# UNIQUE FACTORIZATION AND RELATED TOPICS

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## ABSTRACT

This work can be split into two parts. In the first part we generalize the concept of Unique Factorization by viewing Unique Factorization Domains as integral domains, non zero non units of which can be expressed uniquely (up to associates and order) as products of finitely many mutually co-prime associates of prime powers. Our working rule consists of taking a subset Q of the set P of all properties of a general prime power and investigating integral domains, whose non zero non units are expressible uniquely as as pro- products of finitely many non units satisfying the properties in Q. For example we take Q consisting of only one property: of any two factors of a prime power one divides the other and call a non unit x rigid if for each h,k dividing x one divides the other. We find that in a Highest Common Factor domain a product of finitely many rigid elements is expressible uniquely as the product of mutually co-prime rigid elements. And a Highest Common Factor domain with the set of non zeros generated by rigid elements and units is the resulting generalization of a Unique Factorization Domain.

We consider three different Q's which, for suitable integral domains give distinct generalizations of Unique Factorization domains. In each case we provide examples to prove their existence, discuss their points of difference with UFD's and study their behaviour under localization and adjunction of indeterminates. We also study these integral domains in terms of the valuations of their fields of fractions and show that these integral domains are generalizations of Krull

domains.

The second part is mainly a study of ideal transforms in generalized Krull domains and some of the results are generalizations of results known for Krull domains.

CONTENTS

# CHAPTER 0

## INTRODUCTION AND CONTENTS

1. Introduction . The main purpose of this work is to study Unique Factorization and its generalizations in commutative integral domains. A Unique Factorization Domain is defined to be an integral domain in which every non zero non unit element x is expressible as the product of a finite number of principal primes i.e.

$$x = p_1 p_2 \dots p_n$$

where a principal ideal (p) is a principal prime if $p \mid ab$ implies that $p \mid a$ or $p \mid b$.

It is well known that

(1) a Unique factorization domain (UFD) is an HCF domain i.e. every two elements have a highest common factor.

(2) a UFD is a Krull domain i.e an integral domain R such that

$K_1$. every non zero non unit of R is contained in only a finite number of minimal non zero prime ideals of R

$K_2$. for every non zero minimal prime ideal P of R, $R_P$ the localization at P is a discrete rank one valuation ring.

$K_3$. $R = \cap R_P$ where P ranges over all minimal non zero primes of R .

(3) every non zero non unit x of a UFD can be written as $x = u p_1^{a_1} p_2^{a_2} \dots p_n^{a_n}$; where u is a unit $a_i > 0$ and $p_i$ , $p_j$ are are co-prime if $i \neq j$ (cf [30] Theorem 5.3 (g)).

We observe that if $x = u p_1^{a_1} p_2^{a_2} \dots p_n^{a_n}$ as in (3) above it is expressible as a product of a finite number of mutually co-prime elements $u_i p_i^{a_i}$ ($i = 1,2,\dots,n$) where $u_i p_i^{a_i}$ are such that (1) for every non unit $x_i \mid u_i p_i^{a_i}$ there exists a positive integer $n_i$ such that $u_i p_i^{a_i} \mid x_i^{n_i}$.

(2) for every n and for every pair $x_i, y_i \mid u_i p_i^{na_i}$ ; $x_i \mid y_i$ or $y_i \mid x_i$.

(3) if $u_i p_i^{na_i}$ is non co-prime to ab then for every n and for every non unit $y \mid u_i p_i^{na_i}$, which divides ab, $y = y_1 y_2$ where $y_1 \mid a$ and $y_2 \mid b$.

This observation gives rise to the

Question. If an element x in an integral domain R is expressible as $x = q_1 q_2 \ldots q_n$ (A)
where $q_i$ are non units, $(q_i, q_j) = 1$ if $i \neq j$ and for each $q_i$ it is true that

$Q_1$. for each non unit $h_i \mid q_i$ there exists $n_i$ such that $q_i \mid h_i^{n_i}$.

$Q_2$. for each n and for each pair $h_i, k_i \mid q_i^n$ ; $h_i \mid k_i$ or $k_i \mid h_i$.

$Q_3$. if $q_i$ is non co-prime to ab then for every n and for every $y \mid q_i^n$ which divides ab; $y = y_1 y_2$ where $y_1 \mid a$ and $y_2 \mid b$.

Is the factorization (A) unique up to associates and order of $q_i$ even if $q_i$ are not powers of primes ?

The main part of this work is the result of an effort to find an answer to the above question. We in fact find out a number of different generalizations of Unique Factorization Domains.

2. Notations and Notions. We explain the notations and notions when ever we use them except for those in common use e.g. (1) we use $a \mid b$ to indicate, <u>a divides b</u>

(2) (a,b) is used to denote the highest common factor of a and b as well as the ideal generated by a,b and the context determines the meaning of (a,b). More over we use $(a,b) \neq 1$ to denote that a and b have at least one non unit common factor

(3) by x <u>is an associate of</u> y we mean
x = uy ; y = vx where u and v are units.

