ON $*$-SEMI HOMOGENEOUS DOMAINS

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ABSTRACT. Let $*$ be a finite character star operation defined on an integral domain D. Call a nonzero $*$-ideal I of finite type a $*$-homogeneous ($*$-homog) ideal, if $I \subsetneq D$ and $(J+K)^* \neq D$ for every pair $D \supsetneq J, K \supseteq I$ of proper $*$-ideals of finite type. Call an integral domain D a $*$-Semi Homogeneous Domain ($*$-SHD) if every proper principal ideal xD of D is expressible as a $*$-product of finitely many $*$-homog ideals. We show that a $*$-SHD contains a family $\mathcal{F}$ of prime ideals such that (a) $D = \cap_{P\in\mathcal{F}} D_P$, a locally finite intersection and (b) no two members of $\mathcal{F}$ contain a common non zero prime ideal. The $*$-SHDs include h-local domains, independent rings of Krull type, Krull domains, UFDs etc. We show also that we can modify the definition of the $*$-homog ideals to get a theory of each special case of a $*$-SH domain.

Dedicated to the memory of Professor P.M. Cohn

1. Introduction

Let $*$ be a finite character star operation defined on an integral domain D.(We will, in the following, introduce terminology necessary for reading this article.)

Call a nonzero $*$-ideal of finite type a $*$-homogeneous ($*$-homog) ideal, if $I \subsetneq D$ and $(J+K)^* \neq D$ for every pair $D \supsetneq J, K \supseteq I$ of proper $*$-ideals of finite type. To fix the ideas the simplest example of a $*$-homog ideal is an ideal generated by some positive power of a principal prime. The initial aim of this article is to show that if a $*$-ideal A is expressible as a $*$-product of finitely many $*$-homog ideals, then A is uniquely expressible as a $*$-product of finitely many mutually $*$-comaximal $*$-homog ideals. Call an integral domain D a $*$-Semi Homogeneous Domain ($*$-SH domain or $*$-SHD) if for every nonzero non unit x of D the ideal xD is expressible as a $*$-product of finitely many $*$-homog ideals. The purpose of this paper is then to show that a $*$-SHD is a $\mathcal{F}$-IFC domain of [11] that is a $*$-SHD contains a family of prime ideals $\mathcal{F}$ such that (a) $D = \cap_{P\in\mathcal{F}} D_P$ and the intersection is locally finite and (b) no two members of $\mathcal{F}$ contain a nonzero prime ideal. It turns out that the $*$-SHDs contain as special cases the h-local domains of Matlis [27], see also an important paper [28] about them by Olberding), independent rings of Krull type of Griffin [23], Krull domains, weakly Krull domains see [8], UFD's etc. What is special with our approach is that for each kind of domains we can modify the definition of the $*$-homog ideals to give a theory of that kind of domains. But before we explain that, let's bring in the above promised introduction to the terminology.

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Since this work is steeped in and dependent upon the unifying quality of star operations, it seems pertinent to give the reader a working knowledge of some of the notions involved. But before we delve into that, let's indicate the quite a few theories that we can generate, if we choose to ignore all mention of star-operations.

What we have developed here is a kind of "theory schema". Let us explain: You may want to give star-operations a wide berth, for some reason (s), but you want to have some idea of what we have developed. Here's what we can say: If you ignore the "*", completely and everywhere, you would get the main theory of SH domains that characterizes h-local domains of Matlis and if you change the definition of a homog ideal a little you would get the theory of unique factorization characterizing 1-dimensional h-local domains. Another change of definition of homog ideals gives you h-local Prufer domains. Proceeding this way and changing definitions of homog ideals judiciously, or as presented in this paper, you can get to the theories of Dedekind domains and PID's. If you have time and inclination, do look into the paper that way. Finally if you know of the easy to define star operation, called the t-operation, you would get another round of theories leading to Krull domains and various kinds of Krull domains and eventually to UFDs.

Let D be an integral domain with quotient field K. Let $F(D)$ (resp., $f(D)$) be the set of nonzero fractional ideals (resp., nonzero finitely generated fractional ideals) of D. A star operation $*$ on D is a function on $F(D)$ that satisfies the following properties for every $I, J \in F(D)$ and $0 \neq x \in K$:

(i) $(x)^* = (x)$ and $(xI)^* = xI^*$,
(ii) $I \subseteq I^*$, and $I^* \subseteq J^*$ whenever $I \subseteq J$, and
(iii) $(I^*)^* = I^*$.

Now, an ideal $I \in F(D)$ is a $*$-ideal if $I^* = I$, so a principal ideal is a $*$-ideal for every star operation $*$. Moreover $I \in F(D)$ is called a $*$-ideal of finite type if $I = J^*$ for some $J \in f(D)$. To each star operation $*$ we can associate a star operation $*_s$ defined by $I^{*_s} = \bigcup\{\, J^* \mid J \subseteq I \text{ and } J \in f(D)\,\}$. A star operation $*$ is said to be of finite type, if $I^* = I^{*_s}$ for all $I \in F(D)$. Indeed for each star operation $*$, $*_s$ is of finite character. Thus if $*$ is of finite character $I \in F(D)$ is a $*$-ideal if and only if for each finitely generated subideal J of I we have $J^* \subseteq I$. For $I \in F(D)$, let $I_d = I$, $I^{-1} = (D :_K I) = \{\, x \in K \mid xI \subseteq D\,\}$, $I_v = (I^{-1})^{-1}$, $I_t = \bigcup\{\, J_v \mid J \subseteq I \text{ and } J \in f(D)\,\}$, and $I_w = \{\, x \in K \mid xJ \subseteq I \text{ for some } J \in f(D)$ with $J_v = D\,\}$. The functions defined by $I \mapsto I_d$, $I \mapsto I_v$, $I \mapsto I_t$, and $I \mapsto I_w$ are all examples of star operations A v-ideal is sometimes also called a divisorial ideal. Given two star operations $*_1, *_2$ on D, we say that $*_1 \leq *_2$ if $I^{*_1} \subseteq I^{*_2}$ for every $I \in F(D)$. Note that $*_1 \leq *_2$ if and only if $(I^{*_1})^{*_2} = (I^{*_2})^{*_1} = I^{*_2}$ for every $I \in F(D)$. The d-operation, t-operation, and w-operation all have finite character, $d \leq \rho \leq v$ for every star operation ρ, and $\rho \leq t$ for every star operation ρ of finite character. We will often use the facts that (a) for every star operation $*$ and $I, J \in F(D)$, $(IJ)^* = (IJ^*)^* = (I^*J^*)^*$, (the $*$-product), (b) $(I + J)^* = (I + J^*)^* = (I^* + J^*)^*$ (the $*$-sum) and (c) $I_v = I_t$ for every $I \in f(D)$. An $I \in F(D)$ is said to be $*$-invertible, if $(II^{-1})^* = D$. If I is $*$-invertible for $*$ of finite character, then both I^* and I^{-1} are $*$-ideals of finite type. An integral domain D is called a Prufer $*$-Multiplication Domain (P$*$MD), for a general star operation $*$, if A is $*_s$-invertible for every $A \in f(D)$. Now let D be a P$*$MD. Because in a P$*$MD D, $F^* = F_v$ for each $F \in f(D)$, we have $A^{*_s} = A_t$ for each $A \in F(D)$. (When $*$ is of finite character, $* = *_s$ and so in such a P$*$MD D, we have $A^* = A_t$

for each $A \in F(D)$ and so $* = t$. Moreover, in a PdMD $d = t$, making a PdMD a Prufer domain.) A PvMD is often written as PVMD. A reader in need of more introduction may consult [37] or [22, Sections 32 and 34].

For a star operation $*$, a maximal $*$-ideal is an integral $*$-ideal that is maximal among proper integral $*$-ideals. Let $*$-Max(D) be the set of maximal $*$-ideals of D. For a star operation $*$ of finite character, it is well known that a maximal $*$-ideal is a prime ideal; every proper integral $*$-ideal is contained in a maximal $*$-ideal; and $*$-Max$(D) \neq \emptyset$ if D is not a field. For a star operation $*$ two ideals A, B may be called $*$-comaximal if $(A, B)^* = D$. Indeed if $*$ is of finite character then two ideals A, B are $*$-comaximal if, and only if, A, B do not share (being in) any maximal $*$-ideal M. Thus integral ideals $A_1, A_2, ..., A_n$ are $*$-comaximal to an ideal B if and only if $(A_1A_2...A_n, B)^* = D$. Next, $I_w = \bigcap_{M \in t\text{-Max}(D)} ID_M$ for every $I \in F(D)$ and $I_wD_M = ID_M$ for every $I \in F(D)$ and $M \in t$-Max(D). A $*$-operation that gets defined in terms of maximal $*$-ideals is denoted by $*_w$ and it is defined as follows: For $I \in F(D)$, and $I^{*_w} = \bigcap_{M \in *_s\text{-Max}(D)} ID_M$. This operation was introduced in [4] where it was established that for any star operation $*$, $*_w$ is a star operation of finite character and $*_w$-Max$(D) = *_s$-Max(D) and $*_w \leq *$, according to, again, [4]. An integral domain D is a P$*$MD if and only if D_M is a valuation domain for every maximal $*$-ideal M of D, [24]. Next, as the $*$-product $(IJ)^*$ of two $*$-invertible $*$-ideals is again $*$-invertible it is easy to see that $Inv_*(D) = \{I : I$ is a $*$-invertible $*$-ideal of $D\}$ is a group under $*$-multiplication with $P(D)$ the group of nonzero principal fractional ideals of D as its sub group. The quotient group $Inv_*(D)/P(D)$ is called the $*$-class group of D, denoted by $Cl_*(D)$. The $*$-class groups were introduced and studied by D.F. Anderson in [12] as a generalization of the t-class groups introduced in [14], [34] and further studied in [15]. It was shown in [12], in addition to many other insightful results, that if $*_1 \leq *_2$ are two star operations then $Cl_{*_1}(D) \subseteq Cl_{*_2}(D)$.

In section 2 we discuss and establish the main features of the general theory as described in the introduction and in section 3 we discuss the various examples or special cases of the $*$-SH domains, while in section 4 we discuss weaker or restricted theories such as weakly factorial domains and almost weakly factorial domains etc. where the $*$-homog ideals have certain properties under special circumstances. In this section we also give examples, as those examples do not frequent the general scene as often as those discussed in section 3.

2. Main Theory

For a start, to save on space, let us agree that throughout hence, $*$ will denote a star operation of finite character, defined on D, and that D will be reserved for an integral domain. Let's also recall from the introduction that a domain D is a $*$-SH domain, if for every nonzero non unit x of D the ideal xD is expressible as a $*$-product of $*$-homog ideals of D. We start with explaining what an ideal being $*$-homog means.

Let's, for a start, recall that we call a nonzero $*$-ideal of finite type a $*$-homogeneous ($*$-homog) ideal, if $I \subsetneq D$ and $(J + K)^* \neq D$ for every pair $D \supsetneq J, K \supseteq I$ of proper $*$-ideals of finite type.

Proposition 1. *Let I be a $*$-homog ideal of D. Define $M(I) = \{x \in D : (x, I)^* \neq D\}$. Then $M(I)$ is the unique maximal $*$-ideal of D containing I.*

Proof. Clearly $M(I)$ is an ideal. For, let $x, y \in M(I)$. Then $(x, I)^* \neq D \neq (y, I)^*$. But then, by definition, $((x, I)^* + (y, I)^*)^* \neq D$. Now $((x, I)^* + (y, I)^*)^* = ((x, y) + I)^*$. Also as $((x, y) + I)^* = ((x, y)^* + I)^*$, we conclude that for each $a \in (x, y)^*$ we have $(a, I)^* \neq D$ and so $x, y \in M(I)$ implies $(x, y)^* \subseteq M(I)$. Next, let $x_1, x_2, ..., x_n \in M(I)$. Then as we have already seen $(x_1, x_2)^* \subseteq M(I)$. Suppose that we have shown that for $x_1, x_2, ..., x_{n-1} \in M(I)$ we have $(x_1, x_2, ..., x_{n-1})^* \subseteq M(I)$. Then by definition $(((x_1, x_2, ..., x_{n-1}), I) + (x_n, I))^* \neq D$ and so $D \neq ((x_1, x_2, ..., x_n), I)^* = ((x_1, x_2, ..., x_n)^*, I)^*$ which of course means that for each $\alpha \in (x_1, x_2, ..., x_n)^*$, $(\alpha, I)^* \neq D$, i.e. for each positive integer n, $x_1, x_2, ..., x_n \in M(I)$ implies that $(x_1, x_2, ..., x_n)^* \subseteq M(I)$. That is, $M(I)$ is a $*$-ideal, because $*$ is of finite character. Indeed as for each $x \in I$ $(x, I)^* = I \neq D$ we have $I^* \subseteq M(I)$. Now let P be a maximal $*$-ideal containing $M(I)$, then since for each $x \in P$ we must have $(x, I)^* \neq D$, $P = M(I)$ and so $M(I)$ is the unique maximal $*$-ideal containing I. □

In [3] a finitely generated nonzero ideal I was called rigid if I belonged to exactly one maximal t-ideal, and the maximal ideal containing a rigid ideal was in turn called potent in [5]. Taking a cue from that a finitely generated ideal I was called $*$-rigid, in [25], if I belongs to exactly one maximal $*$-ideal. (In [20] a $*$-rigid ideal was called homogeneous.) Let us call $M(I)$, defined in the above proposition, the maximal $*$-ideal spawned by I.

Corollary 1. *Let I be a $*$-homog ideal. If I is contained in a prime ideal Q that is contained in some $*$-ideal then $Q \subseteq M(I)$.*

For the record we state and prove the following easy to prove result.

