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# ON SUPER $v$-DOMAINS

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ABSTRACT. An integral domain $D$, with quotient field $K$, is a $v$-domain if for each nonzero finitely generated ideal $A$ of $D$ we have $(AA^{-1})^{-1} = D$. It is well known that if $D$ is a $v$-domain, then some quotient ring $D_S$ of $D$ may not be a $v$-domain. Calling $D$ a super $v$-domain if every quotient ring of $D$ is a $v$-domain we characterize super $v$-domains as locally $v$-domains. Using techniques from factorization theory we show that $D$ is a super $v$-domain if and only if $D[X]$ is a super $v$-domain if and only if $D + XK[X]$ is a super $v$-domain and give new examples of super $v$-domains that are strictly between $v$-domains and P-domains, domains that are essential along with all their quotient rings.

An integral domain $D$, with quotient field $K$, is called a $v$-domain if for every finitely generated nonzero ideal $A$ of $D$, $A$ is $v$-invertible, i.e., we have $(AA^{-1})_v = D$ or equivalently $(AA^{-1})^{-1} = D$. Essentially, $v$-domains are modeled after Prufer domains (every nonzero finitely generated ideal is invertible). Yet using an example of Heinzer's, [15], one can show that if $D$ is a $v$-domain and $S$ a multiplicative set of $D$, then $D_S$ need not be a $v$-domain, see section 3 of [10] for a discussion on this example. This raises the following questions. If $D$ is a $v$-domain, under what conditions on a multiplicative set $S$, or on $D$, can we be sure that $D_S$ is a $v$-domain? Also, what are the $v$-domains whose quotient rings are again $v$-domains and that are not any of the known examples of $v$-domains all of whose quotient rings are $v$-domains. Finally, if every proper quotient ring of $D$ is a $v$-domain, must $D$ be a $v$-domain? Let's call a $v$-domain $D$ a super $v$-domain if $D_S$ is a $v$-domain for each multiplicative set of $D$. One purpose of this article is to discuss some conditions that will ensure that a quotient ring of a $v$-domain is a $v$-domain. We show for instance that if $D$ is a $v$-domain and $S$ is a splitting or a $t$-splitting set of $D$ then $D_S$ is a $v$-domain. Here a multiplicative set $S$ of $D$ is a splitting set if $S$ is saturated and each $d \in D\backslash\{0\}$ is expressible as $d = rs$ where $s \in S$ and $rD \cap tD = rt$ for all $t \in S$, ($t$-splitting sets are defined in a slightly complicated yet similar fashion). Using results on splitting and $t$-splitting sets, in conjunction with other results, we show that $D$ is a super $v$-domain if and only if $D + XK[X]$ is a super $v$-domain, where $X$ is an indeterminate over $K$. We also show that if $X$ is an indeterminate over $D$, then $D$ is a super $v$-domain if and only if $D[X]$ is. (The answer to question (c) is that for a one dimensional quasi local domain $D$ a proper quotient ring is the field of fractions of $D$ and hence a $v$-domain. But a one dimensional quasi local domain need not be a $v$-domain.)

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It seems pertinent to let the reader in on the terminology that we have used above and that we are going to use when we prove our results. Let $D$ be an integral domain with quotient field $K$ and let $F(D)$ be the set of nonzero fractional ideals of $D$. A star operation is a function $A \mapsto A^*$ on $F(D)$ with the following properties:

If $A, B \in F(D)$ and $a \in K\backslash\{0\}$, then
(i) $(a)^* = (a)$ and $(aA)^* = aA^*$.
(ii) $A \subseteq A^*$ and if $A \subseteq B$, then $A^* \subseteq B^*$.
(iii) $(A^*)^* = A^*$.

We may call $A^*$ the $*$-image ( or $*$-envelope ) of $A$. An ideal $A$ is said to be a $*$-*ideal* if $A^* = A$. Thus $A^*$ is a $*$-ideal (by (iii)). Moreover (by (i)) every principal fractional ideal, including $D = (1)$, is a $*$- ideal for any star operation $*$.