Finally we mention that all rings considered are commutative with 1.

3. Contents . In Chapter 1, we prove that the answer to the above question is in the affirmative. And from this arises the concept of a Generalized Unique Factorization Domain (GUFD). We show that a GUFD is a generalized Krull domain (GKD) where a GKD is an integral domain satisfying $K_1, K_3$ of the definition of a Krull domain along with:

($K_2'$). for every minimal prime P, $R_P$ is a rank one valuation domain. We also show that an HCF-GKD is a GUFD.

In Chapter 2, we consider the properties of a non unit $x \neq 0$ satisfying

(R). for every pair of factors h,k of x ; $h|k$ or $k|h$.

Elements satisfying (R) are already known and are called <u>rigid</u> elements (cf [ 6 ] page 129). We restrict our study of rigid elements to those in HCF domains and show that if in an HCF domain R an element x is expressible as the product of a finite number of mutually co-prime non unit rigid elements i.e.

$$x = r_1 r_2 \dots r_n \ ; \ r_i \text{ rigid and } (r_i, r_j) \neq 1 \text{ for } i \neq j$$

then this expression is unique up to associates of and up to a permutation of $r_i$. We shall call an HCF domain R a <u>Semi-rigid Domain</u> if each non zero non unit of R is expressible as a product of a finite number of mutually co-prime rigid non units. We also show that if R is a Semirigid Domain then there exists a family $F = \{P_\alpha\}_{\alpha \in I}$ of prime ideals of R such that

$S_1$. every non zero non unit of R is contained in only a finite number of elements of $F$ .

$S_2$. $P_\alpha \cap P_\beta$ does not contain a non zero prime ideal, $\alpha \neq \beta$, $\alpha, \beta \in I$

$S_3$. $R_{P_\alpha}$ is a valuation domain for each $\alpha \in I$

$S_4$. $R = \cap R_{P_\alpha}$ $\alpha \in I$.

Obviously if F consists of minimal primes only, the above four conditions define a GKD i.e. Semirigid Domains are another generalization of Krull domains.

In Chapter 3, we consider the factorization of an arbitrary non zero non unit in an HCF domain of Krull type and use this study to define Unique Representation Domains.

Chapter 4, is mainly concerned with the study of ideal transforms in a GKD and a part of it consists of extensions of results proved in [15].

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# CHAPTER 1

## GENERALIZED UNIQUE FACTORIZATION DOMAINS

0. Introduction . The theory of Unique Factorization Domains is well known and the most part of the theory is covered by [30],[31],[32] and by [23].

To start with, we mention that if R is a UFD then every non zero non unit x in R can be expressed as

$$x = up_1^{a_1} p_2^{a_2} \ldots p_r^{a_r} \quad \text{-------} \quad \text{-------} \quad (A)$$

where u is a unit and $p_i^{a_i}$ are powers of primes such that $(p_i^{a_i}, p_j^{a_j}) = 1$ if $i \neq j$ and the expression (A) is unique up to associates of the prime powers and up to a suitable permutation (cf [30] page 16).

We call a non zero non unit a an atom if $a = a_1 a_2$ implies that $a_1$ or $a_2$ is a unit and an integral domain is called atomic if every element in it is expressible as a product of a finite number of atoms. A prime is defined to be a non zero non unit p such that $p|ab$ , implies that $p|a$ or $p|b$. Obviously if $p = ab$ and $a = a'p$ ; $p = a'bp$ i.e. $1 = a'b$ , that is b is a unit, similarly we could take $b = b'p$ and show that a is a unit. In other words a prime is an atom and a UFD is an atomic integral domain.

Our main aim in this chapter is to replace the prime powers by the more flexible non units; prime quanta which behave like prime powers but are not products of atoms, and to work out a generalized theory of factorization which does not require a generalized unique factorization domain to be atomic.

Section 1, of this chapter mainly deals with the definition of a prime quantum, its properties and with the definition of a Generalized Unique Factorization Domain(GUFD) as an integral domain in which every non zero non unit is is expressible as the product of a finite number of mutually co-prime, prime quanta. In section 2 we give examples to ensure the existence of notions introduced in section 1, and of course to justify their introduction. Section 3, establishes analogues of some results about UFD's, while in section 4, we study the stability properties of the GUFD's. In section 5, we study the ideal theory of GUFD's and related integral domains and at the end of this section we prove that if a proper ideal A in a Prüfer domain R has a primary decomposition then this decomposition is unique.

1. Definition and properties of Prime quanta.

We split our task of defining a prime quantum into two parts, that is we give the generalization of the concept of atom first and state the

Definition 1. A non zero non unit element h in an integral domain R will be called a quantum if for each non unit $h_1 | h$ there exists a positive integer n such that $h | h_1^n$.