Corollary 2. *A nonzero $*$-ideal I of finite type is a $*$-homog ideal if and only if I is a $*$-rigid ideal.*

Proof. That a $*$-homog ideal I is $*$-rigid follows from the fact that $M(I)$ is the unique maximal $*$-ideal containing I. Conversely I is rigid because I is contained in a unique maximal ideal P and let A be a $*$-ideal containing I such that $A^* \neq D$, then A must be in P and in no other maximal $*$-ideal because A contains I. So if J and K are two proper $*$-ideals of finite type containing I then both J, K are contained in P and hence $(J + K)^* \neq D$. So, by the definition, every $*$-rigid ideal is $*$-homog. □

Remark 1. *(1) The converse in Corollary 2 gives the reason why this is the case that if an ideal A contains a power of a principal prime pD then $A_t \neq D$ only if $A \subseteq pD$. This is because each positive power of a principal prime indeed generates a rigid ideal. It also indicates that if two $*$-homog ideals are such that $M(I) \neq M(J)$ then $(I + J)^* = D$ that is I and J do not share a maximal $*$-ideal, which is obvious. (2). The idea of a $*$ homog ideal comes from* [20]. *(3) There may be a question as to why use $*$-homog when we already have $*$-rigid. My reason is partly choice and partly the fact that when we say, "I is $*$-rigid" we have to declare the maximal $*$-ideal M it belongs to. On the other hand, when we say,"I is $*$-homog" we do not have to worry about that, as I determines its own maximal $*$-ideal $M(I)$.*

Proposition 2. *Let I, J be two $*$-homog ideals and let K be a $*$-homog ideal such that $K \subseteq M(I) \cap M(J)$. Then $M(I) = M(J) = M(K)$. Consequently if I and J spawn two distinct maximal $*$-ideals then $M(I) \cap M(J)$ does not contain a $*$-homog ideal.*

Proof. Since $K \subseteq M(I)$ and $M(I)$ is a $*$-ideal, for each $x \in M(I)$, $(x, K)^* \subseteq M(I)$ and so for each $x \in M(I)$ $(x, K)^* \neq D$. But then $M(I) \subseteq M(K)$. But as $M(I)$ is a maximal $*$-ideal we have $M(I) = M(K)$. Similarly for $M(J)$. The consequently part is obvious. □

We shall call two or more $*$-homog ideals similar if they spawn (are contained in) the same maximal $*$-ideal. Indeed it is easy to deduce from the criterion of similarity that similarity of $*$-rigid ideals is an equivalence relation. If A and B are $*$-homog ideals spawning distinct maximal $*$-ideals we may call A, B dissimilar. Also by Remark 1, dissimilar means $*$-comaximal.

Corollary 3. *If I and J are two similar $*$-homog ideals then $(IJ)^*$ is a $*$-homog ideal similar to them. Consequently any $*$-product of mutually similar $*$- homog ideals is similar to each of them.*

Proof. Suppose that $(IJ)^*$ is contained in another maximal $*$-ideal P then as, say, $I \subseteq P$ we have $P = M(I)$. Similarly if there is a maximal $*$-ideal Q containing, say, J then $Q = M(J) = M(I)$. We can deal with the consequently part using induction. □

Proposition 3. *Let I be a $*$-homog ideal of D. Then $ID_{M(I)} \cap D = I$.*

Proof. Note that since I is a $*$-ideal, $I = I^{*_w} = \bigcap_{P \in *\text{-}\max(D)} I^* D_P$, because $*_w \leq *$. But since I is $*$-homog $ID_P = D_P$ for all maximal $*$-ideals P other than $M(I)$, $I = (I)^{*_w} = ID_{M(I)} \cap (\bigcap_{P \in *\text{-}\max(D) \setminus M(I)} D_P)$. So, $I = ID_{M(I)} \cap D \neq D$ as $I \subseteq M(I)$. □

Corollary 4. *Let I be a $*$-homog ideal of D and let A, B be ideals of D such that A, B are $*$-comaximal, i.e. $(A+B)^* = D$, (i) If $AB \subseteq I$ then $A \subseteq I$ or $B \subseteq I$. (ii) If B is $*$-homog and $(A+B)^* = D$, then $(AB)^* D_{M(B)} \cap D = B$.*

Proof. (i) Clearly if $AB \subseteq I$ then $AB \subseteq M(I)$ which implies $A \subseteq M(I)$ or $B \subseteq M(I)$ but not both, because A, B are $*$-comaximal. Now if $A \subseteq M(I)$ then $AB \subseteq I$ implies $ABD_{M(I)} = AD_{M(I)} \subseteq ID_{M(I)}$. This in turn implies that $A \subseteq AD_{M(I)} \cap D \subseteq ID_{M(I)} \cap D = I$. (ii) As $AB \subseteq (AB)^*$ and as $A \nsubseteq M(B)$ we conclude that $BD_{M(B)} = ABD_{M(B)} \subseteq (AB)^* D_{M(B)}$ and so $B = BD_{M(B)} \cap D \subseteq (AB)^* D_{M(B)} \cap D$. But already $AB \subseteq B$ and so $(AB)^* \subseteq B$ we have $(AB)^* D_{M(B)} \cap D \subseteq BD_{M(B)} \cap D = B$. □

Proposition 4. *If an ideal A is expressible as a $*$-product of $*$-homog ideals, then A is expressible, uniquely, up to order, as a $*$-product of mutually $*$-comaximal $*$-homog ideals.*

Proof. Let $A = (I_1 I_2 ... I_m)^*$. Pick a $*$-homog factor say I_1 and collect all the $*$-homog factors of A that are similar to I_1. Next suppose that by a relabeling $I_1, I_2, ..., I_{n_1}$ are all similar to $I_1$and all the remaining ideals are dissimilar to I_1 and hence to all of $I_1, I_2, ..., I_{n_1}$.Then by Corollary 3, $J_1 = (I_1 I_2 ... I_{n_1})^*$ is a $*$-homog ideal. So $A = (J_1 I_{n_1+1} ... I_m)^*$ where none of the I_{n_1+i} spawns the same maximal $*$-ideal as $M(J_1)$. Now collect all the factors similar to I_{n_1+1} and suppose, by a relabeling, that those factors are all of $I_{n_1+1}, I_{n_1+2}, ..., I_{n_2}$ and the rest are all dissimilar to I_{n_1+1}. Then by setting $J_2 = (I_{n_1+1} I_{n_1+2} ... I_{n_2})^*$ we have $A =$

$(J_1J_2I_{n_2+1}...I_m)^*$ where J_1, J_2 are $*$-comaximal to each other and all remaining I_i. Continuing thus we can end up with $A = (J_1J_2...J_r)^*$ where J_i are mutually $*$-comaximal. Since J_i are mutually $*$-comaximal, and so no two can be in the same maximal $*$-ideal, $AD_{M(J_i)} \cap D = J_i$, by Corollary 4. Now suppose that A has another expression $A = (K_1K_2...K_s)^*$ as a $*$-product of mutually $*$-comaximal $*$-homog ideals. Then $A = (J_1J_2...J_r)^* = (K_1K_2...K_s)^*$. As $A \subseteq J_1 \subseteq M(J_1)$, we must have $K_i \subseteq M(J_1)$ for some i, and as K_j are mutually$*$-comaximal, $K_j \nsubseteq M(J_1)$ for $j \neq i$. But then $M(K_i) = M(J_1)$. Next by Corollary 4, $AD_{M(J_1)} \cap D = J_1$ and $AD_{M(K_i)} \cap D = K_i$. But as $M(K_i) = M(J_1)$, we conclude that $J_1 = K_i$. Thus for each J_i there is a K_j such that $J_i = K_j$ and $r \leq s$. Indeed as K_j are mutually $*$-comaximal, there is a unique K_j to each J_i. Similarly, starting with Ks from the right side we can show that $s \leq r$, thus establishing that $r = s$. □

Corollary 5. *Let D be a $*$-SH domain, then each principal ideal xD generated by a nonzero non unit is uniquely expressible as a $*$-product of mutually $*$-comaximal $*$-homog ideals, each of which is $*$-invertible. Also, if M is a maximal $*$-ideal containing x then $xD_M \cap D$ is a $*$-invertible $*$-homog ideal.*

Proof. Since D is a $*$-SHD, for every nonzero non unit $x \in D$, $xD = (J_1J_2...J_s)^*$ where each J_i is a $*$-homog ideal. By Proposition 4 we can write $xD = (I_1I_2...I_r)^*$ where I_i are mutually $*$-co-maximal $*$-homog ideals. Also as xD is $*$-invertible, each of I_i is $*$-invertible. Finally if M is a maximal $*$ containing x then because I_i are mutually $*$-comaximal M contains exactly one of the I_i, say I_k. So $M = M(I_k)$. But then by Corollary 4 applies and we get $xD_M \cap D = I_k$. □

Recall that for a finite character star operation $*$ defined on D, D is of finite $*$-character if every nonzero non unit element of D belongs to at most a finite number of maximal $*$-ideals. Recall also that if P and Q are two prime ideals of D such that no nonzero prime ideal is contained in $P \cap Q$ then $D_PD_Q = K$ the quotient field of D [11, Lemma 4.1].

Theorem 1. *The following are equivalent for an integral domain D. (1) D is a $*$-SH domain, (2) D is of finite $*$-character and for every pair P, Q of distinct maximal $*$-ideals $P \cap Q$ does not contain a nonzero prime ideal.*

Proof. (1) ⇒ (2). Suppose that D is a $*$-SH domain and let x be a nonzero non unit of D. Then $xD = (I_1I_2...I_r)^*$ where I_i are $*$-homog ideals. Since each $I_i \subseteq M(I_i)$ which is a unique maximal $*$-ideal, by Proposition 1, we conclude that x belongs to at most r maximal $*$-ideals. Also if P is a maximal $*$-ideal of D then for $x \in P\backslash\{0\}$ $xD = (I_1I_2...I_r)^*$ and so at least one of I_i, say $I_j \subseteq P$. But then by Corollary 1, $P \subseteq M(I_j)$. Since P is a maximal $*$-ideal $P = M(I_j)$. Thus for each maximal $*$-ideal P of a $*$-SHD D, there is a $*$-homog ideal I such that $P = M(I)$. Now let P and Q be two distinct maximal $*$-ideals in a $*$-SH domain D. As we have established above, there exist $*$-homog ideals I, J such that $P = M(I)$ and $Q = M(J)$. Now suppose that there is a nonzero prime ideal $m \subseteq P \cap Q$. Then as m is a nonzero prime ideal, m contains a nonzero element and hence a $*$-homog ideal A, which is impossible by Proposition 2, because $P = M(I)$ and $Q = M(J)$ are distinct. We next show (2) ⇒ (1). Suppose that D is of finite $*$ character and that no two maximal $*$-ideals P, Q contain a nonzero prime ideal. Let x be a nonzero non unit element of D. Let $T = \{P_1, P_2, ..., P_r\}$ be the set of all the maximal $*$-ideals containing

x. Then $(x) = \bigcap_{P\in *\text{-}Max(D)} xD_P = xD_{P_1} \cap xD_{P_2} \cap ... \cap xD_{P_r} \cap (\bigcap_{Q\in *\text{-}Max(D)\setminus T} D_Q)$ $= (xD_{P_1} \cap xD_{P_2} \cap ... \cap xD_{P_r}) \cap D = \bigcap_{i=1}^{r}(xD_{P_i} \cap D)$. We now proceed to show that $xD_{P_i} \cap D$ is contained in P_i and to no other maximal $*$ -ideal for each $i = 1, .., r$. Indeed for any maximal $*$-ideal Q other than P_i we have $D_{P_i}D_Q = K$ [11, Lemma 4.1] we have $(xD_{P_i} \cap D)D_Q = xD_{P_i}D_Q \cap D_Q = K \cap D_Q = D_Q$. So $xD_{P_i} \cap D$ is not contained in any maximal $*$-ideal other than P_i. Using this piece of information we see that $(\prod_{i=1}^{r}(xD_{P_i} \cap D))^{*_w} = (\bigcap_{i=1}^{r} xD_{P_i} \cap D)^{*_w} = \bigcap_{i=1}^{r} xD_{P_i} \cap D = (x)$ and as $*_w \leq *$ we have $(x) = (\prod_{i=1}^{r}(xD_{P_i} \cap D))^{*}$. That is each $(xD_{P_i} \cap D)$ is $*$-invertible and hence of finite type and consequently is a $*$-homog ideal, being also a $*$-ideal. Thus for each nonzero non unit x of D, xD is expressible as a finite $*$-product of $*$-homog ideals and D is a $*$-SH domain. □

The proof of (2) ⇒ (1) of Theorem 1 shows that Theorem 1 could have been replaced by another interesting result, if we were to use the terminology of [11]. The terminology can be described as follows. Let $\mathcal{F} = \{P_\alpha : \alpha \in I\}$ be a family of nonzero prime ideals of D. $\mathcal{F}$ is called a defining family of D if $D = \bigcap_{\alpha\in I} D_{P_\alpha}$. The defining family $\mathcal{F}$ is of finite character if no nonzero non unit of D belongs to infinitely many members of $\mathcal{F}$. We may call the defining family $\mathcal{F}$ independent if no two members of $\mathcal{F}$ contain a common nonzero prime ideal. The function $*_{\mathcal{F}}$ on $F(D)$ defined by $A \mapsto A^{*_{\mathcal{F}}} = \bigcap_{\alpha\in I} AD_{P_\alpha}$ is called a star operation induced by the family $\{D_{P_\alpha}\}$ of localizations at members of $\mathcal{F}$. (We shall, in what follows, introduce concepts to facilitate reading of the paper.) In [11], an integral ideal A of D was called unidirectional if A belongs to a unique member of the defining family $\mathcal{F}$ of primes. With this terminology it was shown in [11, Theorem 2.1] that if D, $\mathcal{F}$, $*_{\mathcal{F}}$ are defined as above and if $*_{\mathcal{F}}$ is of finite character then the family $\mathcal{F}$ is independent of finite character if and only if every nonzero non unit element x of D, xD is expressible as a $*_{\mathcal{F}}$-product of a finite number of unidirectional ideals.