For all $A, B \in F(D)$ and for each star operation $*$, $(AB)^* = (A^*B)^* = (A^*B^*)^*$. These equations define what is called $*$-multiplication *( or* $*$-product*).*

Define $A_v = (A^{-1})^{-1}$ and $A_t = \bigcup\{J_v | \ 0 \neq J$ is a finitely generated subideal of $A\}$. The functions $A \mapsto A_v$ and $A \mapsto A_t$ on $F(D)$ are more familiar examples of star operations defined on an integral domain. A $v$-ideal is better known as a divisorial ideal. The identity function $d$ on $F(D)$, defined by $A \mapsto A$ is another example of a star operation. There are of course many more star operations that can be defined on an integral domain $D$. But for any star operation $*$ and for any $A \in F(D)$, $A^* \subseteq A_v$. Some other useful relations are: For any $A \in F(D)$, $(A^{-1})^* = A^{-1} = (A^*)^{-1}$ and so, $(A_v)^* = A_v = (A^*)_v$. Using the definition of the $t$-operation one can show that an ideal that is maximal w.r.t. being a proper integral $t$-ideal is a prime ideal of $D$, each ideal $A$ of $D$ with $A_t \neq D$ is contained in a maximal $t$-ideal of $D$ and $D = \cap D_M$, where $M$ ranges over maximal $t$-ideals of $D$. For more on $v$- and $t$-operations the reader may consult sections 32 and 34 of Gilmer [12]. It was shown in [17] that $D$ is a $v$-domain if and only if every 2-generated nonzero ideal of $D$ is $v$-ivertible. An integral domain $D$ is called a Prufer $v$-Multiplication domain (PVMD) if for every finitely generated $A \in F(D)$ we have $(AA^{=1})_t = D$. It can be shown that every quotient ring of a PVMD is a PVMD. Our terminology essentially comes from [12]. We define any terms that did not appear in [12].

Call a multiplicative set $S$ of $D$ a splitting set if $S$ is saturated and for each $d \in D\backslash\{0\}$ we can write $d = d's$ where $s \in S$ and $d' \in D$ such that $(d', t)_v = D$ for all $t \in S$. For more on splitting sets look up [3]. On the other hand a multiplicative set $S$ of $D$ is a $t$-splitting set if for all $d \in D\backslash\{0\}$ we can write $dD = (AB)_t$ where $B_t \cap S \neq \phi$ and $(A, s)_v = D$ for all $s \in S$. The $t$-splitting sets were introduced and applied in [4].

As mentioned above, $v$-domains are a generalization of Prufer domains. So, some generalizations of Prufer domains such as GCD domains, PVMDs, essential domains are $v$-domains and some, such as some integrally closed integral domains, are not. Here $D$ is essential if $D$ has a family $\mathcal{F}$ of prime ideals such that $D_P$ is a valuation domain for each $P \in \mathcal{F}$ and $D = \cap_{P \in \mathcal{F}} D_P$. As indicated in [10] an essential domain is a $v$-domain and so is the so-called "P-domain". A P-domain here is an essential domain, each of whose quotient rings is essential. It was shown in [18] that $D$ is a P-domain if and only if $D$ is locally essential. Perhaps that is why they were called locally essential domains in [9], and in later literature. While PVMDs and P-domains are super $v$-domains, an essential domain may not be a super $v$-domain. As a matter of fact P domains arose from an example by Heinzer and Ohm [16] of an essential domain that was not a PVMD, yet happened to be a P-domain. When Heinzer found out about the work in [18], he wrote [15].

Let us be clear about what we are looking for, when we study "super $v$-domains" as there do exist super $v$-domains in the form of the P-domains and Prufer domains and the so-called Prufer $v$-Multiplication domains or PVMDs. While we prove general results about super $v$-domains, we are also looking for $v$-domains $D$ that are not P-domains yet have the property that $D_S$ is a $v$-domain for each multiplicative set $S$ of $D$.

The first thing that seems to prevent a $v$-domain from having a quotient ring that is a $v$-domain seems to be that while for a nonzero finitely generated ideal $I$ we have $(ID_S)^{-1} = I^{-1}D_S$ we have no such general formula for a nonzero ideal $I$. One way of dealing with a situation like this is to bring in a new definition. Call a quotient ring $D_S$ of $D$ super extending if for each nonzero ideal $I$ of $D$ we have $(ID_S)^{-1} = I^{-1}D_S$. An immediate consequence is that if $D_S$ is super extending, then $(ID_S)_v = I_vD$.