We note that the semigroup $R^* = R - \{0\}$ is preordered by $a|b$ ( divisibility) and if U is the set of all the units of R then the semigroup $R^*/U$ is partially ordered by $aU < bU$ iff $a|b$, and obviously by h is a quantum we mean that for every $U \neq h_1U < hU$ there exists a positive integer n such that $hU < h_1^nU$. In view of the partial order we may call a quantum $h_2$ higher than another quantum $h_1$ if $h_1U < h_2U$.

Definition 2. If in an integral domain R a quantum h divides an element a such that there exists no other quantum $h_1$ with $hU < h_1U < aU$, then h will be said to divide a completely.

Now to make a quantum behave more like a prime power we impose some more conditions on it by

Definition 3. A quantum $q$ in an integral domain $R$ will be called a <u>prime quantum</u> if

(1) for every $n$ and for every $q_1, q_2 | q^n$, $q_1 | q_2$ or $q_2 | q_1$

(2) if $q$ is non co-prime to $ab$ then for every $n$ and for every $q_1 | q^n$ which divides $ab$, $q_1 = q_r q_s$ such that $q_r | a$ and $q_s | b$ i.e. every factor of $q^n$ is primal.

We recall that an element $x$ in an integral domain is called primal if $x | ab$ implies that $x = yz$ ; $y | a$ and $z | b$ and an integrally closed integral domain in which every non zero element is primal is a <u>Schreier domain</u>. More over an HCF domain is a Schreier domain (cf [5] p.254).

Looking back at the Definitions 1 and 3, we note that an atom vacuously satisfies the condition for an element to be a quantum, while a prime $p$ is a prime quantum because every factor of $p^n$ is primal and this marks the basic difference between the concepts of a quantum and of a prime quantum.

Definition 4. Two prime quanta will be called <u>similar</u> if they are non co-prime and dissimilar or distinct otherwise.

Lemma 1. In any integral domain R.

(1) Any non unit factor of a prime quantum is a prime quantum.

(2) If $q_1, q_2$ are similar prime quanta then $q_1 | q_2$ or $q_2 | q_1$.

(3) If $q_1, q_2$ are similar prime quanta then $q_1 q_2$ is a prime quantum similar to them.

(4) If a prime quantum $q$ divides $ab$ completely, that is there is no prime quantum $q' | ab$ such that $q | q'$ properly; then $q = q_1 q_2$ where $a = a_1 q_1$ , $b = b_1 q_2$ and $(a_1, q) = 1 = (b_1, q)$.

(5) The relation of similarity between prime quanta is

an equivalence relation.

Remark 1. Statements (1) - (3) can be equivalently replaced by the following comprehensive statement:

" The prime quanta in an integral domain similar to a given one, with units form a multiplicative set which is saturated and totally ordered by divisibility."

Proof. (1) Let q be a prime [redacted] quantum and $q_1$ be a non unit factor of q. To prove that $q_1$ is a prime quantum we have to show that $q_1$ satisfies (1) and (2) of Definition 3,(obviously $q_1$ is a quantum). Now for some n

$q_r, q_s | q_1^n$ then $q_r, q_s | q^n$ and so $q_r | q_s$ or $q_s | q_r$ i.e. (1) of Definition 3,is satisfied.

Further if $q_1$ is non co-prime to ab then so is q, and every factor $q_t$ of $q_1^n$ which divides ab, being also a factor of $q^n$ can be written as $q_t = q_u q_v$ where $q_u | a$ and $q_v | b$, which is (2) of Definition (3).

(2) If $q_1, q_2$ are similar prime quanta then let $q_3$ be a non unit common factor of $q_1$, $q_2$. By (1) above $q_3$ is a prime quantum. So there exist m,n such that $q_1 | q_3^n$ , $q_2 | q_3^m$ and thus $q_1 q_2 | q_3^{n+m}$ and by (1) of Definition 3, $q_1 | q_2$ or $q_2 | q_3$ .

(3) We establish that if q is a prime quantum then $q^m$ is again a prime quantum (for every positive integral m). By (1) of Def. 3, if $x,y | q^m$ then $x|y$ or $y|x$. So if a non unit $h | q^m$ , $h|q$ or $q|h$. If $h|q$ then there is n such that $q | h^n$ and so $q^m | h^{nm}$, and if $q|h$ then $q^m | h^m$. Hence $q^m$ is a [redacted] quantum. Further if $h_1, h_2$ are factors of an integral power of $q^m$, $h_1, h_2$ are factors of a power of q and so $h_1 | h_2$ or $h_2 | h_1$. Similarly if $q^m$ is non co-prime to ab then so is q and it is easy to see that $q^m$ satisfies (2) of Def. 3 .

Finally if $q_1, q_2$ are similar prime quanta and if $q_3$ is is a non unit common factor then there exists an integer m such that $q_1 q_2 \mid q_3^m$ i.e. $q_1 q_2$ is a factor of a prime quantum and hence is a prime quantum.