Now, if we match $\mathcal{F}$ with $*$-$Max(D)$, $*_{\mathcal{F}}$ gets matched with $*_w$ and unidirectional ideal with $*$-homog ideal we can restate Theorem 3.3 of [11] as the following result.

Theorem 2. *Let $*$ be a star operation of finite character defined on an integral domain D. Then the following are equivalent: 1. D is of finite $*$-character and for any two distinct $P, Q \in *$-$Max(D)$ $P \cap Q$ does not contain a nonzero prime ideal, 2. every nonzero prime ideal of D contains an element x such that xD is a $*_w$-product of $*$-homog ideals (Note that as $*_w \leq *$, "$*_w$-product" can be replaced by "$*$-product", here.), 3. every nonzero prime ideal of D contains a $*$-homog $*$-invertible $*$-ideal, 4. for $P \in *$-$Max(D)$ and $0 \neq x \in P$, $xD_P \cap D$ is a $*$-invertible and $*$-homog ideal (In* [11] *$0 \neq x \in D$ was mistakenly typed in place of $0 \neq x \in P$.), 5. no pair of distinct maximal $*$-ideals contains a nonzero prime ideal and for any nonzero ideal A of D, A^{*_w} is of finite type. whenever AD_P is finitely generated for all P in $*$-$Max(D)$.*

Note that Theorem 1 proves the equivalence of (1) and (2) of Theorem 2 and that is grounds enough to include Theorem 2 as part of this paper. On the other hand the theory developed in [11] is not enough to take care of the more general approach in this paper. There is, of course, another important difference. While [11] takes care of independent rings of Krull type and Krull domains by requiring that for each $P \in \mathcal{F}$, D_P is a valuation domain, and requiring for Krull domains that D_P is a rank one DVR for each $P \in \mathcal{F}$, the theory presented here lets us define a $*$-homog ideal to fit the picture. For instance we can define, as we show in the following, a $*$-homog ideal to establish the theory of independent rings of Krull type, or of Krull domains etc. In each case, obviously, Theorem 1 and Theorem 2 are ready proved and all we need show is that the resulting theory has the distinctive feature that we claim it has. In what follows, in the next section we make a few demonstrations that list the variations of the definition of $*$-homog ideals and the domains they lead to.

In view of a comment, at the start of section 3, in [11] we may also call the domains characterized in Theorems 1, 2 $*$-h-local, as the domains whose principal ideals generated by nonzero non units are expressible as $*$-products of finitely many $*$-homog ideals, noting that when $* = d$ we have the usual definition of h-local domains of Matlis [27] and when $* = t$ we have what we termed as weakly Matlis domains in [11]. The interesting part of this approach is that, as we demonstrate below, we can redefine the $*$-homog ideals to fit the various special cases of $*$-SH domains.

3. Clones or examples of $*$-SHDs

Let's call D a $*$-weakly Krull domain ($*$-WKD) if D is a $*$-SHD such that each maximal $*$-ideal P of D is of height 1. These domains are known as weakly Krull domains and were first studied in [8].

Definition 1. *Call a $*$-homog ideal I, $*$-homog of type 1, if for every $x \in M(I)\backslash\{0\}$ there is a positive integer n such that $x^n D_{M(I)} \cap D \subseteq I$. Also call a domain D a $*$-SH domain of type 1 if for every nonzero non unit x of D, xD is a $*$-product of finitely many $*$-homog ideals of type 1.*

Indeed if I and J are $*$-homog of type 1 then so is $(IJ)^*$. This is because $(IJ)^*$ is a $*$-homog ideal similar to both I and J, to start with. So, $M(I) = M(J)$. Now let $x \in M(I)\backslash\{0\}$. Then for some positive integers m, n we have $x^m D_{M(I)} \cap D \subseteq I$ and $x^n D_{M(I)} \cap D \subseteq J$. This gives $(x^m D_{M(I)} \cap D)(x^n D_{M(I)} \cap D) \subseteq IJ \subseteq (IJ)^*$. But then $(IJ)^* = (IJ)^* D_{M(I)} \cap D \supseteq (x^m D_{M(I)} \cap D)(x^n D_{M(I)} \cap D))D_{M(I)} \cap D$ $= x^{m+n} D_{M(I)} \cap D$, for each $x \in M(I)$.

Theorem 3. *Let D be an integral domain and suppose that D is a $*$-SHD of type 1. Then D is a $*$-WKD. Conversely if D is a $*$-WKD, then every nonzero proper principal ideal of D is expressible as a $*$-product of finitely many $*$-homog ideals of type 1, i.e. D is a $*$-SHD of type 1.*

Proof. All we need prove is that if every nonzero proper principal ideal of D is expressible as a finite $*$-product of $*$-homog ideals of type 1 then for each maximal $*$-ideal M we have $ht(M) = 1$. For this let us first observe that if D is a $*$-SHD with M a maximal $*$-ideal of D then by Corollary 5, $xD_M \cap D$ is a $*$-invertible $*$-homog ideal. Indeed as in the proof of Corollary 5, $xD = (I_1 I_2 ... I_r)^*$ where I_i are mutually

$*$-comaximal and of type 1, because x is a product of $*$-homog ideals of type 1. Also as M is a maximal $*$-ideal, $xD_M \cap D = I_i$ for some i, by the proof of Corollary 5. Now let aD_M, bD_M be two nonzero non units in D_M. We can assume that $a, b \in D$. So that $aD_M \cap D$ is of type 1 and $b \in M$ and so by definition $b^m D_M \cap D \subseteq aD_M \cap D$, for some positive integer m. But then $(b^m D_M \cap D)D_M \subseteq (aD_M \cap D)D_M$ and this means $b^m D_M \subseteq aD_M$. Since a and b are arbitrary, we conclude that MD_M is of height one. (We have $b^m D_M \subseteq aD_M$, for some m, so bD_M belongs to every nonzero prime ideal aD_M belongs to. Similarly the other way around and indeed this forces M to be of height one.) This makes the, otherwise, $*$-h-local domain a $*$-WKD. Conversely if D is a $*$-WKD. Then as we know from Theorems 1 and 2 every ideal generated by a nonzero non unit of D is a $*$-product of $*$-homog ideals and we need to show these $*$-homog ideals are of type 1. Now all we need do is show that for each nonzero x and a maximal $*$-ideal M, containing x, the ideal $xD_M \cap D$ is of type 1. For this let $a \in M \backslash (0)$. Then aD_M and xD_M are nonzero non units of MD_M which is of height one and so $a^m D_M \subseteq xD_M$ for some positive integer m. But then $a^m D_M \cap D \subseteq xD_M \cap D$ which makes $xD_M \cap D$ of type 1. □

Recall that, as we hinted in relation with Theorem 2 that D is a Krull domain if $D = \bigcap D_P$ where the intersection is locally finite and each D_P is a discrete valuation domain. Let's call a $*$-WKD a $*$-Krull domain if for each maximal $*$-ideal P of D, the localization D_P is a discrete rank one valuation domain. Now note that a $*$-WKD D is $*$-Krull if and only if every maximal $*$-ideal of D is $*$-invertible. Since if $*$ is of finite type then every $*$-invertible $*$-ideal is a t-invertible t-ideal [37, Theorem 1.1] and for a t-invertible prime t-ideal P of height 1, D_P is a discrete valuation domain, because t-invertible extends to t-invertible in localizations [37, page 436, consequence (a)] PD_P is t-invertible and because PD_P is of height one, PD_P is a t-ideal and in a t-local domain (i.e. maximal ideal is a t-ideal.) t-invertible is principal [5, Proposition 1.12]. So, in view of the definitions of a Krull domain, a $*$-Krull domain is a Krull domain. It appears that a definition that links the $*$-homog ideal with this fact can be worded as below.

Definition 2. *Call a $*$-homog ideal I, of type* 2 *if for some positive integer n, $I = ((M(I))^n)^*$. Also call a domain D a $*$-SH domain of type* 2 *if for every nonzero x in D, xD is expressible as a $*$-product of a finite number of $*$-homog ideals of type* 2.

Indeed if I and J are both $*$-homog of type 2 then $(IJ)^* = ((M(I))^n(M(I))^p)^*$ and that makes $(IJ)^*$ of type 2.

Theorem 4. *Let D be an integral domain and suppose that D is a $*$-SH domain of type* 2*. Then D is a $*$-Krull domain. Conversely if D is a $*$-Krull domain, then every nonzero proper principal ideal of D is expressible as a $*$-product of finitely many $*$-homog ideals of type* 2.

Proof. Indeed a $*$-homog ideal I that is of type 2 is of type 1 as well, because if $x \in M(I)$, then $xD_{M(I)} \cap D \subseteq M(I)$ and so $x^n D \subseteq M(I)^n \subseteq I$. But then $x^n D_M \cap D \subseteq ID_{M(I)} \cap I = I$, by Proposition 3. So, D is a $*$-WKD. Next, let M be a maximal $*$-ideal and let x be a nonzero element in M. Because $xD = (I_1 I_2 ... I_n)^*$ where each of the I_i is a $*$-homog ideal of type 2, and each of I_i is $*$-invertible. Also at least one of I_i, say I_j, is contained in M. But as I_j is $*$-homog and as M is a maximal $*$-ideal, $M = M(I_j)$. Finally as I_j is of type 2, $I_j^* = (M^n)^*$. This makes M $*$-invertible.

But as $*$ is of finite type, M being $*$-invertible means M is t-invertible and so is MD_M, [37, page 436, consequence (a)]. Next as MD_M is of height one, MD_M is a t-ideal and D_M is a t-local ring. But in a t-local ring t-invertible is principal, [5, Proposition 1.12]. But this makes D_M a one dimensional quasi local domain with maximal ideal principal and so a rank one DVR. Now since M was arbitrary and D is of finite t-character D is $*$-Krull, as defined above. Conversely, note that a $*$-Krull domain D is a Krull domain, as we have already established and every maximal $*$-ideal of D is $*$-invertible and hence a t-invertible t-ideal. Now it is well known that D is a Krull domain if and only if every proper principal ideal of D is a t-product of prime t-ideals [8, Corollary 3.2]. So, $xD = (P_1...P_n)_t$. Moreover, as a $*$-Krull domain is a P$*$MD, because D_M is a valuation domain for every maximal $*$-ideal M, $* = t$ and thus $(P_1...P_n)^* = (P_1...P_n)_t = xD$. Now as in a Krull domain each prime t-ideal is a maximal t-ideal which is a maximal $*$-ideal we can say that in a $*$-Krull domain every proper $*$-ideal is a $*$-product of maximal $*$-ideals. Finally as each maximal $*$-ideal P in a $*$-Krull domain is a $*$-ideal of finite type, being $*$-invertible, it is obviously $*$-homog of type 2. □

Definition 3. *A nonzero integral $*$-ideal I of finite type is called $*$-super homogeneous ($*$-super homog) if (1) if each integral $*$-ideal of finite type containing I is $*$-invertible and (2) For every pair of proper integral $*$-ideals A, B of finite type containing I, $(A+B)^* \neq D$.*

Remark 2. *Note that since every $*$-ideal of finite type is $*$-invertible in a P$*$MD a $*$-homog ideal is $*$-super homog in a P$*$MD. Indeed, as the definition indicates, a $*$-super homog ideal I is a $*$-homog ideal such that each integral$*$-ideal of finite type containing I is $*$-invertible, in particular a $*$-super homog ideal is $*$-invertible. Note that a $*$-super homog ideal is a $*$-super rigid ideal of* [25]. *Some properties of $*$-super rigid (i.e. $*$-super homog) ideals are given in* [25, Theorem 1.10]. *We list them, with one addition (2'), here in the language of the present paper, even though some of them have been proved more in more general setting above.*

Theorem 5. *Let I be a $*$-super homog ideal of D and suppose that $M = M(I)$.*

(1) If A is a proper finitely generated ideal for which $A \supseteq I$, then A is $*$-super homog.

(2) If J is a $*$-super homog ideal contained in M, then $I \subseteq J^*$or $J \subseteq I^*$.

(2') If J is a $*$-homog ideal contained in M such that $I \nsubseteq J$ then $J \subseteq I^*$

(3) If J is a $*$-super homog ideal contained in M, then IJ is also a $*$-super homog ideal.

(4) I^n is $*$-super homog for each positive integer n.

(5) If D is local with maximal ideal M, then I is comparable to each ideal of D, and $\cap_{n=1}^{\infty} I^n$ is prime.

(6) $I^* = ID_M \cap D$.

(7) $\cap_{n=1}^{\infty}(I^n)^*$ is prime.

(8) If P is a prime ideal of D with $P \subseteq M$ and $I \nsubseteq P$, then $P \subseteq \cap_{n=1}^{\infty}(I^n)^*$.

Proof. of (2). Let J be a $\star$-super homog ideal contained in M, and set $C := I + J$. Then C is $\star$-invertible, and we have $(IC^{-1} + JC^{-1})^\star = R$. Note that $IC^{-1} \supseteq I$ and $JC^{-1} \supseteq J$, and hence $IC^{-1} \nsubseteq M$ or $JC^{-1} \nsubseteq M$. Since IC^{-1}, JC^{-1} can be contained in no maximal $\star$-ideal of R other than M, we must have $(IC^{-1})^\star = R$ or $(JC^{-1})^\star = R$, that is, $C^\star = I^\star$ or $C^\star = J^\star$. The conclusion follows easily.

Proof of (2'). Let J be a $\star$- homog ideal contained in M such that $I \nsubseteq J$, and set $C := I + J$. Then C is $\star$-invertible, and we have $(IC^{-1} + JC^{-1})^\star = R$. Note that $IC^{-1} \supseteq I$ and $JC^{-1} \supseteq J$, and hence $IC^{-1} \nsubseteq M$ or $JC^{-1} \nsubseteq M$. Since IC^{-1}, JC^{-1} can be contained in no maximal $\star$-ideal of R other than M, we must have $(IC^{-1})^\star = R$ or $(JC^{-1})^\star = R$. Now $(JC^{-1})^\star \neq R$ for $(JC^{-1})^\star = R$ would give us $J^* = C^* \supseteq I$ which is not the case. Hence $(IC^{-1})^\star = R$. This gives $I^* = C^*$ forcing $J \subseteq I^*$.

