ if and only if D_P is a valuation ring for every maximal t-ideal of D.

Corollary 1. *Let D be locally a v-domain. Suppose that for every maximal t-ideal M of D we have MD_M divisorial then D is a PVMD.*

Proof. For every maximal t-ideal M we have D_M a v-domain and MD_M a divisorial ideal. Then by Proposition 3 we have that D_M is a valuation domain with maximal ideal principal.

Alternative proof: Let J be a nonzero ideal of D. We claim that JJ^{-1} is not in any maximal t-ideal of D. For if $JJ^{-1} \subseteq M$. Then $(JJ^{-1})D_M = JD_M J^{-1}D_M = JD_M(JD_M)^{-1} \subseteq MD_M$. Since D_M is a v-domain, $D_M = ((JD_M(JD_M)^{-1})_v$. Yet as MD_M is divisorial and $JD_M J^{-1}D_M = JD_M(JD_M)^{-1} \subseteq MD_M$ we get $D_M = ((JD_M(JD_M)^{-1})_v \subseteq MD_M$ a contradiction. Now JJ^{-1} not being in any maximal t-ideals means that $(JJ^{-1})_t = D$. Thus every nonzero finitely generated ideal of D is t-ivertible and this is another characteristic property of PVMDs. □

Recall that a prime ideal P of a domain D is called strongly prime if $x, y \in K$ and $xy \in P$ imply that $x \in P$ or $y \in P$. According to [14], D is a pseudo valuation domain PVD if every prime ideal of D is strongly prime. It turns out that a PVD is a valuation domain or a quasi local domain (D, M) such that $M^{-1} = V$ a valuation ring. This makes the maximal ideal of a non-valuation PVD a divisorial ideal.

Corollary 2. *In a non-valuation PVD D, every v-invertible ideal is principal. Consequently a non-valuation PVD can never be a v-domain.*

Proof. Suppose that a non-valuation PVD D is a v-domain. Then D is a v-local v-domain and hence a valuation domain by Proposition 3, a contradiction. □

Remark 0.6. Using the fact that the set of prime ideals in a PVD is linearly ordered it is shown in [14] that a GCD PVD is a valuation domain. However a non-valuation PVD D can never be a GCD domain, because a GCD domain is a v-domain. We can also say that a non-valuation PVD can never be a PVMD, because a PVMD is a v-domain as well.

Let S be a multiplicative set of D. Following [4] we say that $d \in D\backslash\{0\}$ is t-split by S if there are two integral ideals A, B of D such that $(d) = (AB)_t$ where $B_t \cap S \neq \phi$ and $(A, s)_t = D$ for all $s \in S$. As in [4] we call S a t-splitting set if S t-splits every $d \in D\backslash\{0\}$. By Lemma 2.1 of [4] if S is a t-splitting set of D, then $dD_S \cap D = A_t$ is a t-invertible t-ideal and hence a v-ideal and of course $B_t = dA^{-1}$.

Theorem 0.7. *Let S be a t-splitting set of an integral domain D. If D is a v-domain, then so is D_S.*

Proof. Suppose that D_S is not a v-domain. That is, there is a pair a, b of D_S such that $(((a, b)(a, b)^{-1})\ D_S)_v \neq D_S$. Since $(r, s)^{-1}D_S = ((r, s)D_S)^{-1}$ for all $r, s \in D\backslash\{0\}$, we can take $a, b \in D$ and regard $(a, b)(a, b)^{-1}$ as an ideal of D. Since $(((a, b)(a, b)^{-1})\ D_S)_v \neq D_S$, $(a, b)(a, b)^{-1} \cap S = \phi$. Again since $(((a, b)(a, b)^{-1})\ D_S)_v \neq D_S$ there exist $x, y \in D_S$ such that $((a, b)(a, b)^{-1})\ D_S \subseteq \frac{x}{y}D_S$ where $x \nmid y$ in D_S and we can take x, y in D. This gives $y((a, b)(a, b)^{-1})D_S \subseteq xD_S$ and $y((a, b)(a, b)^{-1}) \subseteq y((a, b)(a, b)^{-1})\ D_S \cap D \subseteq xD_S \cap D$. Now as $y((a, b)(a, b)^{-1}) \subseteq xD_S \cap D$ and $xD_S \cap D$ is divisorial, we have $y((a, b)(a, b)^{-1})_v \subseteq xD_S \cap D$, which forces $yD \subseteq xD_S \cap D$. But then $yD_S \subseteq (xD_S \cap D)D_S = xD_S$ which contradicts the assumption that $x \nmid y$ in D_S. □