(4) Let q be a prime quantum such that $q \mid ab$ completely,. By (2) of Def.3, $q = q_1 q_2$ such that $q_1 \mid a$ and $q_2 \mid b$, so that $ab = a_1 b_1 q_1 q_2$ . Suppose that $(a_1 , q) \neq 1$, and let $q_3$ be a non unit common factor i.e. $a_1 = a_2 q_3$. Thus

$ab = a_2 b_1 q_1 q_2 q_3$, but then $q_1 q_2 q_3 = qq_3$ is a prime quantum higher than q with respect to ab, a contradiction and hence $(a_1 , q) = 1$. Similarly $(b_1 , q) = 1$.

(5) Reflexivity and symmetry are obvious. For transitivity let $q_1 , q_3$ and $q_2$ be prime quanta such that (a) $q_1$ is similar to $q_2$ and (b) $q_2$ is similar to $q_3$.

Here (a) implies that $q_1$ and $q_2$ have a non unit common factor $q_{12}$ say. Now $q_2$ and $q_3$ are similar and so by (3) above $q_2 \mid q_3$ or $q_3 \mid q_2$. If $q_2 \mid q_3$ then $q_{12} \mid q_3$ and so $q_1$ and $q_3$ are similar. Further if $q_3 \mid q_2$ then since $q_{12}$ and $q_3$ both divide a prime quantum $q_2$, $q_{12} \mid q_3$ or $q_3 \mid q_{12}$, that is $q_1$ and $q_3$ are similar.

Corollary 1. A quantum is a prime quantum iff it has a prime quantum as a factor.

Proof. If q is a quantum and $q_1$ is a prime quantum dividing it then there exists a positive n such that $q \mid q_1^n$. Now $q_1$ being a prime quantum the result follows from (1) and (3) of the above lemma. The converse is obvious.

Corollary 2. If a prime quantum $q \mid ab$ and $(q,a) = 1$ then $q \mid b$.

Proof. By (2) of Def. 3, if $q \mid ab$ then $q = q_1 q_2$ such that $q_1 \mid a$ and $q_2 \mid b$, but since $(q,a) = 1$, $q_1$ is a unit and hence $q \mid b$.

Proposition 2. If an element in an integral domain R is expressible as the product of a finite number of distinct dissimilar prime quanta then the expression is unique up to the permutation of distinct prime quanta and up to their associates.

Proof. Let x be a non zero non unit element in an integral domain R and let x be a product of prime quanta $q_i$ i.e.

$x = q_1 q_2 \dots q_n$ , $q_i , q_j$ dissimilar if $i \neq j$

Suppose that x can also be written as

$x = p_1 p_2 \dots p_m$ ; $p_j$ prime quanta, $p_i , p_j$ dissimilar if $i \neq j$. Now

$$q_1 q_2 \dots q_n = p_1 p_2 \dots p_m$$

Since $q_1$ is a factor of the L.H.S.

$$q_1 \mid p_1 p_2 \dots p_m$$

and similarity between prime quanta being an equivalence relation, $q_1$ can be similar to ~~only one~~ at most one of the $p_i$ ( $i = 1 \dots m$ ) while from the definition of a prime quantum it follows that $q_1$ is similar to at least one of the $p_i$ . That is there exists a unique $p_t$ such that $q_1 \mid p_t$.

We claim that $q_1$ and $p_t$ are associates, because reversing the process, that is taking $p_t \mid q_1 q_2 \dots q_n$ , we get $p_t \mid q_1$. And combining the two results confirms the claim.

Now we are left with

$$q_2 q_3 \dots q_n = p_1 p_2 \dots p_{t-1} p_{t+1} \dots p_m$$

and repeating the above procedure we conclude that n = m and each $q_i$ is an associate of some $p_i$ for a suitable permutation of $p_1, p_2, \dots p_n$.

Definition 5. An integral domain R will be called a Generalized Unique Factorization Domain (GUFD for short) if every non zero non unit element x in R can be expressed as the product of a finite number of distinct prime quanta.

The proof of Proposition 2, depends heavily on the assumption that we can write $x = q_1 q_2 \dots q_n$, where (1) $q_i$ are prime quanta (i = 1,2,..n) and (2) $q_i , q_j$ are dissimilar if $i \neq j$.

In the case of an element x which is a product of primes we do not need the assumption (2) above, while proving the uniqueness of the factorization because of the fact that a prime is an atom . But as it can be easily verified that every positive integral power of a prime is a prime quantum we can easily achieve the form

$x = up_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$ ; where u is a unit and $p_i, p_j$ are non associate primes for $i \neq j$, and hence $p_i^{a_i}, p_j^{a_j}$ are distinct prime quanta. But before accepting the above two restrictive assumptions as a price of generalization we have to be sure that there do exist (1) quanta (2) prime quanta (3) quanta which are not prime quanta (4) Generalized Unique Factorization Domains.