**Lemma 0.1.** *If $D_S$ is super extending and $D$ is a $v$-domain, then $D_S$ is a $v$-domain.*

*Proof.* Let $\alpha, \beta \in D_S$. Then $\alpha = \frac{a}{s}, \beta = \frac{b}{t}$ for some $a, b \in D$ and $s, t \in S$ and $(\alpha, \beta)D_S((\alpha, \beta)D_S)^{-1} = (a, b)D_S((a, b)D_S)^{-1} = ((a, b)(a, b)^{-1})D_S$. Now as $D_S$ is super extending we conclude that $((\alpha, \beta)D_S((\alpha, \beta)D_S)^{-1})^{-1} = (((a, b)(a, b)^{-1})D_S)^{-1} = (((a, b)(a, b)^{-1}))^{-1}D_S = D_S$ because in $D$ we have $(((a, b)(a, b)^{-1}))^{-1} = D$. $\square$

But the drawback of Lemma 0.1 is that if $D_S$ happens to be such that $(a, b)^{-1}D_S$ is a finitely generated ideal of $D_S$ for each pair $a, b$ of $D$, then Lemma 0.1 would be an overkill. Though $D_S$ would have to be a stronger form of a PVMD. Also, in some domains, $(ID_S)_v = I_vD$ may not hold for some nonzero ideals $I$ and multiplicative sets $S$ of some domains $D$. For example, it can be shown that for every prime ideal $\wp$ of height greater than one in the ring $\mathcal{E}$ of entire functions $\wp_v = \mathcal{E}$. Yet if $\wp$ is non-maximal and $\mathcal{M}$ is a maximal ideal containing $\wp$, then $(\wp\mathcal{E}_{\mathcal{M}})_v = \wp\mathcal{E}_{\mathcal{M}} \neq \mathcal{E}_{\mathcal{M}} = \wp_v\mathcal{E}_{\mathcal{M}}$; because $\mathcal{E}_{\mathcal{M}}$ is a valuation domain and every non-maximal nonzero prime ideal in a valuation domain is divisorial. All this beside, super extending is too much even for our needs. So let's call $D_S$ simple extending if $(((a, b)(a, b)^{-1})D_S)^{-1} = (((a, b)(a, b)^{-1}))^{-1}D_S$. We do seem to have disadvantages of super extending when working with simple extending and simple extending is sort of too obvious a ploy, but it may work in some interesting ways.

**Proposition 1.** *Let $D$ be an integral domain and let $\{S_\alpha\}$ be a family of multiplicative sets of $D$ such that $D = \cap D_{S_\alpha}$. If, for each $\alpha \in I$, $D_{S_\alpha}$ is a simple extending quotient ring of $D$ and a $v$-domain, then $D$ is a $v$-domain.*

*Proof.* Note that, as the inverse of an ideal is divisorial, we have $(((a, b)(a, b)^{-1}))^{-1} = \cap(((a, b)(a, b)^{-1}))^{-1}D_{S_\alpha} = \cap((((a, b)(a, b)^{-1}))D_{S_\alpha})^{-1} = \cap D_{S_\alpha} = D$. $\square$

If on the other hand $D_S$ is a $v$-domain, then $D_S$ is simple extending anyway. This follows from the observation that if $D_S$ is a $v$-domain, then for all $a, b \in D\backslash\{0\}$ we have $(((a, b)(a, b)^{-1})D_S)^{-1} = D_S$, (because $D_S$ is a $v$-domain) $(((a, b)(a, b)^{-1}))^{-1}D_S \supseteq D_S$, (because $((a, b)(a, b)^{-1}) \subseteq D$) and as for each nonzero ideal $I$ of $D$, $I^{-1}D_S \subseteq (ID_S)^{-1}$ we have $(((a, b)(a, b)^{-1}))^{-1}D_S \subseteq (((a, b)(a, b)^{-1})D_S)^{-1} = D_S$. Thus we have the following corollary.

**Corollary 1.** *Let $D$ be an integral domain and let $\{S_\alpha\}$ be a family of multiplicative sets of $D$ such that $D = \cap D_{S_\alpha}$. If, for each $\alpha \in I$, $D_{S_\alpha}$ is a $v$-domain, then $D$ is a $v$-domain.*

*Remark* 0.2. Of course generally $D_S$ is not simple extending. For example if $D$ is a $v$-domain such that $D_S$ is not a $v$-domain then for some $a, b \in D$ we must have $(((a,b)(a,b)^{-1})D_S)^{-1} \neq D_S$. On the other hand $(((a,b)(a,b)^{-1}))^{-1}D_S = D$, because $D$ is a $v$-domain.