Let X be an indeterminate over D, let $R = D[X]$ and let $G = \{f \in D[X] | (A_f)_v = D\}$. It was shown in [6, Proposition 3.7] that G is a t-complemented t-lcm t-splitting set of $D[X]$. Here a t-splitting set S is a t-lcm t-splitting set if for all $s \in S$ and for all $x \in D\backslash\{0\}$, $sD \cap xD$ is t-invertible. The following result was proved, as Theorem 3.4 in [6].

Proposition 4. *Let D be an integral domain with quotient field K, S a t-splitting set of D, and $\mathcal{S} = \{A_1 \cdots A_n | A_i = d_i DS \cap D$ for some $0 \neq d_i \in D\}$. Then the following statements are equivalent. (1) S is a t-lcm t-splitting set, (2) every finite type integral v-ideal of D intersecting S is t-invertible and (3) $D_{\mathcal{S}} = \{x \in K | xC \subseteq D$ for some $C \in T\}$ is a PVMD.*

A t-splitting set S is called t-complemented if $D_{\mathcal{S}} = D_T$ for some multiplicative set T of D.

Corollary 3. *Let X be an indeterminate over D, let $R = D[X]$ and let $G = \{f \in D[X] | (A_f)_v = D\}$. Then D is a v-domain if and only if $D[X]_G$ is.*

Proof. Indeed as D is a v-domain, then so is $D[X]$ [9, Theorem 4.1]. Since G is a t-splitting set, Theorem 0.7 applies. For the converse, note that according to Proposition 3.7 of [6], G is a t-complemented t-lcm t-splitting set of $D[X]$. So, $D[X]_{\mathcal{S}}$ is a PVMD and there is a multiplicative set N of $D[X]$ such that $D[X]_{\mathcal{S}} = D[X]_N$. So $D[X] = D[X]_G \cap D[X]_N$ where $D[X]_N$ is a PVMD. Thus if $D[X]_G$ is a v-domain, then so is $D[X]$. But then D is a v-domain, [9, Theorem 4.1]. □

Corollary 3 can be put to an interesting use, but for that we need some preparation. Let's first note that if (D, M) is a t-local domain and X an indeterminate over D, then $G = \{f \in D[X] | (A_f)_v = D\}$ is precisely $H = \{f \in D[X] | A_f = D\}$, because the maximal ideal of D is a t-ideal. In other words if D is a t-local domain, then $D[X]_G = D[X]_H = D(X)$, the Nagata extension of D. For description and properties of $D(X)$ the reader may consult [1].

Corollary 4. *(to Corollary 3)Let D be a t-local domain. Then D is a v-domain if and only if $D(X)$ is a v-domain.*

Next, according to Corollary 8 of [5], if $\mathcal{P}$ is an associated prime of a nonzero polynomial of $D[X]$, then $\mathcal{P} \cap D = (0)$ or $\mathcal{P} = (\mathcal{P} \cap D)[X]$ where $(\mathcal{P} \cap D)$ is an associated prime of a principal ideal of D.