2. Examples.

(1) Quanta: Example 1. Every atom is a quantum.

Obviously every non unit factor of an atom a is an associate of a, and so an atom satisfies the condition of being a quantum.

Example 2. Let R be a quasi-local domain of Krull dimension 1. It is well known that if a,b are two non zero non units of R then there exists a positive integer n such that $b|a^n$ (cf Theorem 108 [23]). And of course the result is symmetric, that is $a|b^m$ for some positive integral m. So if x is a non zero non unit in R and h is a non unit factor of x then there exists n such that $x|h^n$. Thus we conclude that every non zero non unit element of R is a quantum. This example also establishes the existence of quanta which are not atoms e.g. when R is non Noetherian.

(2) Prime quanta: Example 3. A prime is a prime quantum.

As we have mentioned before this fact can be easily verified. It can also be verified that an atom is a prime quantum iff it as a prime.

Example 4. Let R be a rank one valuation ring. Obviously R is a quasi-local ring of Krull dimension 1. So that by Example 2, above every non zero non unit of R is a quantum. Further, R being a valuation ring if x is a non zero non unit of R then for every positive integer n and for every $x_1, x_2 | x^n$ , $x_1 | x_2$ or $x_2 | x_1$ (holds vacuously). And if x is non co-prime to ab then at least one of a,b is a non unit and so is non co-prime to x . Moreover if $y | x^n$ for some n such that $y | ab$ then $y = y_1 y_2$ where $y_1 | a$ , $y_2 | b$ ( follows from the fact that a valuation ring is HCF). So we have verified that x satisfies (1) and (2) of Def. 3, and thus is a prime quantum. It may be noted that x is an arbitrary non unit of R.

(3) *Comparing* Examples (1), (2) and (4) we see that any atom which is not a prime can serve as an example of a quantum which is not a prime quantum. Also since there exist non Noetherian integral domains of Krull dimension 1, which are not valuation domains we have our examples of non atomic quanta which are not prime quanta.

• (4) Generalized Unique Factorization Domains:

Example 5. A UFD is a GUFD. This follows from the fact that a prime is a prime quantum.

Example 6. A rank 1, valuation domain. Each non zero non unit of a rank one valuation domain is a prime quantum (Ex.4) and so the statement that, "Every non zero non unit is a product of a finite number of distinct prime quanta." is satisfied.

Example 7. Let S be the product of two copies of positive rationals i.e.

$S = \{ x^{\alpha} y^{\beta} \;;\alpha,\beta$ rationals $\geqslant 0 \}$ where x,y are indeterminates over the field of reals. Let R be the field of real numbers and consider the algebra $R[S] = L$ say. It is not difficult to prove that L is an integral domain. Let

$T = \{ t \in L \mid t$ is co-prime to x and y both$\}$.

The set T has elements of the type:

$$\left.\begin{aligned} t_1 &= r_1 + ax^{\alpha} \\ t_2 &= r_2 + by^{\beta} \\ t_3 &= ax^{\alpha} + by^{\beta} \end{aligned}\right\} \begin{aligned} &r_1,\, r_2 \in R - \{0\} \\ &a,b \in R[S] \\ &;(y^{\beta},a) = 1 = (x^{\alpha},\, b) \end{aligned}$$

$t_4 = \ell + ax^{\alpha} + by^{\beta}$ where $\alpha,\beta > 0$, $\ell \in R - \{0\}$ and $a, b \in R[S]$.

The forms of these elements show that T is a multiplicative set, and is saturated(cf Sec.3). Now in the localization, $(R[S])_T = D$ , every element d can be written as

$d = ux^{\alpha}y^{\beta}$ ; where u is a unit and obviously this expression is unique. It can also be verified that $x^{\alpha}, y^{\beta}$ are prime quanta ($\alpha$ ,$\beta$ rational $\geqslant 0$).

Example 7, above ensures the existence of GUFD's and as we develop the theory further we shall see that there exists a sufficiently large class of integral domains which are GUFD's but are not UFD's.

## 3. Some Results analogous to Classical theorems.

First we recall that in a ring R a set S is said to be multiplicative if a,b $\in$ S implies that ab $\in$ S and S is saturated if ab $\in$ S implies that a,b $\in$ S. Further it is well known that in an integral domain R a set S generated by primes is multiplicative and saturated. Analogously we prove

Proposition 3. Let R be an integral domain and H the set generated multiplicatively by units and prime quanta then H is multiplicative and saturated.

Proof. The hypothesis implies that if x$\in$ H then

$x = q_1 q_2 \ldots q_n$ where each $q_i$ is a prime quantum or a unit for each $i = 1,2,\ldots n$. From the fact that the product of two similar prime quanta is a prime quantum similar to them we deduce that if x is a non unit we can write

$$x = p_1 p_2 \ldots p_m \quad , \; p_i, p_j \text{ dissimilar if } i \neq j.$$

That H is multiplicative is quite obvious. To prove that H is saturated let $ab \in H$,.