In any case there is a better result available on the market in the form of Proposition 3.1 of [10]. This result says.

**Proposition 2.** *Let $\{D_\lambda | \lambda \in \Lambda\}$ be a family of flat overrings of $D$ such that $D = \cap_{\lambda\in\Lambda} D_\lambda$ . If each of $D_\lambda$ is a $v$-domain, then so is $D$.*

Let us recall that a prime ideal $P$ is called an associated prime of a principal ideal $(a)$ if $P$ is minimal over an ideal of the form $0 \neq (a) : (b) = \{r \in D | rb \in (a)\} \neq D$. Associated primes of principal ideals, or simply associated primes, of $D$ have been studied by quite a few authors, but our reference in this regard is [5]. According to Proposition 4 of [5], if $S$ is a multiplicative set of $D$ and $\{P_\alpha\}$ is the family of associated primes of principal ideals of $D$ disjoint from $S$, then $D_S = \cap_\alpha D_{P_\alpha}$.

With Proposition 2, or Corollary 1, at hand, we can state and prove the following characterization of super $v$-domains.

**Theorem 0.3.** *([10, Proposition 3.4])The following are equivalent for an integral domain $D$. (1) $D_S$ is a $v$-domain for every multiplicative set $S$ of $D$, (2) $D_P$ is a $v$-domain for every prime ideal $P$ of $D$ and (3) $D_P$ is a $v$-domain for every associated prime $P$ of $D$.*

*Proof.* That (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is obvious. For (3) $\Rightarrow$ (1), let $S$ be a multiplicative set of $D$ and let $\mathcal{F} = \{P_\alpha\}$ be the family of associated primes disjoint from $S$. Then by (3) each of $D_{P_\alpha}$ is a $v$-domain and by [5, Proposition 4] $D_S = \cap_{P_\alpha \in \mathcal{F}} D_{P_\alpha}$. Thus by Proposition 2, or Corollary 1, $D_S$ is a $v$-domain. □

My reason for proving Theorem 0.3 all over again is that its proof can now be carried out via Corollary 1 rather than via Proposition 2 which was proven in [10], using the star operation theoretic approach. Now, however much fulfilling Theorem 0.3 may appear, it does not give us a clue as to how to find/construct super $v$-domains. This makes us look for multiplicative sets $S$ for which $D_S$ is a $v$-domain, whenever $D$ is. As we shall see below this happens when the multiplicative set $S$ in $D$ is a splitting set. If $S$ is a splitting set, the set $T = \{t \in D | \ (t,s)_v = D$ for all $s \in S\}$ often denoted as $S^\perp$ is called the $m$-complement of $S$. Indeed if $S$ is a splitting set and $T = S^\perp$, then $D = D_S \cap D_T$ and $dD_S \cap D = tD$ where $t \in T$ such that $d = ts$ for some $s \in S$. A splitting set $S$ of $D$ is an lcm splitting set if $sD \cap xD$ is principal for all $s \in S$ and for all $x \in D\backslash\{0\}$.

**Theorem 0.4.** *Let $S$ be a splitting multiplicative set of $D$ and let $T = S^\perp$. If $D$ is a $v$-domain, then so is $D_S$. Moreover if $S$ is an lcm splitting set then $D_S$ is a $v$-domain if and only if $D$ is a $v$-domain.*

*Proof.* Suppose that $D_S$ is not a $v$-domain. That is, there is a pair $a, b$ of $D_S$ such that $(((a,b)(a,b)^{-1})\ D_S)_v \neq D_S$. Since $(r,s)^{-1}D_S = ((r,s)D_S)^{-1}$, for $r, s \in D\backslash\{0\}$, we can take $a, b \in D$ and regard $(a,b)(a,b)^{-1}$ as an ideal of $D$. Since $(((a,b)(a,b)^{-1})$