Corollary 5. *Let D be an integral domain. Then D is a super v-domain if and only if $D[X]$ is.*

Proof. Let D be a super v-domain. To see that $D[X]$ is a super v-domain let $\wp$ be an associated prime of $D[X]$. Then $\wp$ is an upper to 0, i.e., $\wp \cap D = (0)$ or $\wp = P[X]$ where P is an associated prime of a principal ideal of D. If $\wp$ is an upper to 0 then $D[X]_\wp$ is a rank one DVR and so a v-domain. If, on the other hand, $\wp = P[X]$, where P is an associated prime of a principal ideal of D, then $D[X]_\wp = D[X]_{P[X]} = D_P(X)$. Since D is a a super v-domain, D_P is a v-domain. But, then so is $D_P(X)$, by Corollary 4; because D_P is t-local [10, Corollary 2.3]. That $D[X]$ is a super v-domain, now follows from Theorem 0.2. For the converse note that if P is a minimal prime of $(a) : (b)$ then $P[X]$ is minimal over $aD[X] : bD[X]$, making $P[X]$ an associated prime of a principal ideal of $D[X]$. Since $D[X]$ is a super v-domain, $D[X]_{P[X]} = D_P(X)$ is a v-domain. Now as D_P is t-local, Corollary 4 applies to give the conclusion that D_P is a a v-domain. Now P being any associated prime of D we conclude, by Theorem 0.2, that D is indeed a super v-domain. □

References

[1] Anderson, D.D., Anderson, D.F. and Markanda, R.: The rings $R(X)$ and $R < X >$, J. Algebra 95 (1985) , 96-115.

[2] Anderson, D.D., Anderson, D.F., Fontana, M. and Zafrullah, M.: On v-domains and star operations, Comm. Algebra, 37 (2009) 3018–3043.

[3] Anderson, D.D., Anderson, D.F. and Zafrullah, M.: Splitting the t-class group, J. Pure Appl. Algebra 74(1991) 17-37.

[4] Anderson, D.D., Anderson, D.F. and Zafrullah, M.: The ring $D + XDS[X]$ and t-splitting sets, Commutative Algebra Arabian J. Sci. Eng. Sect. C Theme Issues 26 (1) (2001) 3–16.

[5] Brewer, J. and Heinzer, W.: Associated primes of principal ideals, Duke Math. J. 41(1974) 1-7.

[6] Chang, G.W., Dumitrescu, T. and Zafrullah, M.: t-Splitting sets in integral domains, J. Pure Appl. Algebra 187 (2004) 71–86.

[7] Costa, D.L., Mott, J.L. and Zafrullah, M.: The construction $D+XD_S[X]$, J. Algebra 53(1978) 423-439.

[8] Costa, D.L., Mott, J.L. and Zafrullah, M.: Overrings and dimensions of general $D + M$ constructions, J. Natur. Sci. and Math. 26 (2) (1986), 7-14.

[9] Fontana, M. and Zafrullah, M.: On v-domains: a survey. In: Fontana, M., Kabbaj, S., Olberding, B., Swanson, I. (eds.) Commutative Algebra: Noetherian and Non-Noetherian Perspectives, pp. 145–180. Springer, New York (2011)

[10] Fontana, M. and Zafrullah, M.: On t-local domains and valuation domains, in "Advances in Commutative Algebra" Editors: Badawi, Ayman, Coykendall, Jim, Trends in Mathematics, Birkhäuser 2019, pp. 33-62.

[11] Gabelli, S.: On divisorial ideals in polynomial rings over Mori domains, Comm. Algebra 15(11)(1987) 2349-2370.

[12] Gilmer, R.: Multiplicative Ideal Theory, Marcel-Dekker, New York, 1972.

[13] Griffin, M.: Some results on v-multiplication rings, Canad. J. Math.19(1967) 710-722.

[14] Hedstrom, J. and Houston, E.: Pseudo-valuation domains, Pacific J. Math. 75 (1978), 137–147.

[15] Heinzer, W.: An essential integral domain with a nonessential localization, Can. J. Math. 33, 400–403 (1981).

[16] Mott, J., Nashier, B., and Zafrullah, M.: Contents of polynomials and invertibility. Comm. Algebra18 (1990), 1569–1583.

[17] Mott, J. and Zafrullah, M.: On Prüfer v-multiplication domains, Manuscripta Math. 35(1981)1-26.

[18] Nagata, M.: On Krull's conjecture concerning valuation rings, Nagoya Math. J. (4)(1952) 29-33.

[19] Nagata, M.: Correction to my paper "On Krull's conjecture concerning valuation overrings, Nagoya Math.J. (9)(1955) 209-212.

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