First suppose that $ab = q$ a single prime quantum. Either, one of them is a unit or both are similar prime quanta, and in both cases $a,b \in H$.

Further let $ab = q_1 q_2$ where $q_1, q_2$ are distinct prime quanta. Now as $q_1, q_2$ are distinct $q_1 | ab$ completely and so $q_1 = q_{1r} q_{1s}$ where $q_{1r} | a$ , $q_{1s} | b$ such that $a = a_1 q_{1r}$, $b = b_1 q_{1s}$ and $(a_1, q_1) = 1 = (b_1, q_1)$ (cf (4) of Lemma 1). Consequently $q_2 = a_1 b_1$ implying that $a_1 | b_1$ or $b_1 | a_1$ i.e. one of them is a unit or both are prime quanta. In other words a and b both are products of prime quanta and hence are in H.

Applying induction on the number of distinct prime quanta involved we can prove that if

$$ab = q_1 q_2 \ldots q_n \; ; \quad q_i, q_j \text{ distinct for } i \neq j,$$

then a,b are products of prime quanta and hence are in H i.e. H is saturated.

An integral domain in which every two elements a,b have the highest common factor is called an <u>HCF domain</u>. It is well known that a UFD is an HCF domain and in analogy to this we state the

Proposition 4. A GUFD is an HCF domain.

Proof. Let R be a GUFD and let $x,y \in R$ if one of them is a unit then obviously they have a highest common factor; a unit. If one of them say y is zero then x is the highest common factor. thus we can assume x and y to be non zero non units. Now let

$x = q_1 q_2 \dots q_n$ , $q_i$ prime quanta and all distinct

$y = p_1 p_2 \dots p_m$ , $p_i$ prime quanta and all distinct

Now for every prime quantum $q_i | x$ ( $i = 1,2,\dots n$) $q_i$ has a common factor with y or does not. Also if $q_i$ does have a common factor with y then $q_i$ is similar to one and only one of $p_j | y$ (Def. 3). Now select out of $q_1, q_2, \dots q_n$ all those prime quanta $q_1', q_2', \dots q_r'$ such that $(q_i', y) \neq 1$. Similarly select out of $p_1, p_2, \dots p_m$ all those $p_1', p_2', \dots p_s'$ such that $p_j'$ are non co-prime to x. By the above assertion $r = s$ and we can form pairs $\{ q_i', p_i' \}$ of similar prime quanta for a suitable permutation of $p_i'$ say.

Let $d_i = (p_i', q_i')$ where $d_i = p_i'$ if $p_i' | q_i'$ and $d_i = q_i'$ if $q_i' | p'$. Obviously as $p_i'$ and $q_i'$ are similar in pairs, $d_i$ exists for each $i = 1,2,\dots r$. And it is easy to see that in each case $d_i$ is the HCF of $p_i', q_i'$.

Let $d = d_1 d_2 \dots d_r$ ; that d is a common factor of x and y is obvious. To prove that d is the highest common factor we have to show that every common factor d' of x and y divides d. We first note that d' is a product of prime quanta that is $d' = \pi_1 \pi_2 \dots \pi_t$ ; $\pi_i$ distinct prime quanta dividing x and y. That is each $\pi_i$ is similar to one of $d_1, d_2, \dots d_r$ and so divides it. And it is easy to see that $d' | d$ and that d is the highest common factor.

Remark 2. Many notions in the classical theory of Unique Factorization are taken as granted; for example we hardly need to state the fact that if in a UFD, x is a non unit factor of y then there exists a positive integer n such that $x^n \nmid y$. If on the other hand we need to stress this fact we content ourselves by saying that a UFD is atomic. In case of a GUFD the above mentioned property holds but needs an explanation:

Let $y = p_1 p_2 \dots p_n$ where $p_i$ are distinct prime quanta. And let x be a non unit factor of y, then by Proposition 3, x is expressible as a product of distinct prime quanta, that is $x = p_1' p_2' \dots p_s'$ where $p_i'$ are distinct prime quanta each dividing one ( and hence only one) of $p_1, \dots, p_n$. Suppose that for a suitable permutation of $p_j$, $p_i' | p_i$. And by the definition of a quantum, there exists a positive integer n such that $p_i | p_i'^n$ (properly) that is $x^n$ has at least one prime quantum as a factor which does not divide the prime quantum factor of y which is similar to it and hence $x^n \nmid y$.

Before proceeding further with the analogy, we need an auxiliary arrangement of some new notions and facts. As our first step we introduce the notion of a prime ideal associated to a prime quantum.

Let q be a prime quantum in an integral domain R and put $Q_q = \{ x \in R \mid (x,q) \neq 1 \}$.