$D_S)_v \neq D_S$, $(a,b)(a,b)^{-1} \cap S = \phi$. Again since $(((a,b)(a,b)^{-1})\ D_S)_v \neq D_S$ there exist $x, y \in D_S$ such that $((a,b)(a,b)^{-1})\ D_S \subseteq \frac{x}{y} D_S$ where $x \nmid y$ in $D_S$. As $S$ is a splitting set, we can take $x, y \in T$. But then $y((a,b)(a,b)^{-1})\ D_S \subseteq xD_S$ and $y((a,b)(a,b)^{-1}) \subseteq y((a,b)(a,b)^{-1})\ D_S \cap D \subseteq xD_S \cap D$. As $x \in T$, we have $xD_S \cap D = xD$ ([3], Theorem 2.2). Thus we have $y((a,b)(a,b)^{-1}) \subseteq xD$. Applying the $v$-operation throughout and noting that $D$ is a $v$-domain we conclude that $yD \subseteq xD$. But then $yD_S \subseteq xD_S$, a contradiction. Whence $D_S$ is a $v$-domain. For the moreover part note that $D = D_S \cap D_T$ where $D_T$ is a GCD domain, by Theorem 2.4 of [3]. Thus if $S$ is lcm splitting $D_S$ is a $v$-domain and so is $D_T$, being a GCD domain, forcing $D = D_S \cap D_T$ to be a $v$-domain, by Proposition 2. □

**Theorem 0.5.** *Let $D$ be an integral domain with quotient field $K$ and let $X$ be an indeterminate over $D$. Then $D$ is a super $v$-domain if and only if $D + XK[X]$ is a super $v$-domain.*

*Proof.* Let $D$ be a super $v$-domain. Then by Theorem 4.42 of [7] $T = D + XK[X]$ is a $v$-domain. Also by Proposition 2.2 of [8], every overring $S$, and hence every quotient ring $S$, of $T$ is a quotient ring of $S \cap K + XK[X]$. According to the proof of Proposition 2.2 of [8] the elements of $S$ are of the form $\frac{\alpha + Xf(X)}{1+Xg(X)}$ where $\alpha \in S \cap L$. Let $U = \{u \in D | u$ is a unit in $S\}$. Then $D_U \subseteq S \cap K$. Let $h \in S$. Then $h = \frac{a+Xf(X)}{b+Xg(X)}$ where, $a, b \in D$ and, $b + Xg(X)$ is a unit in $S$. This gives $b = b(1 + \frac{X}{b}g(X)(1 + \frac{X}{b}g(X)^{-1}$ and so $b$ is a unit in $S \cap K$, whence $b \in U$. But then $a/b = h(0) \in D_U$. Noting that $h(0) \in S \cap K$ we conclude that $D_U = S \cap K$. This leads to the conclusion that $S$ is a quotient ring of $D_U + XK[X]$. Since $D$ is a super $v$-domain $D_U$ is a $v$-domain and so is $D_U + XK[X]$. Next, by the proof of Proposition 2.2 of [8], denoting by $U(S)$ the set of units of $S$ we have $U(S) = \{f \in D_U + XK[X] | f = u + Xg(X)$, where $u$ is a unit in $D_U\}$ and as elements of the form $1 + Xg(X)$ are finite products of height one primes of $D_U + XK[X]$ ([7], Theorem 4.21) we conclude that $U(S)$ is a splitting set generated by primes. But then, by Theorem 0.4, $S = (D_U + XK[X])_{U(S)}$ is a $v$-domain. For the converse note that if $T$ is a multiplicative set in $D$, then $(D + XK[X])_T = D_T + XK[X]$ which is a $v$-domain if and only if $D_T$ is a $v$-domain. Thus if $D + XK[X]$ is a super $v$-domain, then so is $D$. □

Some super $v$-domains such as the P-domains have the property that $D_P$ is a valuation domain for every associated prime of a principal ideal of $D$. Now if $P$ is an associated prime of a principal ideal, one can easily show that $D_P$ is $t$-local, i.e., $PD_P$ is a $t$-ideal [10]. This may lead one to ask if a $t$-local super $v$-domain is close to a valuation domain. The answer is: Close but not too close, as there does exist a one dimensional completely integrally closed integral domain $\mathcal{N}$, due to Nagata [19] and [20], that is not a valuation domain and a one dimensional quasi local domain is $t$-local. (Of course a completely integrally closed domain is a $v$-domain.) Now, trivially, $\mathcal{N}$ has the property that every quotient ring of $\mathcal{N}$ is $\mathcal{N}$ or $qf(\mathcal{N})$. Thus, albeit trivially, $\mathcal{N}$ serves as an example of a super $v$-domain. This gives us the following example.