Proof of (3). We note that $(IJ)^*$ is $*$-homog, being a $*$-product of two similar $*$-homog ideals. Let K be an integral $*$-ideal of finite type such that $K \supseteq (IJ)^*$. If K contains either of I, J then by definition K is $*$-invertible. So, let's suppose K contains neither of I, J. Then, by (2'), $K \subseteq I, J$. But then, as J is $*$-invertible, being $*$-super homog $D \supseteq (KJ^{-1})^* \supseteq I$. Again as I is $*$-super homog and $(KJ^{-1})^*$ is an integral $*$-ideal of finite type we conclude that $(KJ^{-1})^*$ and hence K is $*$-invertible. So, the $*$-homog ideal $(IJ)^*$ is such that each integral $*$-ideal of finite type containing $(IJ)^*$ is $*$-invertible and hence is a $*$-super homog ideal. □

Using the proof of (3) we can show that if $I_1, I_2, ..., I_r$ are $*$-super homog ideals similar to each other then $(I_1 I_2 ... I_r)^*$ is a $*$-super homog ideal similar to each of I_i. Now Let $A = (J_1 J_2 ... J_n)^*$ be a $*$-product of a finite number of $*$-super homog ideals. Then as we can regroup them into classes of similar $*$-super homog ideals as in the proof of Proposition 4, we can write $A = (K_1 K_2 ... K_m)^*$ where K_i^* are mutually $*$-comaximal. But this expression is unique being a $*$-product of mutually $*$-comaximal $*$-homog ideals as shown in the proof of Proposition 4. We have thus proved the following proposition.

Proposition 5. *Let $J_1, J_2, ..., J_n$ be a set of $*$-super homog ideals of a domain D. Then the $*$-product $(J_1 J_2 ... J_n)^*$ can be expressed uniquely, up to order, as a $*$-product of mutually $*$-comaximal $*$-super homog ideals.*

To make an efficient use of the material we have put together let us recall that an integral domain D an independent ring of Krull type (IRKT) if (1) There is a family $F = \{P_\alpha\}$ of prime ideals such that D_{P_α} is a valuation domain for each $P_\alpha \in F$. (2) $D = \cap D_{P_\alpha}$ and the intersection is locally finite and (3) No two members of F contain as a subset a nonzero prime ideal of D. Independent rings of Krull type were studied by Griffin [23]. Let us call D a $*$-independent ring of Krull type ($*$-IRKT), for a star operation $*$ of finite type, if (i) D_P is a valuation domain for each maximal $*$-ideal P, (ii) $D = \cap D_P$, the intersection is locally finite and P ranges over maximal $*$-ideals of D, (3) No two distinct maximal $*$-ideals P and Q contain a nonzero prime ideal in common. In other words a $*$-IRKT is a $*$-SH domain such that D_P is a valuation domain for each maximal $*$-ideal P of D. Now recall, again, that a domain D is called a P$*$MD if every finitely generated nonzero ideal of D is $*$-invertible and one of the characterizations of a P$*$MD is that D_P is a valuation domain for each maximal $*$-ideal P of D [24] and indeed a $*$-IRKT is a P$*$MD, as we have noted above. Let's also note that a $*$-IRKT is an IRKT and there is mixed opinion on whether there are any $*$-IRKTs, for finite type $*$ different from d and t. If $* = d$ the $*$-IRKT is indeed a Prufer domain. The situation gets complicated in view of the fact that for any finite type star operation $*$ a $*$-invertible ideal is t-invertible [37, Theorem 1.1]. In any case, even d and t causing two different kinds of domains makes the case for the use of a general $*$-operation approach sufficiently strong. We shall call a $*$-SH domain whose $*$-homog ideals are $*$-super homog a

$*$-super SH domain. In general a $*$-homog ideal I is a $*$-super homog ideal in a $\dot{\text{P}}*$MD, with $*$ of finite type, because every $*$-ideal F of finite type containing I is $*$-invertible.

We now proceed to show that if for every nonzero non unit x of a domain D, xD is a $*$-product of $*$-super homog ideals then D is a $*$-IRKT. Note that since a $*$-super homog ideal is $*$- homog, a domain D whose principal ideals generated by nonzero non units are $*$-products of $*$-super homog ideals is $*$-h-local to start with. All we have to do is show that for each maximal $*$-ideal P of D the localization D_P is a valuation domain. For this all we need show is that xD_P and yD_P are comparable for every pair of nonzero non units x, y in D_P. As we can assume that $x, y \in D$, we have that $xD_P \cap D, yD_P \cap D$ are $*$-homog by Corollary 5. Indeed $xD = (I_I I_2 ... I_r)^*$ where I_i are mutually $*$-comaximal $*$-super homog ideals in the current situation. Then the maximal $*$-ideal P contains exactly one of the $*$-super homog ideals I_i, say $I_1 \subseteq P$. That is $P = M(I_1)$. But then by Corollary 4 $I_1^* = xD_{M(I_1)} \cap D = xD_P \cap D$ is a $*$-super homog ideal. Similarly $yD_P \cap D$ is a $*$-super homog ideal. By (2) of Theorem 5, $xD_P \cap D \subseteq yD_P \cap D$ or $xD_P \cap D \supseteq yD_P \cap D$, because $xD_P \cap D, yD_P \cap D$ are $*$-super homog ideals contained in the same maximal $*$-ideal. Now "$xD_P \cap D, yD_P \cap D$ comparable" translates to $xD_P = (xD_P \cap D)D_P$, $yD_P = (yD_P \cap D)D_P$ comparable for each pair x, y of nonzero non units of D_P. That is, D_P is a valuation domain. Conversely if D is a $*$-IRKT, then using Theorem 2 we can establish that for every nonzero non unit x of D, xD expressible as a $*$-product of finitely many $*$-homog ideals of the form $xD_P \cap D$. But as a $*$-IRKT is a P$*$MD, every $*$ ideal of finite type is $*$-invertible, so every $*$-ideal F of finite type containing $xD_P \cap D$ is $*$-invertible making $xD_P \cap D$ a $*$-super homog ideal. In other words we have the following result.

Proposition 6. *The following are equivalent for an integral domain D : (1) D is a $*$-super SH domain, i.e., every nonzero non unit x of the domain D, xD is expressible as a $*$-product of finitely many $*$-super homog ideals (2) D is a $*$-IRKT).*

Note that in a $*$-IRKT every maximal $*$-ideal M contains at least one $*$-super homog ideal and so must be spawned by a $*$-super homog ideal. So, for $* = d$, d-IRKT is a $*$-IRKT in which every maximal $*$-ideal is a maximal ideal. Now, by Proposition 6, in a $*$-IRKT D, we have D_P a valuation domain for every maximal $*$-ideal P. So a d-IRKT is a Prufer domain.

Recall that a domain D is called a generalized Krull domain (GKD) if there is a family $\mathcal{F}$ of height one primes such that (1) $D = \bigcap_{P \in \mathcal{F}} D_P$ where the intersection is locally finite and D_P is a valuation ring for each $P \in \mathcal{F}$. Indeed a GKD is an IRKT. Following the pattern we can say that a $*$-IRKT whose maximal $*$-ideals are of height one is a $*$-GKD. Indeed a d-GKD is a Prufer domain

Definition 4. *Call a $*$-super homog ideal I a $*$-super homog ideal of type* 1*, if I is also a $*$-homog ideal of type* 1.

Indeed as the $*$-product of two $*$-homog ideals of type 1 is $*$-homog of type 1, the $*$-product of two $*$-super homog ideals of type 1 is a $*$-super homog ideal of type 1 and the theory runs along lines parallel to the theory based on $*$-homog ideals of type 1.

Definition 5. *Call a domain D a $*$-super SH domain of type* 1 *if for every nonzero non unit x of D the principal ideal xD is expressible as a $*$-product of $*$-super homog ideals of type* 1.

Proposition 7. *The following are equivalent for an integral domain D : (1) For every nonzero non unit x of the domain D, xD is expressible as a $*$-product of finitely many $*$-super homog ideals of type* 1*,i.e. D is a $*$-super SH domain of type* 1 *(2) D is a $*$-GKD.*

Proof. By Proposition 6, D is a $*$-super SH domain ($*$-IRKT) and by Theorem 3 D is $*$-WKD. Thus D is a $*$-GKD. The converse can be proved in the same manner as the converse of Proposition 6 was. That is by assuming that D is a $*$-GKD, then using Theorem 2 we can establish that for every nonzero non unit x of D, xD expressible as a $*$-product of finitely many $*$-homog ideals of the form $xD_P \cap D$. But as D_P is a rank one valuation domain for each maximal $*$-ideal P, every finite type $*$-ideal I containing $xD_P \cap D$ would have to be $*$-invertible because a $*$-GKD is a P$*$MD, so $xD_P \cap D$ is a $*$-super homog ideal. □

Proposition 8. *For each pair a, b of nonzero non units in a $*$-IRKT (i.e. a $*$-super SH domain) D, $(a, b)^* = D$ or $(a, b)^* =$ a $*$-product of $*$-super homog ideals I, each containing both a and b such that $(a, b)D_{M(I)} = aD_{M(I)}$ or $(a, b)D_{M(I)} = bD_{M(I)}$.*

Proof. Because a, b are nonzero non units of a $*$-IRKT D, we can write $(a) = (I_1I_2....I_l)^*$, where I_i are mutually $*$-comaximal $*$-super homog ideals and similarly we can write $(b) = (J_1J_2....J_m)^*$, where $m \geq n$ and J_j are mutually $*$-comaximal $*$-super homog ideals. If for some i, I_i is similar to some J_j, then (I_i, J_j) is a unique pair in that I_i is $*$-comaximal with each of the other $*$-super homog ideals appearing in the expression for (a) above and similarly for J_j. We conclude that $(I_i, b)^* \neq D$ and similarly $(J_j, a)^* \neq D$ and that there are exactly the same number of I_is that have $(I_i, b)^* \neq D$ as J_js that have $(J_j, a)^* \neq D$. Suppose that, by a relabeling if necessary, $I_1, I_2, ..., I_r$ are all the I_i such that $(I_i, b)^* \neq D$, and $J_1, J_2, ..., J_r$ are all the J_j such that $(J_j, a)^* \neq D$ Thus $(a) = (I_1I_2...I_rI_{r+1}...I_l)^*$ such that each of the $I_1, ..., I_r$ is similar to some, and hence exactly one, of the J_j. Similarly we can write, relabeling if necessary, $(b) = (J_1J_2...J_r...J_m)^*$ such that each of the J_i is similar to I_i for $i = 1, 2, ..., r$. Now as I_i, J_i are similar, i.e. $M(I_1) = M(J_1)$, we conclude that $(I_i, J_i)^* = K_i^*$, for $i = 1, ..., r$, where $K_i^* = J_i$ if $I_i \subseteq J_i$ and $K_i^* = I_i$ if $J_i \subseteq I_i$. Obviously, in either case, $(a, b)^* \subseteq K_i^*$ and as K_i are mutually $*$-comaximal $(a, b)^* \subseteq (K_1K_2...K_r)^* = H$. As K_i^* are the only $*$-super homog ideals containing both a and b we conclude that $(a, b)^* = (K_1K_2...K_r)^*$. This can be seen as follows: Since $(a, b)^* = ((K_1...K_r)((K_1^{-1}...K_r^{-1})a, (K_1^{-1}...K_r^{-1})b))^*$. This is because K_i are $*$-invertible and $(a, b)^* \subseteq (K_1K_2...K_r)^*$. So, $(a, b)^* = ((K_1...K_r)^*\ (K_1^{-1}...K_r^{-1}I_1I_2 ...\ I_rI_{r+1}...I_l)^*,\ (K_1^{-1}...K_r^{-1}b))^* = ((K_1...K_r)^*\ ((K_1^{-1}I_1)^*...(K_r^{-1}I_r)^*I_{r+1}\ ...\ I_l, (K_1^{-1}J_1)^*...\ (K_r^{-1}J_r)^*J_{r+1}...J_m)^*)^*$. Now note that $(K_i^{-1}I_i)^* = D$ or $(K_i^{-1}I_i)^*$ is a $*$-super homog ideal similar to I_i. Similarly $(K_i^{-1}J_i)^* = D$ or $(K_i^{-1}J_i)^*$ is a $*$-super homog ideal similar to J_i. Moreover, in both cases, $((K_i^{-1}I_i)^*, (K_i^{-1}J_i)^*)^* = D$ and, for $i \neq j$, $((K_i^{-1}I_i)^*, (K_j^{-1}J_j)^*)^* = D$, $i, j = 1, ..., r$. Moreover $((K_i^{-1}I_i)^*, J_t)^* = D$, for $i = 1, ..., r, t = r + 1, ..., m$ anyway and already for each $s = r + 1, ..., l_t = r + 1, ..., m$ $(I_s, J_t)^* = D$. So, each of the factors in $(K_1^{-1}I_1)^*...\ (K_r^{-1}I_r)^*I_{r+1}...I_l = (K_1^{-1}...K_r^{-1})a$ is $*$-comaximal with each of the factors in the $*$-product of $(K_1^{-1}J_1)^*. ..(K_r^{-1}J_r)^*J_{r+1}...J_m = (K_1^{-1}...K_r^{-1})b$. Thus $((K_1^{-1}...K_r^{-1})a,\ (K_1^{-1}...K_r^{-1})b)^* = D$

and that gives $(a,b)^* = ((K_1...K_r)((K_1^{-1}...K_r^{-1})a\ (K_1^{-1}...K_r^{-1})b))^* = (K_1...K_r)^*$.
∎

The above, ab-initio, proof was to stress the idea that there is a kind of GCD at work. Below we provide an alternate statement that seems to get similar results in a different way.