Now $x, y \in Q_q$ implies that there are two prime quanta $q_1, q_2$ such that $x = x_1 q_1$, $y = y_2 q_2$. As similarity between prime quanta is an equivalence relation, $q_1$ and $q_2$ are similar and consequently $q_1 | q_2$ or $q_2 | q_1$. If $q_1 | q_2$ say,

$x + y = x_1 q_1 + y_2 q_2 = q_1 (x_1 + y_2 q_2')$ non co-prime to $q$,

that is $x + y \in Q_q$. And since for every x non co-prime to q rx is non co-prime to q for every r in R, $Q_q$ is an ideal. Moreover $xy \in Q_q$ implies that xy is non co-prime to q and by Def. 3, either x is non co-prime to q or y is i.e. $xy \in Q_q$ implies that $x \in Q_q$ or $y \in Q_q$ and so $Q_q$ is a prime ideal. And this observation provides us the

Definition 6. Let q be a prime quantum in an integral domain R then the prime ideal

$Q_q = \{ x \in R \ (x,q) \neq 1 \}$ will be called the prime ideal associated to q.

Further, it is easy to see that if $q_1, q_2$ are two similar prime quanta then $q_2 \in Q_{q_1}$ and as every element in the integral domain R, non co-prime to $q_2$ is also non co-prime to $q_1$, $Q_{q_2} \subseteq Q_{q_1}$ and similarly $Q_{q_1} \subseteq Q_{q_2}$, that is $Q_{q_1} = Q_{q_2}$ and conversely if $Q_{q_1} = Q_{q_2}$ then $q_2 \in Q_{q_2}$ and so $q_2 \in Q_{q_1}$ which implies that $q_1, q_2$ are non co-prime and hence are similar.

We note that if in an integral domain R, a prime quantum q is contained in a prime ideal P then every non unit factor $q_1$ of q is in P. The proof follows from the fact that q is a quantum. This observation suggests that if a prime quantum q is in a prime P then $Q_q \subseteq P$.

For further references we record the above observations and their easy consequences as the

Proposition 5. Let $q, q_1, q_2$ be prime quanta in an integral domain R then

(1) $Q_{q_1} = Q_{q_2}$ iff $q_1$ and $q_2$ are similar.

(2) If P is a prime ideal in R and $q \in P$ then $Q_q \subseteq P$ and if P is minimal then $Q_q = P$.

(3) If P is a minimal prime ideal and $q \in P$ then $q_1 \in P$ iff $q_1$ is similar to q .

Note . By a minimal prime ideal we mean a minimal non zero prime ideal.

We recall that an integral domain R with quotient field K is called <u>completely integrally closed</u> if for a and u in K with $a \neq 0$, $au^n \in R$ for all n implies that $u \in R$ (cf [23] p.53). From Remark 2, it follows that if x and y are two elements of a GUFD R then $x^n | y$ for all n implies that x has no prime quantum as a factor i.e. x is a unit. Now a GUFD R is an HCF domain and if K is the quotient field of R then for every $u \in K-\{0\}$ , $u = x/y = x_1/y_1$ where $(x_1, y_1) = 1$.

Similarly $0 \neq a \in K$ can be written as $a = x_2/y_2$ where $(x_2, y_2) = 1$.

Now $au^n \in R$ for all n implies that $(x_2/y_2)(x_1/y_1)^n \in R$ for all n. By the HCF property $y_1^n | x_2$ for all n, which by the above observation is possible only if $y_1$ is a unit in R, that is $u \in R$. Thus we have proved the

Proposition 6. A GUFD is a completely integrally closed integral domain.

We go further in our pursuit of analogous results and state the

Proposition 7. An integral domain R is a GUFD iff every non zero prime ideal in R contains a prime quantum.

Proof. Suppose that every prime ideal of R contains a prime quantum and let S be the set generated multiplicatively by prime quanta and units of R. If $S \neq R - \{0\}$ then by Zorn's Lemma, the complement R −S contains a prime ideal and hence a prime quantum, a contradiction and hence $S = R - \{0\}$ i.e. R is a GUFD. Conversely if R is a GUFD and P a prime ideal in R, let x be a non zero element in P. Then $x = q_1 q_2 \ldots q_n$ where $q_i$ are distinct prime quanta. Obviously $q_1 q_2 \ldots q_n \in P$ implies that $q_1 \in P$ or $q_2 q_3 \ldots q_n \in P$, and proceeding in this manner we conclude that at least one of $q_i$ ( $i = 1,2,\ldots n$) is in P.

Corollary 3. If q is a prime quantum in a GUFD R then $Q_q$ the prime ideal associated to q is a minimal prime ideal($\neq 0$)

Proof. Obviously $Q_q$ is non zero. Now suppose that $Q_q$ is not minimal and let P be a non zero prime ideal contained in $Q_q$. By Proposition 7, P contains a prime quantum q' say and by (2) of Prop.5, $Q_{q'} \subset P$. But as $q' \in P \subset Q_q$ ; q' is similar to q and thus by (1) of Prop. 5, $Q_q = Q_{q'}$, so that $Q_q \subseteq P$ i.e. $Q_q = P$.