**Example 0.6.** Let $F$ be the quotient field of $\mathcal{N}$ and let $X$ be an indeterminate on $F$. Then $\mathcal{N}+XF[X]$ is a super $v$-domain.

Illustration: By Theorem 0.5, every quotient ring $S$ of $\mathcal{N}+XF[X]$ is a quotient ring $(\mathcal{N}+XF[X])_U$ of $\mathcal{N}+XF[X]$, by a multiplicative set $U$ generated by elements of the form $1+Xg(X)$, or a quotient ring of $F[X]$. Since $\mathcal{N}+XF[X]$ is a $v$ domain and elements of the form $1+Xg(X)$ being products of height one primes, $U$ is a splitting set and by Theorem 0.4, $(\mathcal{N}+XF[X])_U$ is a $v$-domain. Also since $F[X]$ is a PID every quotient ring of $F[X]$ is a PID and hence a $v$-domain. So, every quotient ring of $\mathcal{N}+XF[X]$ is indeed a $v$-domain.

Indeed $\mathcal{N}+XF[X]$ provides a "non-trivial" example of a super $v$-domain and Theorem 0.5 provides a scheme for producing super $v$-domains of any Krull dimension. And these super $v$-domains are not essential and hence not P-domains.

Next call a domain $D$ a $v$-local domain if $D$ is quasi local such that the maximal ideal $M$ of $D$ is divisorial. Of course, the situation can drastically change if we relax "$t$-local" to "$v$-local".

**Proposition 3.** *An integral domain $D$ is a $v$-local $v$-domain if and only if $D$ is a valuation domain with maximal ideal $M$ principal.*

*Proof.* Let $D$ be a $v$-local $v$-domain and let $A$ be a nonzero finitely generated ideal of $D$. Then $AA^{-1}=D$. For if $AA^{-1}\neq D$ we must have $AA^{-1}\subseteq M$. But as $M$ is a $v$-ideal and $D$ a $v$-domain we have $D=(AA^{-1})_v\subseteq M_v=M$ a contradiction. Whence every nonzero finitely generated ideal of $D$ is invertible and hence principal, because $D$ is $v$-local and hence quasi local. Thus $D$ is a valuation domain. Now the maximal ideal being divisorial means $M_v\neq D$ which means that there is a pair of elements $a,b$ of $D$ such that $M\subseteq(a/b)D$ where $a\nmid b$. Since $a\nmid b$ and $D$ is a valuation domain $M\subsetneq(a/d)D$ a principal ideal of $D$. But then $M$ is principal because $M$ is the maximal ideal. The converse is obvious. $\square$

Let's recall from Griffin [13, Theorem 5] that $D$ is a PVMD if and only if for every finitely generated nonzero ideal $I$ of $D$ we have $(II^{-1})_t=D$ if and only if $D_P$ is a valuation ring for every maximal $t$-ideal of $D$.

**Corollary 2.** *Let $D$ be locally a $v$-domain. Suppose that for every maximal $t$-ideal $M$ of $D$ we have $MD_M$ divisorial then $D$ is a PVMD.*

*Proof.* For every maximal $t$-ideal $M$ we have $D_M$ a $v$-domain and $MD_M$ a divisorial ideal. Then by Proposition 3 we have that $D_M$ is a valuation domain with maximal ideal principal.

Alternative proof: Let $J$ be a nonzero ideal of $D$. We claim that $JJ^{-1}$ is not in any maximal $t$-ideal of $D$. For if $JJ^{-1}\subseteq M$. Then $(JJ^{-1})D_M=JD_MJ^{-1}D_M=JD_M(JD_M)^{-1}\subseteq MD_M$. Since $D_M$ is a $v$-domain, $D_M=((JD_M(JD_M)^{-1})_v$. Yet as $MD_M$ is divisorial and $JD_MJ^{-1}D_M=JD_M(JD_M)^{-1}\subseteq MD_M$ we get $D_M=((JD_M(JD_M)^{-1})_v\subseteq MD_M$ a contradiction. Now $JJ^{-1}$ not being in any maximal $t$-ideals means that $(JJ^{-1})_t=D$. Thus every nonzero finitely generated ideal of $D$ is $t$-invertible and this is another characteristic property of PVMDs. $\square$

Recall that a prime ideal $P$ of a domain $D$ is called strongly prime if $x,y\in K$ and $xy\in P$ imply that $x\in P$ or $y\in P$. According to [14], $D$ is a pseudo valuation domain PVD if every prime ideal of $D$ is strongly prime. It turns out that a PVD is a valuation domain or a quasi local domain $(D,M)$ such that $M^{-1}=V$ a valuation ring. This makes the maximal ideal of a non-valuation PVD a divisorial ideal.