Corollary 6. *Given two nonzero elements a,b in a $*$-super SH domain D, the following hold: (1) if there is no maximal $*$-ideal P of D that contains both a and b, then $(a,b)^* = D$, (2) if either of a,b is a unit, then $(a,b)^* = D$, (3) if P is a maximal $*$-ideal containing both a,b then $(a,b)^*D_P \cap D = aD_P \cap D$ if $a|b$ in D_P and $(a,b)^*D_P \cap D = bD_P \cap D$ if $b|a$ in D_P, (4) if $P_1,P_2,...,P_n$ are all the maximal $*$-ideals of D that contain both a and b and if $I_i = (a,b)^*D_{P_i} \cap D$ for $i = 1,...,n$, then $(a,b)^* = (I_1I_2...I_n)^*$, (5) for every pair $a,b \in D\backslash\{0\}, (a,b)$ is $*$-invertible and so, D is a P$*$MD and (6) a d-super SH domain is a Prufer domain.*

Proof. (1) and (2) are straight forward. For (3) let P be a maximal $*$-ideal containing both a,b. Then as $(a,b)^* = (a,b)^{*_w}$ we have $(a,b)^*D_P \cap D = (a,b)^{*_w}D_P \cap D = (a,b)D_P \cap D$. Now as D_P is a valuation domain $a|b$ or $b|a$ in D_P and so $(a,b)D_P = aD_P$ if $a|b$ and $(a,b)D_P = bD_P$ if $b|a$ in D_P. For (4) note that for any maximal $*$-ideal P such that P does not contain at least one of a or b, $(a,b)D_P = D_P$. Now $(a,b)^{*_w} = \bigcap_{P\in t\text{-}Max(D)} (a,b)D_P = (\bigcap_{i=1}^{i=n}(a,b)D_{P_i}) \cap D = \bigcap_{i=1}^{i=n}((a,b)D_{P_i} \cap D) = \bigcap_{i=1}^{i=n} I_i$ $= (I_1I_2...I_n)^{*_w}$, I_i being $*$-super homog. So $(a,b)^{*_w} = (I_1I_2...I_n)^{*_w}$ and applying $*$ to both sides we get $(a,b)^* = (I_1I_2...I_n)^*$. Finally, for (5) and (6), note that as $(a,b)D_P$ is principal for each maximal $*$-ideal P, because D_P is a valuation domain, we conclude that if D is a $*$-super SH domain and if, for $a,b \in D$ with $(a,b) \neq (0)$ then (a,b) is $*_w$-invertible and hence $*$-invertible. This makes the $*$-super SHD D a P$*$MD and d-super SHD a PdMD which is Prufer. ∎

Part (5) of Corollary 6 is sort of tongue in the cheek in that for every maximal $*$-ideal M of a $*$-IRKT D we have that D_M is a valuation domain, a necessary and sufficient condition for D to be a P$*$MD.

An integral domain D is called an almost GCD (AGCD) domain if for every pair $a,b \in D\backslash\{0\}$ there is a positive integer n such that $(a^n) \cap (b^n)$ is a principal ideal. Equivalently, D is an AGCD domain if (and only if) for every pair $a,b \in D\backslash\{0\}$ there is a positive integer n such that $(a^n,b^n)_v$ is a principal ideal. Now we can write $(a^n,b^n)_v$ as $(a^n,b^n)_t$ because the number of generators is finite. AGCD domains have been studied in [34] and in [10] as a generalization of GCD domains. Here D is a GCD domain if every pair a,b of nonzero elements of D has a greatest common divisor GCD. It is well known that D is a GCD domain if and only if for every pair of nonzero elements a,b the ideal $aD \cap bD$ is principal (i.e. if and only if $(a,b)_v$ is principal).

Since a $*$-IRKT is a P$*$MD, and hence integrally closed $(a^n,b^n)_t = ((a,b)^n)_t$. Also since a P$*$MD is a P$t$MD we have $((a,b)^n)^* = ((a,b)^n)_t = (a^n,b^n)_t$.

Proposition 9. *A $*$-IRKT D is an AGCD domain if and only if for every $*$-super homog ideal A of D we have $(A^n)^*$ principal for some positive integer n.*

Proof. Let D be a $*$-IRKT. Suppose that for every $*$-super homog ideal A we have $(A^n)^*$ principal for some n. Let a,b be two nonzero non units of D. By Proposition

8 we have $(a,b)^* = (J_1J_2...J_r)^*$ where J_i are mutually ∗-comaximal ∗-super homog ideals, each dividing out a or b. Now let n_i be the positive integers such that $(J_i)^{n_i} = (d_i)$. Let $m = LCM(\{n_i\})$. Then $((a,b)^m)^* = (J_1^mJ_2^m...J_r^m)^*$, as $(J_i^m)^* = ((J_i^{n_i})^{m/n_i})^* = ((d_i)^{m/n_i}) = (D_i)$, say. But then $((a,b)^m)^* = (D_1...D_r)$ a principal ideal. Applying the v-operation to both sides we have $((a,b)^m)_v = (D_1...D_r)$ as a ∗-IRKT is integrally closed, we have $((a,b)^m)_v = (a^m,b^m)_v$. Now $(a^m,b^m)_v$ being principal leads to $a^mD \cap b^mD$ Indeed if one of a or b is a unit, or if a,b are ∗-comaximal, $(a,b)^*$ is principal which leads to $aD \cap bD$ principal and so D is an AGCD domain. Conversely, let the ∗-IRKT D be an AGCD domain. Then, as every ∗-homog ideal I is such that I^* is of finite type and as D is a ∗-IRKT, we have $(I^n)^*$ principal for some n [34, Theorem 3.9]. □

Definition 6. *A ∗-homog ideal I will be called a ∗-almost factorial homog (∗-af-homog) ideal if for each integral ∗- ideal J of finite type containing I we have $(J^n)^*$ principal, for some positive integer n. Also, a domain D will be called ∗-af-SH domain if for every nonzero non unit x of D, xD is expressible as a ∗-product of finitely many ∗-af-homog ideals.*

Indeed a ∗-af-homog ideal I is ∗-super homog, as $(J^n)^*$ principal implies J is ∗-invertible for each ∗- ideal J of finite type containing I.

Proposition 10. *The ∗-product of a finite number of ∗-af-homog ideals is uniquely expressible as a ∗-product of mutually ∗-comaximal ∗-af-homog ideals.*

Proof. We first show that the ∗-product of two similar ∗-af-homog ideals K,L is a ∗-af-homog ideal similar to I,J. For this let J be an integral ∗-ideal of finite type containing KL, i.e. $J \supseteq KL$. Then as K,L are both ∗-af-homog and hence ∗-super homog, J is a ∗-super homog ideal. If J contains either of K,L then by definition J is ∗-af-homog. If not then $J \subseteq K,L$. So say $(JL^{-1})^*$ an integral ∗-ideal of finite type containing K. But then, by definition, there a positive integer m such that $((JL^{-1})^m)^* = dD$, or $(J^m)^* = (L^m)^*dD$. Now as, for some positive integer n, we have $(L^n)^*$ is principal we conclude that $(J^{mn})^*$ is principal. Now $(KL)^*$ is ∗-homog similar to K and L because K and L are similar and because for each ∗-ideal J of finite type containing $(KL)^*$ there is a positive integer r such that $(J^r)^*$ is principal we conclude that $(KL)^*$ is indeed a ∗-af-homog ideal, similar to K and L. That a ∗-product of finitely many ∗-af-homog ideals similar to each other is a ∗-af-homog ideal similar to them can be shown by doing it taken two at a time. Next ∗-af-homog ideals being ∗-homog we can express the ∗-product uniquely as a ∗-product of mutually ∗-comaximal ∗-homog ideals obtained by taking ∗-products of similar ∗-homog ideals. Now in this case the ∗-products of those mutually similar ∗-homog ideals are ∗-af-homog ideals, as we found in the proof of Proposition 6, by noting that if xD is a ∗-product of ∗-af-homog ideals then $xD \cap D$ is one of those ∗-af-homog ideals. □

Theorem 6. *The following are equivalent for an integral domain D : (1) for every nonzero non unit x of the domain D, xD is expressible as a ∗-product of finitely many ∗-af-homog ideals (2) D is an AGCD ∗-IRKT.*

Proof. D is a ∗-IRKT by Proposition 6 and by Proposition 9, supported by the definition of ∗-af-ideals, D is an AGCD ∗-IRKT. For the converse note that, as we have already observed, every principal ideal xD generated by a nonzero non unit x can be expressed as a ∗-product of ∗-homog ideals, each of which is, ∗-invertible and, expressible as $xD_P \cap D$ where D_P is a valuation domain. Now a $*$-ideal J of finite type containing $xD_P \cap D$ is a $*$-ideal of finite type of an AGCD domain in which $* = t$ and so there must be a positive integer n such that $(J^n)^*$ is principal. Thus each of $xD_P \cap D$ is $*$-af-homog. □

We can define $*$-af-homog ideals of type 1 and type 2 and prove obvious results about AGCD $*$-GKD and AGCD $*$-Krull.

Call a $*$-homog ideal I a $*$-af-homog ideal of type 1, if I is a $*$-af-homog ideal and a $*$-homog ideal of type 1. Now let I, J be two $*$-af-homog ideals of type 1, Then IJ is a $*$-af-homog ideal, by the proof of Proposition 10 and of type 1 by the remark before Theorem 3

As we have done in other cases let's call an integral domain D a $*$-af-SH domain of type 1 if for every nonzero non unit $x \in D$, xD is expressible as a $*$-product of $*$-af-homog ideals of type 1.

Indeed a $*$-af-homog ideal of type 1 is a $*$-super homog ideal of type 1 and so a $*$-af-SH domain of type 1 is at least a $*$-GKD. Next, an AGCD $*$-GKD is at least a $*$-GKD. So the proof of the following statement will run along lines similar to that of Proposition 7.

Proposition 11. *The following are equivalent for an integral domain D : (1) D is a $*$-af- domain of type* 1*, i.e. for every nonzero non unit x of the domain D, xD is expressible as a $*$-product of finitely many $*$-af-homog ideals of type* 1*, (2) D is an AGCD $*$-GKD.*

Next call a $*$-homog ideal I, a $*$-af-homog ideal of type 2 if I is $*$-af-homog and every $*$-homog ideal J containing I is $*$-af-homog such that $J^* = (M(J)^n)^*$ for some positive integer n. It is easy to see that a $*$-af-homog ideal of type 2 is a $*$-af-homog of type 1. Now we can, indeed, call D a $*$-af-SH domain of type 2 if for every nonzero non unit $x \in D$ the ideal xD is expressible as a $*$-product of finitely many $*$-af-homog ideals of type 2. Thus a $*$-af-SH domain of type 2 is a $*$-af-SH domain of type 1 and so a $*$-GKD.

Theorem 7. *Let D be an integral domain and suppose that every nonzero proper principal ideal of D is expressible as a finite $*$-product of $*$-af-homog ideals of type* 2*. Then D is an AGCD $*$-Krull domain. Conversely if D is an AGCD $*$-Krull domain, then every nonzero proper principal ideal of D is expressible as a $*$-product of finitely many $*$-af-homog ideals of type* 2*.*

The proof should be somewhat simpler than that of Theorem 4 because we have assumed I $*$-af-homog and that makes $M(I)$ $*$-invertible for each $*$-af-homog ideal I, making D a $*$-Krull domain right away.

The AGCD $*$-Krull domains were first studied by U. Storch in [30]. The easiest to access these domains is taking Dedekind domains with torsion class groups.

Definition 7. *Call a $*$-homog ideal $*$-factorial homog ($*$-f-homog) if every integral $*$-ideal of finite type containing I is principal.*

In other words, repeating Definition 3, a nonzero integral $*$-ideal of finite type is called $*$-f-homog if (1)S for each $*$-ideal of finite type containing I is principal and (2)S For every pair of proper integral $*$-ideals A, B of finite type containing I, $(A + B)^* \neq D$. Indeed a $*$-f-homog ideal I is $*$-super homog and so has all the properties listed in Proposition **??**. In particular as a $*$-f-homog ideal is principal, we can use "$*$-f-homog element x" instead of "$*$-f-homog ideal xD". Consequently, we can say that, the set of all factors of $*$-f-homog elements is totally ordered under inclusion of the principal ideals generated by them, i.e. $*$-f-homog element is a rigid element. To be exact we have the following result linking "rigid element" with "$*$-f-homog element".

Proposition 12. *For the generator of the principal ideal rD the following are equivalent. (1) rD is $*$-f-homog, (2) r is a rigid element that belongs to a unique maximal t-ideal and every $*$-ideal of finite type containing r is principal.*

Proof. (1) $\Rightarrow$ (2) Suppose rD is $*$-f-homog then, as already mentioned, r is rigid i.e. for each pair x, y of factors of r, $x|y$ or $y|x$. The condition (2)S ensures that r belongs to a unique maximal $*$-ideal the remainder is taken care of by condition (1)S. For (2) $\Rightarrow$ (1) Let I be a $*$-ideal of finite type containing rD. Then I contains r and hence must be principal, this takes care of (1)S. Next let A and B be two integral $*$-ideals of finite type containing rD. Then we have seen that $A = xD$ and $B = yD$ for some factors x and y of r. But as $x|y$ or $y|x$ we have $(A + B)^* \neq D$ which is (2)S.

It may be noted that merely saying "r is a rigid element belonging to a maximal $*$-ideal P", is not enough. We need to make sure that every finite type proper $*$-ideal containing r is principal. This is because in the Dedekind domain $\mathbb{Z}[\sqrt{-5}]$, where, of course, $* = d$. For the prime ideal $P = (2, 1 + \sqrt{-5})$ we have $P^2 = (2)$ where 2 is irreducible in $\mathbb{Z}[\sqrt{-5}]$ and so fits the definition of a rigid element, but 2 is not $*$-f-homog because P contains 2 yet P is not principal.

Note that the $*$-product $(IJ)^*$ of two similar $*$-f-homog ideals I, J is a $*$-f-homog ideal similar to both I and J. (I, J are similar $*$-super homog, so $(IJ)^*$ is $*$-super homog, similar to I and J and ($(IJ)^* \subseteq I \subseteq J$ or $(IJ)^* \subseteq J \subseteq I$), say $I \subseteq J$. Now let C be a an integral $*$-ideal of finite type containing $(IJ)^*$. As in the other two cases C would be principal if C contains either of I, J. Otherwise $C \subseteq I, J$ and so $CI^{-1} \subseteq D$. Since I is principal so is I^{-1} and so CI^{-1} is a $*$-ideal of finite type and $CI^{-1} \supseteq (IJ)^*I^{-1} = J$. So CI^{-1} is principal which forces C to be principal.) Consequently a product of finitely many $*$-f-homog ideals/elements is expressible, uniquely, up to associates and order, as a product of mutually $*$-comaximal $*$-f-homog ideals/elements.

Theorem 8. *Suppose that every nonzero non unit x of D generates xD that is expressible as a $*$-product of finitely many $*$-f-homog ideals. Then D is a GCD $*$-IRKT whose nonzero non units are uniquely expressible as products of mutually co prime $*$-f-homog elements. Conversely if D is a GCD $*$-IRKT then every proper principal ideal of D is expressible as a finite $*$-product of $*$-f-homog ideals.*

Proof. Because every $*$-f-homog ideal is $*$-super homog ideal, D is a $*$-IRKT by Proposition 6. It is also well known that if D is a $*$-IRKT then D is a P$*$MD and so $* = t$. We have already established, in Proposition 4, that if $xD = (I_1I_2...I_n)^*$, where I_i are $*$-homog ideals then $xD = (J_1J_2...J_r)^*$ where J_j are mutually $*$-comaximal $*$-homog ideals and this expression is unique up to order etc. Indeed as a $*$-f-homog ideal is $*$-homog the statement holds here too. Now let a, b be two nonzero elements of D. We can assume that $aD = (A_1...A_r)^*$ where A_i are mutually $*$-comaximal $*$-f-homog ideals. $(A_1^*...A_r^*)^* = a_1D...a_rD$, here $A_i^* = a_iD$ because A_i is $*$-f-homog . Similarly, $bD = b_1D...b_sD$ where b_i are mutually $*$-comaximal. Let $P_1, P_2...P_n$ be all the maximal $*$-ideals that contain both a and b. By rearranging we can assume that $a_iD, b_iD \in P_i$, for $i = 1, 2, ...n$. Now as a_iD, b_iD are $*$-f-homog belonging to the same maximal $*$-ideal P_i $a_i|b_i$ or $b_i|a_i$. Thus $(a_i, b_i)^* = a_iD$ or b_iD according as $a_i|b_i$ or $b_i|a_i$. Let's denote $(a_i, b_i)^*$ by d_iD. Proceeding as in Proposition 8 we have $(a, b)^* = (\prod_{i=1}^{n}(a_i, b_i)^*)^* = (\prod d_iD)^* = (d_1d_2...d_nD)^* = d_1d_2...d_nD$. Thus $(a, b)^* = d_1d_2...d_nD$ a principal ideal. Applying the v-operation on both sides we get $(a, b)_v = d_1d_2...d_nD$ a principal ideal. As a, b were arbitrary we conclude that D is a GCD-domain. Also as we have already established that D is a $*$-IRKT, we are done. For the converse note that in a $*$-IRKT $* = t, d$ a t-IRKT is an IRKT and a d-IRKT is a Prufer domain. That a GCD IRKT is semirigid (every nonzero non unit expressible as a product of finitely many rigid elements) was established in [33], where IRKT was dubbed as IKT domain or use the following lemma. □

Lemma 1. *The following are equivalent for a $*$-ideal I of finite type in a GCD domain D : (1) I is $*$-homog, (2) I is $*$-super homog, (3) I is $*$-f-homog and (4) $I = rD$ where r is a rigid element.*

Proof. Note that in a GCD domain $* = t$. Now (1) $\Rightarrow$ (3) because I is $*$-homog such that every $*$-ideal of finite type of D is principal, being t-ideal of finite type of a GCD domain and one that fits the definition of a $*$-f-homog ideal (3) $\Rightarrow$ (2) because principal is invertible and (2) $\Rightarrow$ (1) is obvious. Now, in a GCD domain a rigid element r belongs to a unique maximal t-ideal $M = \{x, (r, x)_v \neq D\}$. This is because $(r, x)_v \neq D$ implies that r has a non unit common factor $r_x$with x So for each $x \in M\backslash\{0\}$ we have $x = r_x(x/r_x)$ where r_x is a non unit factor of r. Now let $x_1, ..., x_n \in M\backslash\{0\}$. Then $x_i = r_{x_i}(x/r_{x_i})$ where r_{x_i} are non unit factors of the rigid element r. Since for all $a, b|r$ we have $a|b$ or $b|a$ we have $(x_1, x_2, ..., x_n) \subseteq (r_{x_j})$ which means $(x_1, x_2, ..., x_n)_v \subseteq (r_{x_j})$ for some $1 \leq j \leq n$. But as $(r_{x_j}, x)_v = (r_{x_j}) \neq D$ we have $(x_1, x_2, ..., x_n)_v \subseteq (r_{x_j}) \subseteq M$ and M is a t-ideal and there is no t-ideal not contained in M that contains r. For if N were such a t-ideal, then there is say $\alpha \in N\backslash M$. But then $(\alpha, r)_v = D$. Whence any t-ideal containing r must be contained in M . Thus $I = rD$ is $*$-homog and (4) $\Rightarrow$ (1). Now a $*$-f-homog ideal that is principal must be a generated by a rigid element by Proposition 12 and this establishes (3) $\Rightarrow$ (4). □

We can call a $*$-f-homog ideal $I = xD$ a $*$-f-homog ideal of type 1 if, in addition, 1I is $*$-homog of type 1. Indeed $I = xD$ a $*$-f-homog ideal is of type 1 if and only if for every $*$-f-homog ideal $A = yD$ containing xD, i.e. $y|x$ in D, there is a positive integer n such that $x|y^n$. If we develop a theory of factorization on it we will get a theorem like the following.

Theorem 9. *The following are equivalent for an integral domain D : (1) For every nonzero non unit x of the domain D, xD is expressible as a $*$-product of finitely many $*$-f-homog ideals of type 1 (2) D is a GCD $*$-GKD.*

Proof. By (1), using Theorem 8, D is a GCD $*$-IRKT, because every $*$-f-homog ideal of type 1 is a $*$-f-homog ideal. But a $*$-f-homog ideal of type 1 is also a $*$-super homog ideal of type 1 and so Proposition 7 applies to give that D is a GCD $*$-GKD. For the converse the reader may refer to [2] or just note that a $*$-GKD is a $*$-IRKT whose maximal $*$-ideals are of height 1. □

The domains of Theorem 9 were studied in [2] under the name of Generalized Unique Factorization Domains (GUFDs).

We can call a $*$-f-homog ideal xD of type 2 if $xD = (M(xD))^n$ and get a theory of UFD's. Of course that is too well known to repeat here.

4. Restricted or weak theories

Before we get down to explaining the restricted theories let us take care of a topic that is in a way essential to them. The topic is that of (integral) $*$-invertible $*$-ideals. It is often noted that an integral invertible ideal behaves like a principal ideal in many respects, for example an invertible ideal is locally principal. In fact a nonzero finitely generated ideal I is invertible if and only if I is locally principal, i.e., ID_M is principal for every maximal ideal M. In a similar manner a $*$-invertible $*$-ideal may be characterized by, "a $*$-ideal of finite type I such that ID_P is principal for each maximal $*$-ideal P of D". (We plan to use this feature in the following in a somewhat indirect manner. But first we must talk about another important property of integral $*$-invertible $*$-ideals, in the context of $*$-SH domains.

Theorem 10. *Let D be a $*$-SH domain and I a $*$-invertible $*$-ideal of D. Then I is uniquely expressible, up to order, as a $*$-product of mutually $*$-comaximal $*$-homog ideals.*

Proof. Indeed as D is of finite $*$-character, I is contained in at most a finite number of maximal $*$-ideals $P_1, P_2, ..., P_n$ we have $I = \cap_{i=1}^{n}(ID_{P_i} \cap D)$. Now because P_i shares no nonzero prime ideal with any other maximal t-ideal we conclude that none of $I_i = (ID_{P_i} \cap D)$ is contained in any maximal $*$-ideal other than P_i for $i = 1, ..., n$. Next as D is of finite $*$-character each of I_i is a $*$-ideal of finite type. Thus each of I_i is $*$-homog. Also I_i are mutually $*$-comaximal by Remark 1. So, $I = \cap_{i=1}^{n} I_i = (I_1 I_2 ... I_n)^{*_w} = (I_1 I_2 ... I_n)^*$. That this expression is unique, up to order, follows from proofs of similar results in earlier sections such as Proposition 4.

Corollary 7. *Let D be a $*$-SH domain. Then the $*$-class group of D is* 0 *if and only if every $*$-invertible $*$-homog ideal of D is principal.*

Taking a cue from the above result we make the following definition.

Definition 8. *An integral $*$-ideal I is a $*$-weakly factorial homogeneous ($*$-wf-homog) ideal if I is $*$-homog such that I is principal when $*$-invertible.*

So a $*$-homog ideal I is $*$-wf-homog if I is principal, in the event that I is $*$-invertible. Otherwise it is just a $*$-homog ideal. We may call the generator of a principal $*$-wf-homog ideal a $*$-wf-homog element. Indeed a $*$-homog ideal I is $*$-wf-homog if $I = ID_{M(I)} \cap D$ is principal whenever $ID_{M(I)}$ is principal.

If I, J are two similar $*$-wf-homog ideals then $(IJ)^*$ is $*$-wf-homog, because if $(IJ)^*$ is invertible then so are both I and J and hence principal.

Definition 9. *Call an integral domain D a $*$- weakly factorial SH ($*$-wf-SH) domain, if every nonzero non unit of D is expressible as a finite product of $*$-wf-homog elements.*

Proposition 13. *A $*$-SH domain with trivial $*$-class group is a $*$-wf-SH domain. Conversely a $*$-wf-SH domain is a $*$-SH domain with trivial $*$-class group.*

Proof. Indeed if the $*$-class group of D is zero, every $*$-invertible $*$-ideal of D is principal. Now for every nonzero non unit x in a $*$-SH domain $xD = (I_1 I_2 ... I_n)^*$, where I_i are mutually $*$-comaximal $*$-invertible $*$-ideals. With the added restriction of trivial $*$-class group, each of $I_i = x_i D$ is a principal ideal which fits the definition of a $*$-wf-homog element. Hence, as the $*$-operation is ineffective on principal ideals, $xD = d_1 d_2 ... d_n D$, or $x = ed_1 d_2 ... d_n$ is a product of $*$-wf-homog elements, where e is a unit. Conversely, it is obvious that (a) D is of finite $*$-character and (b) every prime ideal of D contains a $*$-wf-homog element which generates a $*$-homog ideal. It is now easy to show that every maximal $*$-ideal of D is spawned by a $*$-wf-homog principal ideal and that no two distinct maximal $*$-ideals contain a nonzero prime ideal. So, a $*$-wf-SH domain D is a $*$-SH domain. Next to show that the $*$-class group of D is trivial,

take an integral $*$-invertible $*$-ideal I of D. By Theorem 10, $I = (I_1 I_2 ... I_n)^*$ where each of I_i is a $*$-homog ideal and I_i mutually $*$-comaximal. Pick one, say I_k, and note that I_k is $*$-invertible and that we can assume that I_k is a $* =$-ideal. So $I_k D_{M(I_k)} = ID_{M(I_k)} = a_k D_{M(I_k)}$ where a_k is a nozero nonunit of D. But then $a_k = a_{k1} a_{k2} .. a_{kn}$ where a_{ki} are mutually $*$-comaximal $*$-wf-homog elements and so only one, say a_{kl_k}, belongs to $M(I_k)$. But then $ID_{M(I_k)} = a_k D_{M(I_k)} = a_{kl_k} D_{M(I_k)}$. Thus for each k we have $ID_{M(I_k)} \cap D = I_k D_{M(I_k)} \cap D = I_k^* = a_{kl_k} D_{M(I_k)} \cap D$ $= a_{kl_k} D$, because a_{kl_k} is a $*$-wf-homog element. This forces I to be a principal ideal, since $I = (I_1 I_2 ... I_n)^* = (a_{1l_1} a_{2l_2} ... a_{nl_n} D)^* = a_{1l_1} a_{2l_2} ... a_{nl_n} D$. □

Examples: (a) (When $* = d$ and no restriction on dimension). Let (R, M) be a regular local domain of dimension $n \geq 2$, let L be the quotient field of R and let X be an indeterminate over L. Then the ring $D = R + XL[X]$ is an $n+1$ dimensional d-SH factorial domain such that $Cl_d(D)$, the ideal class group of D is zero.

Illustration: By [19, Cororollary 1.3] D is a GCD domain and so $Cl_t(D) = (0)$ and as $Cl_d(D) \subseteq Cl_t(D)$ we conclude that $Cl_d(D) = (0)$. Also, by [19, Theorem 4.21], the maximal ideals of D are (a) $M + XL[X]$ where M is the maximal ideal of R and (b) ideals of the type $f(X)D$ where $f(X)$ is an irreducible element and hence a prime, of D such that $f(0) = 1$. Next a typical nonzero non unit $f(X)$ of D can be expressed as $\frac{a}{b} X^r (1 + Xg(X))$ where $a, b \in D \backslash \{0\}$, $b = 1$ if $r = 0$ and $g(X) \in K[X]$. Clearly $f(X) = \frac{a}{b} X^r \times (1 + f_1(X))^{r_1} \times ... \times (1 + f_m(X))^{r_m}$ where the $(1 + f_i(X)), i = 1, .., m$, and r, $r_i \geq 0$, are irreducible and hence generate a principal maximal ideal each and of course $\frac{a}{b} X^r \in M + XL[X]$. (Indeed if $r = 0$, $\frac{a}{b} X^r = a$ and if a is a non unit then $a \in M + XL[X]$. Thus $f(X)$ belongs to only finitely many maximal i.e. maximal d-ideals). That no two maximal ideals contain a common nonzero prime ideal is obvious. In sum D is a d-SH factorial domain, which works out to be an h-local domain with zero ideal class group.