**Corollary 3.** *In a non-valuation PVD $D$, every $v$-invertible ideal is principal. Consequently a non-valuation PVD can never be a $v$-domain.*

*Proof.* Suppose that a non-valuation PVD $D$ is a $v$-domain. Then $D$ is a $v$-local $v$-domain and hence a valuation domain by Proposition 3, a contradiction. □

*Remark* 0.7. Using the fact that the set of prime ideals in a PVD is linearly ordered it is shown in [14] that a GCD PVD is a valuation domain. However a non-valuation PVD $D$ can never be a GCD domain, because a GCD domain is a $v$-domain. We can also say that a non-valuation PVD can never be a PVMD, because a PVMD is a $v$-domain as well.

Let $S$ be a multiplicative set of $D$. Following [4] we say that $d \in D\backslash\{0\}$ is $t$-split by $S$ if there are two integral ideals $A, B$ of $D$ such that $(d) = (AB)_t$ where $B_t \cap S \neq \phi$ and $(A, s)_t = D$ for all $s \in S$. As in [4] we call $S$ a $t$-splitting set if $S$ $t$-splits every $d \in D\backslash\{0\}$. By Lemma 2.1 of [4] if $S$ is a $t$-splitting set of $D$, then $dD_S \cap D = A_t$ is a $t$-invertible $t$-ideal and hence a $v$-ideal and of course $B_t = dA^{-1}$.

**Theorem 0.8.** *Let $S$ be a $t$-splitting set of an integral domain $D$. If $D$ is a $v$-domain, then so is $D_S$.*

*Proof.* Suppose that $D_S$ is not a $v$-domain. That is, there is a pair $a, b$ of $D_S$ such that $(((a,b)(a,b)^{-1})\ D_S)_v \neq D_S$. Since $(r,s)^{-1}D_S = ((r,s)D_S)^{-1}$ for all $r, s \in D\backslash\{0\}$, we can take $a, b \in D$ and regard $(a,b)(a,b)^{-1}$ as an ideal of $D$. Since $(((a,b)(a,b)^{-1})\ D_S)_v \neq D_S$, $(a,b)(a,b)^{-1} \cap S = \phi$. Again since $(((a,b)(a,b)^{-1})\ D_S)_v \neq D_S$ there exist $x, y \in D_S$ such that $((a,b)(a,b)^{-1})\ D_S \subseteq \frac{x}{y} D_S$ where $x \nmid y$ in $D_S$ and we can take $x, y$ in $D$. This gives $y((a,b)(a,b)^{-1})D_S \subseteq xD_S$ and $y((a,b)(a,b)^{-1}) \subseteq y((a,b)(a,b)^{-1})\ D_S \cap D \subseteq xD_S \cap D$. Now as $y((a,b)(a,b)^{-1}) \subseteq xD_S \cap D$ and $xD_S \cap D$ is divisorial, we have $y((a,b)(a,b)^{-1})_v \subseteq xD_S \cap D$, which forces $yD \subseteq xD_S \cap D$. But then $yD_S \subseteq (xD_S \cap D)D_S = xD_S$ which contradicts the assumption that $x \nmid y$ in $D_S$. □

Let $X$ be an indeterminate over $D$, let $R = D[X]$ and let $G = \{f \in D[X] | (A_f)_v = D\}$. It was shown in [6, Proposition 3.7] that $G$ is a $t$-complemented $t$-lcm $t$-splitting set of $D[X]$. Here a $t$-splitting set $S$ is a $t$-lcm $t$-splitting set if for all $s \in S$ and for all $x \in D\backslash\{0\}$, $sD \cap xD$ is $t$-invertible. The following result was proved, as Theorem 3.4 in [6].