Sticking with $D = R + XL[X]$, where R is quasi local we note that a maximal ideal (hence a maximal d-ideal) of D is either $M + XL[X]$ where M is the maximal ideal of R or a height one principal prime ideal of the form $f(X)D$ ([19, Theorem 4.21]). Thus a d-homog ideal I of D such that $I \cap R$ is non-trivial is of the form $I \cap R + XL[X]$. On the other hand any d-homog ideal J with $M(J) \cap R = (0)$ would have to be a power of a prime of the form $f(X)D$ where $f(X)$ generates the maximal ideal $M(J)$. For, by part (a) of Proposition 4.12 of [19] $J = h(X)(F + XL[X])$ where F is a D-submodule of L and such that $h(0)F \subseteq D$ and $h(X) \in L[X]$. We claim that $h(0) \neq 0$ for otherwise $h(X)(F + XL[X])$ would be contained in $XL[X]$ and so in $M + XL[X]$. Finally as $J = h(X)(F + XL[X]) \subseteq f(X)D$ where $f(X)D$

is a height one prime ideal, there a positive integer n such that $J \subseteq f(X)^n D$ but $J \nsubseteq f(X)^{n+1} D$ or $J/f(X)^n \subseteq D$ but $J/f(X)^n \nsubseteq f(X)D$. We claim that $J/f(X)^n$ is not contained in any maximal deal. For if $J/f(X)^n$ is contained in a maximal ideal N then J is contained in N, contradicting the assumption that J is d-homog. Thus $J/f(X)^n = D$, making $J = f(X)^n D$. Thus a finite product of d-homog ideals of D is an ideal of the form $l(X)(A + XL[X])$ where $l(X)$ is a polynomial in D with $l(0) = 1$ and A an ideal of R. Also by Lemma 4.41 of [19] $(l(X)(A+XL[X]))_t = l(X)(A_t+XL[X])$.Thus we have proved the following result.

Lemma 2. *Let (R, M) be quasi local, L the quotient field of R, X an indeterminate over L and let $D = R + XL[X]$. Then D is d-SH with d-homog ideals J described by (a) $J = f(X)^n D$, when $J \cap R = (0)$ and (b) $J = J \cap R + XL[X]$, when $J \cap R \neq (0)$. Moreover a finite product of d-homog ideals of D is of the form $J = l(X)(A + XL[X])$ and $J_t = l(X)(A_t + XL[X])$. Finally, every principal ideal of D is of the form $J = al(X)D$ where $a \in D$.*

The construction $D = R+XL[X]$ where L is the quotient field of R, will deliver D with $Cl_d(D) = 0$, when $Cl_d(R) = 0$. This is because in $D = R + XL[X]$ every finitely generated ideal is of the form $F = f(X)JD$ where $f(X) \in T$ and J is a finitely generated ideal of R [19, Proposition 4.12]. So every invertible ideal of D is principal if and only if every invertible ideal of R is principal. Thus if R is quasi local then $D = R + XL[X]$ is a d-SH domain with $Cl_d = 0$.

The above reasoning works in the $Cl_t(D)$ trivial or torsion cases too if we look at it this way: If an ideal G of $R + XL[X]$ is t-invertible then there is a finitely generated ideal $F = f(X)JD$ of D such that $G_{t_D} = (f(X)JD)_{t_D}$, where t_D denotes the t-operation of the domain D. But by Lemma 4.41 of [19] $G_{t_D} = f(X)J_{t_R}D$. So every t-invertible t-ideal of D is principal if and only if every t-invertible t-ideal of R is principal. In other words, $Cl_t(D) = 0 \Leftrightarrow Cl_t(R) = 0$. We have seen that for a t-invertible ideal $G \in R + XL[X]$ we have $G_{t_D} = f(X)J_{t_R}D$. So, $(G^n)_{t_D} = f(X)(J^n)_{t_R}D$. Thus $(G^n)_{t_D}$ is principal if and only if $(J^n)_{t_R}$ is principal. Thus $Cl_t(D)$ is torsion $\Leftrightarrow Cl_t(R)$ is torsion.

Next every t-local domain, i.e. a quasi local domain whose maximal ideal is a t-ideal is an example of a t-wf-SH domain. This is because in a t-local domain (D, M) every t-invertible ideal is invertible and hence principal, [5].

We shall see other examples as we define the $*$-homog ideals defining the various clones of the $*$-wf-SH domains.

Example (b). Let R be a t-local domain and let $D = R+XL[X]$ be as in Lemma 2 then D is an example of a t-wf-domain.

Illustration: Indeed by 2 a t-wf-homog ideal J of D is either principal of the form $J = f(X)^n D$, when $J \cap R = (0)$ or $J_t = (J \cap R)_t + XL[X]$, when $J \cap R \neq (0)$. Here $f(X)^n D$ is principal (in fact a t-f-homog ideal) that satisfies the condition that if t-invertible then every t-invertible t-ideal containing it is principal. Of course the ideal $J_t = (J \cap R)_t + XL[X]$ satisfies the same condition because its being t-invertible or principal depends upon $(J \cap R)_t$ being t-invertible or principal which is a t-ideal of a t-local ring R.

Definition 10. *Call a $*$-homog ideal I $*$-wf-homog of type* 1*, if I is a $*$-homog ideal of type* 1 *such that whenever I is a $*$- invertible $*$- ideal every $*$-invertible $*$-ideal J containing I is principal.*

As above we can call the generator of a principal $*$-wf-homog ideal a $*$-wf-homog element and define a $*$-wf-SH domain of type 1 as the domain whose nonzero non unit elements are expressible as products of $*$-wf-homog elements of type 1.

Proposition 14. *A $*$-wf-SH domain of type* 1 *is a $*$-weakly Krull domain with trivial $*$-class group.*

The proof follows as we chase the definitions. Remarkable here is the abundance of examples. Every one dimensional quasi local domain is indeed an example of a d-wf-SH domain of type 1 and so is every one dimensional local domain with trivial ideal class group. A weakly Krull domain D with trivial t-class group is an example of a t-wf-SH domain of type 1. A weakly Krull domain with zero t-class group is also known as a weakly factorial domain and that, perhaps, is the reason for the abundance of examples. Weakly factorial domains were among the earliest efforts to generalize the notion of factoriality. These domains were initially defined by Anderson and Mahaney in [6] as domains whose nonzero non units were expressible as products of primary elements. Here an element x of D is called primary if xD is a primary ideal. Then it was shown, among other results, in [9], that D is a weakly factorial domain if and only if $D = \bigcap_{P\in X^1(D)} D_P$, the intersection is locally finite, and D has trivial t-class group, another way of saying that D is a weakly Krull domain with trivial t-class group. (At that time we did not have the idea of christening the domains D that are locally finite intersections of localizations at height one primes as weakly Krull domains.) To give the other properties, more important for the purposes of that paper, it was shown that (a) D is a weakly factorial domain if and only if every convex directed subgroup of the group of divisibility of D is a cardinal summand and (b) D is a weakly factorial domain if and only if the following is true: if P is a prime ideal of D minimal over a proper principal ideal xD, then P has height one and $xD_P \cap D$ is principal. Indeed there has been a lot of activity around this concept.

On the other hand, as we come to consider the $*$-super homog ideals and $*$-super SH domains, things fall into the pattern of same old same old. Just to make sure that the readers don't miss anything let's recall that the definition of a $*$-super homog ideal I requires that every $*$-ideal of finite type containing I must be $*$-invertible and the definition of a $*$-wf-homog ideal I requires that if I is $*$-invertible then I must be principal. That is if D is a $*$-IRKT in which every $*$-super homog ideal is $*$-wf-homog then every $*$-ideal of finite type containing I is principal. But that, in case I is $*$-invertible, makes I a $*$-f-ideal, as Definition 7 tells us, and I is $*$-invertible, being $*$-super homog. Conversely if I is $*$-f-homog, then I is obviously a $*$-super homog and a $*$-wf-homog ideal. This gives us the following result.

Proposition 15. *Every $*$-ideal I of finite type of a $*$-IRKT is $*$-f-homog if and only if every $*$-super homog ideal is $*$-wf-homog ideal.*

We already know that a domain whose nonzero non units are products of $*$-f-elements is a GCD $*$-IRKT. All that remains is making links with other related concepts.

Proposition 16. *For a domain D the following statements are equivalent: (1) D is a $*$-IRKT whose $*$-super homog ideals are also $*$-wf-homog, (2) D is a $*$-wf-SH domain whose $*$-wf-homog ideals are also $*$-super homog, (3) D is a GCD*

$*$-IRKT, (4) D is a $*$-IRKT with $Cl_*(D) = 0$, (5) D is a locally GCD $*$-IRKT and $Cl_d(D) = 0$.

Proof. (1) $\Leftrightarrow$ (3) By Proposition 15 and Theorem 8, (1) $\Rightarrow$ (2) A $*$-IRKT is a $*$-super SH domain. Now apply Proposition 15, (2) $\Rightarrow$ (4) A $*$-wf-SH domain has trivial $*$-class group by Proposition 13 and every $*$-wf-homog ideal being $*$-super homog makes D a $*$-IRKT, (4) $\Rightarrow$ (3) Note that if D is a $*$-IRKT with $Cl_*(D) = 0$, then every $*$-invertible $*$-ideal of D is principal and so every $*$-super homog ideal of D is principal. Thus every $*$-super homog ideal of D is $*$-f-homog. Now apply Theorem 8, (3) $\Rightarrow$ (5) D being a GCD domain implies that D is locally GCD and $Cl_t(D) = 0$ and we know that $Cl_t(D) \supseteq Cl_d(D)$, (5) $\Rightarrow$ (3) Let's prove the following result. □

Lemma 3. *Let D be a locally GCD domain and let $\mathcal{F}$ be a family of nonzero primes of D such that $D = \bigcap_{P\in\mathcal{F}} D_P$ is locally finite. If $Cl_d(D) = 0$ then D is a GCD domain.*

Proof. Let τ be the star operation induced by $\{D_P\}_{P\in\mathcal{F}}$. Since D is locally GCD, each of D_P is a GCD domain and so for each pair a, b of non zero elements of D we have $(aD \cap bD)D_P = aD_P \cap bD_P$ principal. Because $D = \bigcap_{P\in\mathcal{F}} D_P$ is locally finite, $aD \cap bD$ is contained in at most a finite number $P_1, P_2, ..., P_n$ of members of $\mathcal{F}$, precisely ones that contain at least one of a, b. That is $(aD \cap bD)D_Q = D_Q$ for all $Q \in \mathcal{F}$ such that $ab \notin Q$. Then $(aD \cap bD)D_{P_i} = x_iD_{P_i}$ where we can take $x_i \in P_i$ for some i and indeed we can take $x_i \in aD \cap bD$, $i = 1, 2, ...n$. Set $A = (ab, x_1, ...x_n)$. Then $A_v \subseteq aD\cap bD$ because $A \subseteq aD\cap bD$ which is a v-ideal. Now $(aD \cap bD)D_{P_i} = x_iD_{P_i} \subseteq AD_{P_i}$ for $i = 1, ..., n$ and $(aD \cap bD)D_Q = D_Q = AD_Q$ for all $Q \in \mathcal{F}$ such that $ab \notin Q$. So $(aD \cap bD)D_P \subseteq AD_P$ for all $P \in \mathcal{F}$. Thus $(aD\cap bD) \subseteq A_t \subseteq A_v$ and this shows that $(aD\cap bD)$ is a v-ideal of finite type. Now as D is locally GCD $(aD \cap bD)$ is locally principal and hence flat. But a flat ideal that is also a v-ideal of finite type is invertible [36, Proposition 1]. Now for each pair $a, b \in D\backslash\{0\}$, $(aD \cap bD)$ is invertible and $Cl_d(D) = 0$ means every invertible ideal of D is principal. Whence for each pair $a, b \in D\backslash\{0\}$ $(aD \cap bD)$ is principal and D is a GCD domain. □

Next from Proposition 15, we conclude that a $*$-f-homog ideal of type 1 is nothing but a $*$-super homog ideal of type 1 that is also a $*$-wf-homog ideal. Again we know that a domain whose nonzero non units are expressible as products of $*$-f- elements of type 1 is a GCD-$*$-GKD (cf Theorem 9) and that these domains were studied in [2] as GUFDs with a totally different set of definitions. We also know that only two values of $*, d$ and t, have any effect. That is a GUFD D is a one dimensional Bezout domain if $* = d$ and a GCD-GKD if $* = t$.

Let's call a $*$-homog ideal I a weak $*$-almost factorial homog ($*$-waf-homog) ideal if whenever I is a $*$-invertible $*$-ideal we have $(I^i)^*$principal for some positive integer i.

Indeed if I and J are $*$-waf-homog ideals of D $(IJ)^*$ is a $*$-waf-homog ideal because if $(IJ)^*$ is $*$-invertible then both I and J are $*$-invertible and so $(I^m)^*$ and $(J^n)^*$ are principal for some positive integers m, n. But then $((IJ)^{mn})^*$ is principal.

So if I is $*$-waf-homog then so is $(I^n)^*$ for every. But then for each positive integer n $(I^n)^* = (I^n)^* D_{M(I)} \cap D$.

Let's start with a clone of Corollary 7.

Proposition 17. *Let D be a $*$-SH domain. Then the $*$-class group of D is torsion if and only if for every $*$-invertible $*$-homog ideal I of D we have $(I^r)^*$ principal for some r.*

Next we have a clone of the definition of weakly factorial domains.