**Proposition 4.** *Let $D$ be an integral domain with quotient field $K$, $S$ a $t$-splitting set of $D$, and $\mathcal{S} = \{A_1 \cdots A_n | A_i = d_i DS \cap D$ for some $0 \neq d_i \in D\}$. Then the following statements are equivalent. (1) $S$ is a $t$-lcm $t$-splitting set, (2) every finite type integral $v$-ideal of $D$ intersecting $S$ is $t$-invertible and (3) $D_{\mathcal{S}} = \{x \in K | xC \subseteq D$ for some $C \in T\}$ is a PVMD.*

A $t$-splitting set $S$ is called $t$-complemented if $D_{\mathcal{S}} = D_T$ for some multiplicative set $T$ of $D$.

**Corollary 4.** *Let $X$ be an indeterminate over $D$, let $R = D[X]$ and let $G = \{f \in D[X] | (A_f)_v = D\}$. Then $D$ is a $v$-domain if and only if $D[X]_G$ is.*

*Proof.* Indeed as $D$ is a $v$-domain, then so is $D[X]$ [10, Theorem 4.1]. Since $G$ is a $t$-splitting set, Theorem 0.8 applies. For the converse, note that according to Proposition 3.7 of [6], $G$ is a $t$-complemented $t$-lcm $t$-splitting set of $D[X]$. So, $D[X]_{\mathcal{S}}$ is a PVMD and there is a multiplicative set $N$ of $D[X]$ such that $D[X]_{\mathcal{S}} = D[X]_N$. So $D[X] = D[X]_G \cap D[X]_N$ where $D[X]_N$ is a PVMD. Thus if $D[X]_G$ is a $v$-domain, then so is $D[X]$. But then $D$ is a $v$-domain, [10, Theorem 4.1]. □

Corollary 4 can be put to an interesting use, but for that we need some preparation. Let's first note that if $(D, M)$ is a $t$-local domain and $X$ an indeterminate over $D$, then $G = \{f \in D[X] | (A_f)_v = D\}$ is precisely $H = \{f \in D[X] | A_f = D\}$, because the maximal ideal of $D$ is a $t$-ideal. In other words if $D$ is a $t$-local domain, then $D[X]_G = D[X]_H = D(X)$, the Nagata extension of $D$. For description and properties of $D(X)$ the reader may consult [1].

**Corollary 5.** *(to Corollary 4)Let $D$ be a $t$-local domain. Then $D$ is a $v$-domain if and only if $D(X)$ is a $v$-domain.*

Next, according to Corollary 8 of [5], if $\mathcal{P}$ is an associated prime of a nonzero polynomial of $D[X]$, then $\mathcal{P} \cap D = (0)$ or $\mathcal{P} = (\mathcal{P} \cap D)[X]$ where $(\mathcal{P} \cap D)$ is an associated prime of a principal ideal of $D$.

**Corollary 6.** *Let $D$ be an integral domain. Then $D$ is a super $v$-domain if and only if $D[X]$ is.*

*Proof.* Let $D$ be a super $v$-domain. To see that $D[X]$ is a super $v$-domain let $\wp$ be an associated prime of $D[X]$. Then $\wp$ is an upper to 0, i.e., $\wp \cap D = (0)$ or $\wp = P[X]$ where $P$ is an associated prime of a principal ideal of $D$. If $\wp$ is an upper to 0 then $D[X]_\wp$ is a rank one DVR and so a $v$-domain. If, on the other hand, $\wp = P[X]$, where $P$ is an associated prime of a principal ideal of $D$, then $D[X]_\wp = D[X]_{P[X]} = D_P(X)$. Since $D$ is a a super $v$-domain, $D_P$ is a $v$-domain. But, then so is $D_P(X)$, by Corollary 5; because $D_P$ is $t$-local [11, Corollary 2.3]. That $D[X]$ is a super $v$-domain, now follows from Theorem 0.3. For the converse note that if $P$ is a minimal prime of $(a) : (b)$ then $P[X]$ is minimal over $aD[X] : bD[X]$, making $P[X]$ an associated prime of a principal ideal of $D[X]$. Since $D[X]$ is a super $v$-domain, $D[X]_{P[X]} = D_P(X)$ is a $v$-domain. Now as $D_P$ is $t$-local, Corollary 5 applies to give the conclusion that $D_P$ is a a $v$-domain. Now $P$ being any associated prime of $D$ we conclude, by Theorem 0.3, that $D$ is indeed a super $v$-domain. $\square$

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