Definition 11. *Call an integral domain D a $*$-SH weakly almost factorial ($*$-waf-SH) domain if every nonzero non unit of D is expressible as a finite $*$-product of $*$-waf-homog ideals.*

Proposition 18. *A $*$-SH domain with torsion $*$-class group is a $*$-waf-SH domain. Conversely a $*$-waf-SH domain is a $*$-SH domain with torsion $*$-class group.*

Proof. Indeed if the $*$-class group of D is torsion, for every $*$-invertible $*$-ideal I of D there is a positive integer n such that $(I^n)^*$ is principal. Now for every nonzero non unit x in a $*$-SH domain $xD = (I_1 I_2 ... I_n)^*$, where I_i are mutually $*$-comaximal $*$-invertible $*$-ideals. With the added restriction of torsion $*$-class group, each of I_i is a $*$-invertible $*$-ideal which fits the definition of a $*$-waf-homog ideal, i.e., whenever I_i is a $*$-invertible $*$-ideal there is a positive integer n_i such that $(I_i)^{n_i}$ is principal. Conversely, it is obvious that (a) D is of finite $*$-character and (b) every prime ideal of D contains a $*$-waf-homog ideal. It is now easy to show that every maximal $*$-ideal of D is spawned by a $*$-waf-homog ideal, because a $*$-waf-homog ideal is a $*$-homog ideal to start with, and that no two distinct maximal $*$-ideals contain a nonzero prime ideal. So, a $*$-waf-SH domain D is a $*$-SH domain. Next to show that the $*$-class group of D is torsion, take an integral $*$-invertible $*$-ideal I of D and let $0 \neq x \in I$. By Theorem 10, $I = (I_1 I_2 ... I_r)^*$ where each of I_i is a $*$-homog ideal and I_i mutually $*$-comaximal. By Theorem 10, $I = (I_1 I_2 ... I_n)^*$ where each of I_i is a $*$-homog ideal and all the I_i are mutually $*$-comaximal. Pick one, say I_k, and note that I_k is $*$-invertible and that we can assume that I_k is a $*$-ideal. So that $I_k D_{M(I_k)} = ID_{M(I_k)} = a_k D_{M(I_k)}$ where a_k is a nozero nonunit of D. But then $a_k = h_{k1} h_{k2} .. h_{kn}$ where h_{ki} are mutually $*$-comaximal $*$-waf-homog ideals and so only one, say h_{kl_k}, is contained in $M(I_k)$. But then $ID_{M(I_k)} = a_k D_{M(I_k)}$ $= h_{kl_k} D_{M(I_k)}$. Thus for each k we have $ID_{M(I_k)} \cap D = I_k D_{M(I_k)} \cap D = I_k^*$ $= h_{kl_k} D_{M(I_k)} \cap D = h_{kl_k}$.

Next since $(h_{kl_k})^{n_k}$ is principal, being $*$-waf-homog, for some positive integer n_k we have that $I^{n_k} D_{M(I_k)} \cap D = I_k^{n_k} D_{M(I_k)} \cap D = (I_k^{n_k})^* = (h_{kl_k})^{n_k}$ is principal and so must be $I^{rn_k} D_{M(I_k)} \cap D = I_k^{rn_k} D_{M(I_k)} \cap D = (I_k^{rn_k})^* = (h_{kl_k})^{rn_k}$ for every positive integer r. But that means that if $n = n_1 n_2 ... n_r$ we have $(I^n)^* = (I_1^n I_2^n ... I_r^n)^*$ principal.

Finally as I was arbitrary, we conclude that for every integral $*$-invertible $*$-ideal I of D, $(I^m)^*$ is principal for some positive integer m and this, indeed, forces $Cl_*(D)$ to be torsion. □

Examples: (c) (When $* = d$ and no restriction on dimension). Let (R, M) be a quasi local almost factorial domain, let L be the quotient field of R and let X be an indeterminate over L. Then the ring $D = R + XL[X]$ is an $n+1$ dimensional d-SH domain such that $Cl_d(D)$, the ideal class group of D is zero but $Cl_t(D)$ is torsion, because all d-homog ideals are either principal or of the form $f(X)(A + XL[X])$ where A is an ideal of R and $((A + XL[X])^r)_t = ((A^r)_t + XL[X])$. Thus if for every finitely generated ideal A of R we have a positive integer r such that $(A^r)_t$ is principal the corresponding ideals of the form $f(X)(A + XL[X])$ have the same property.

Of course Example (c) cannot be used as an example of a t-waf-SH domain, if R is not t-local. For if R has say maximal t-ideals M and N then D has two corresponding maximal t-ideals $M + XL[X]$ and $N + XL[X]$, ensuring that D is not a t-SH domain. Meaning that for D to be a t-SH domain R has to be a domain with a unique maximal t-ideal. But such a domain will have to be t-local. Indeed if R has a unique maximal t-ideal N then every nonzero non unit of R would be contained in N and that forces every maximal ideal of R contained in N. But R being t-local means that $Cl_t(D)$ is zero, slightly more than torsion.

The reason is the following result.

Proposition 19. *Let (D, M) be a t-local domain. Then (a) if A is a t-invertible ideal of D then A is principal and (b) If A is an ideal of D such that $(A^n)_t = D$ for some positive integer n, then A is principal.*

Proof. (a) If A is t-invertible, then AA^{-1} is not contained in any maximal t-ideal and so AA^{-1} is not contained in M. But then AA^{-1} is not contained in any maximal ideal of D, because M is the only maximal ideal of D. Hence. $AA^{-1} = D$. But then A is invertible in a quasi local domain and hence principal, (b) if $(A^n)_t = D$ then A is t-invertible and so, by (a) above, is principal. □

Remark 3. *Part (a) of Proposition 19 is* [5, Proposition 1.12] *but the proof there is not quite clear. Here we have made the necessary clarifications. Thus even if (D, M) is an almost valuation domain, i.e. for each pair of nonzero elements x, y there is a positive integer n such that $x^n | y^n$or , $y^n | x^n$, $Cl_t(D) = (0)$.*

This gives us Example (b) all over again. That means $D = R + XL[X]$ is not of much use in this context.

Indeed if, on the other hand, we consider the ∗-super homog ideals in ∗-super SH domains, things fall into the realm of what we already know.

Proposition 20. *For a domain D the following statements are equivalent: (1) D is a ∗-IRKT whose ∗-super homog ideals are also ∗-wf-homog, (2) D is a ∗ waf-SH domain whose ∗-waf-homog ideals are also ∗-super homog, (3) D is an AGCD ∗-IRKT, (4) D is a ∗-IRKT with $Cl_*(D)$ torsion, (5) D is a locally AGCD ∗-IRKT and $Cl_d(D)$ is torsion.*

Proof. (1) ⇒ (2) By chasing the definitions (2) ⇒ (4) A ∗-waf-SH domain has torsion ∗-class group by Proposition 18 and every ∗-waf-homog ideal being ∗-super homog makes D a ∗-IRKT, (4) ⇒ (3) Note that if D is ∗-IRKT with $Cl_*(D)$ torsion, then every ∗-ideal K of D is is ∗-invertible such that $(K^n)^*$ is principal for some n and so every ∗-homog ideal I of D is ∗-super homog such that $(J^n)^*$is principal for every ∗-ideal J of finite type containing I. Thus every ∗-super homog ideal of D is ∗-af-homog. Now apply Theorem 6, (3) ⇒ (1) is straightforward in that in an AGCD ∗-IRKT every ∗-super homog ideal is ∗-af-homog which meets the requirement for it to be a ∗-waf-homog ideal, (3) ⇒ (5) D being an AGCD domain implies that D is locally AGCD and $Cl_t(D$ is torsion and $Cl_t(D) \supseteq Cl_*(D)$, (5) ⇒ (3) Let's note that D is a locally AGCD domain if for each maximal ideal M

of D we have for each pair $a, b \in D$ a positive integer $n_M = n_M(a, b)$ such that $a^{n_M} D_M \cap b^{n_M} D_M$ is principal, then prove the following result.

Let D be a locally AGCD domain and let D be of finite t- character that is $D = \bigcap_{P \in \mathcal{F}} D_P$ is locally finite. If $Cl_d(D)$ is torsion then D is an AGCD domain. □

Proof. Let w be the usual star operation induced by $\{D_P\}_{P \in t\text{-}\max(D)}$.Since D is locally AGCD, each of D_P is an AGCD domain and so for each pair a, b of non zero elements of D we have for some positive integer n_P, $(a^{n_P} D \cap b^{n_P} D) D_P = a^{n_P} D_P \cap b^{n_P} D_P$ principal (actually, as D_P is t-local and AGCD D_P is an almost Bezout domain.) Now for each P there would be a medley of numbers ($\{n_{M(P)}$ for each maximal ideal M containing $P\}$ but choosing any one would serve our purpose. Because $D = \bigcap_{P \in t\text{-}\max(D)} D_P$ is locally finite, $aD \cap bD$ is contained in at most a finite number $P_1, P_2, ..., P_r$ of members of t-max(D), ones that contain a or b. Choose $n = \text{lcm}(n_{P_1}, n_{P_2}, ..., n_{P_r})$. Now as $(a^{n_{P_i}} D \cap b^{n_{P_i}} D) D_{Pi} = a^{n_{P_i}} D_{P_i} \cap b^{n_{Pi}} D_{P_i}$ is principal, for each i, and as $n_{P_i} | n$ we have $(a^{n_{P_i}} D_{P_i} \cap b^{n_{Pi}} D_{P_i})^{n/n_{p_i}} = ((a^{n_{P_i}})^{n/n_{p_i}} D_{P_i} \cap (b^{n_{P_i}})^{n/n_{p_i}} D_{P_i}) = a^n D_{P_i} \cap b^n D_{P_i} = (a^n D \cap b^n D) D_{P_i} = d_i D_{P_i}$ principal and of course for all those $Q \in \mathcal{F}$ such that none of a, b belong to Q we have $(a^n D \cap b^n D) D_Q = D_Q$ and hence principal we conclude, as in the proof of Lemma 3 that $(a^n D \cap b^n D) = (ab, d_1, ..., d_r)_w = (ab, d_1, ..., d_r)_v$. Going back again and applying the result that if A is a t-ideal of finite type and t-locally principal then A is t-invertible. Now $(a^n D \cap b^n D)$ is t-invertible and so, of finite type and as D is locally AGCD there is for each maximal ideal M a positive integer m_M such that $((a^n D \cap b^n D) D_M)^{m_M})_v = d_M D_M$ or, as $(a^n D \cap b^n D)$ is t-invertible $(a^{n(m_M)} D \cap b^{n(m_M)} D) = d_M D_M$. Thus by Theorem 2.3 of [1] there is a positive integer m such that $((a^n D \cap b^n D)^m)_v = (a^{nm} D \cap b^{nm} D)$ is invertible. But as the d-class group of D is torsion we have $(a^{nm} D \cap b^{nm} D)^r = dD$ for a positive integer r and for $d \in D$. Proving that for for each pair $a, b \in D \backslash 0$ there is a positive integer t such that $a^t D \cap b^t D$ is principal. □

The above results can give us more examples of general $*$-waf-SH domains and indeed it may not be too hard to construct examples of $*$-waf-SH domains of higher dimensions. But, as it stands, most of the available examples are one dimensional. So, for now, we look at one dimensional $*$-waf-SH domains. For that let's start with the definition of $*$-waf-homog ideals. We can say that a $*$-homog ideal of type 1 that is a $*$-waf ideal as well is a $*$-waf-homog ideal of type 1. Similarly we can just breeze through other definitions and results saying that a $*$-waf-SH domain of type 1. In the t-dimension 1 scenario one source that stands out is [7]. In it, Anderson and Mott discuss domains with only finitely many non-associated irreducible elements. These domains are called Cohen Kaplansky domains, because Cohen and Kaplansky were the first to study them in [18]. It turns out that CK-domains are weakly Krull domains with only a finite number of maximal t-ideals and D_P is a CK-domain for each maximal t-ideal P. Indeed each maximal t-ideal is of height one and maximal, this is because of the fact that if D has only a finite number of maximal t-ideals then these maximal t-ideals are precisely the maximal ideals of D. It was also established in [7] that a CK-domain D is an AGCD domain that happens to have $Cl_t(D) = 0$. In other words a CK-domain is weakly factorial domain and an almost weakly factorial domain.

The other important source of examples is [13]. In this paper the authors study under the name of generalized weakly factorial domains the domain whose nonzero non units x have the property that for each x there is a positive integer n such that x^n is a product of primary elements. These are weakly Krull domains with torsion t-class group. (Indeed as $xD = ((xD_{P_1} \cap D)(xD_{P_2} \cap D)...(xD_{P_r} \cap D))_t$ and the t-class group is torsion we get the same result.)

The second author got interested in generalizing the existing notions of unique factorization from Professor P.M. Cohn's work. Perhaps the second author was not too interested in non-commutative algebra, that Cohn was so admirably good at, the second author chose to concentrate on unique factorization in commutative ring theory. His first attempt was the theory of GUFDs. Then he tried to mimic Cohn's rigid factorizations [17] in the commutative rings. Apparently all he had to go on was that if r is rigid in the non-commutative domain R then the lattice $L(Rr, R)$ was a chain and that Paul Cohn used 2-firs for rigid factorizations. Another good yet brief source, if you want to have a quick idea is Cohn's survey on UFDs [16]. Now in the commutative case, r being rigid boils down to a non-unit r such that for all $x, y|r$ we have $x|y$ or $y|x$. But then an irreducible element is also rigid and products of irredible elements produce unique factorization under some very stringent conditions. Now Cohn's 2-firs in the commutative case are Bezout domains. It was easy to show that in a Bezout domain a product of finitely many rigid elements can be uniquely written as a product of mutually coprime rigid elements. So, he tried to see if a product of finitely many rigid elements in a GCD domain D is uniquely expressible as a product of mutually coprime rigid elements. It worked and he wrote his paper on Semirigid GCD domains [32]. But the question was: How to define a rigid element so that in a general commutative domain D a finite product of rigid elements is uniquely expressible as a product of mutually coprime "improved" rigid elements? The definition of ∗-f-homog does that. Now the question is: Can we do something similar to the definition of rigid in the non-commutative case, to get better results?

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