Corollary 4. In a GUFD R every non zero prime ideal contains a minimal (non zero ) prime ideal.

Proof. Immediate from Cor. 3 above.

Corollary 5. In a GUFD every non zero minimal prime ideal P is associated to a prime quantum q i.e. $P = Q_q$.

Proof. By Cor. 4, P contains a prime quantum q and the result follows from Prop. 5.

4. Stability Properties of GUFD's.

In this section we shall establish that the property of being a GUFD remains invariant under localizations and polynomial extensions. For this purpose we need to introduce the concept of a Generalized Krull Domain (GKD).

An integral domain R is called a Generalized Krull Domain if

(1) every non zero non unit x in R is contained in a finite number of minimal prime ideals of R.

(2) for every minimal prime ideal P of R, $R_P$ is a rank one valuation domain.

(3) $R = \cap R_P$ , where P varies over all the minimal prime ideals of R.

It may be noted that a Krull domain is a Generalized Krull Domain. In this section we shall use the facts that (1) every localization of a GKD is a GKD (2) if x is an indeterminate over a GKD R then R[x] is a GKD. For a detailed theory of GKD's the reader is referred to [21],[29] and [9].

As our first step towards the consideration of stability properties of GUFD's we collect some useful facts.

Lemma 8. In an HCF domain a quantum is a prime quantum.

Proof. Let q be a quantum in an HCF domain R and suppose that $x,y|q$. We claim that $x|y$ or $y|x$. For if we suppose on the contrary that $x \nmid y$ and $y \nmid x$ then R being an HCF domain x and y have a highest common factor d say, that is $x = x_1 d$, $y = y_1 d$ and $(x_1, y_1) = 1$. Obviously $x_1$, $y_1$ are non unit factors of a quantum and so by the definition of a quantum there exist m,n such that $q|x^m$ and $q|y^n$, so that $x_1 | y_1^n$ and $y_1 | x_1^m$, which in view of the HCF property implies that $(x_1, y_1) \neq 1$ a contradiction and so for all x,y dividing q, $x|y$ or $y|x$. Further we see that if $x|q^n$ for some n then by the HCF property if x is a non unit then it has a non unit factor d common with q. But $q|d^n$ for some n because q is a quantum and it follows that $q|x^n$ and that $q^n|x^{n^2}$, that is $q^n$ is a quantum for all n and it can be shown on the same lines as above that for each pair $u,v|q^n$, $u|v$ or $v|u$, which is exactly (1) of Def. 3. Moreover since an HCF domain is also Schreier every factor of $q^n$ for each n is primal that is (2) of Def.3, also holds and q is a prime quantum.

Lemma 9. If R is an HCF domain and S is a multiplicative set in R then $R_S$ is an HCF domain.

Proof. It is well known that if A and B are ideals of an integral domain R and S is a multiplicative set in R then

$$(A \cap B)R_S = AR_S \cap BR_S \qquad \text{( cf [9] p 34)}.$$

Moreover the necessary and sufficient condition for an integral domain R to be an HCF domain is that the intersection of every two principal ideals is principal(can be verified easily).

Now let $x,y \in R_S$, where R and S are as in the hypothesis. We can write $x = r_1/s_1$, $y = r_2/s_2$ where $(r_i, s_i) = 1$, and $s_i$ are units in $R_S$.

Consider $xR_S \cap yR_S = (r_1/s_1)R_S \cap (r_2/s_2)R_S$, $s_i$ being units we can write the RHS as $r_1R_S \cap r_2R_S$ but since $r_1R_S \cap r_2R_S = (r_1R \cap r_2R)R_S = [r_1, r_2]R_S$ where $[r_1, r_2]$ is the least common multiple of $r_2, r_1$.

But since x,y are arbitrary and for each pair

$xR_S \cap yR_S = r_1 R_S \cap r_2 R_S = [r_1, r_2] R_S$ a principal ideal, $R_S$ is an HCF domain.

Proposition 10. A quasi local domain with Krull dimension 1 is a valuation domain iff it is an HCF domain.

Proof. If R is a domain as in the hypothesis and is HCF also, the result follows from Example 2 and from Lemma 8. The converse is obvious.

Corollary 6. For every minimal prime ideal P in an HCF domain R, $R_P$ is a rank one valuation domain.

Proof. By Lemma 9 $R_P$ is an HCF domain and since P is minimal, $R_P$ is a one dimensional quasi local domain and so by Proposition 10, the result follows.

A simple but worthy of mention fact is recorded as

Proposition 11. If R is an integral domain in which every non zero non unit is expressible as a product of a finite number of quanta then the sufficient condition for R to be a GUFD is that it is an HCF domain.

Proof. By Lemma 8 above, every quantum of R in the hypothesis is a prime quantum. Thus every element x in R (other than zero or a unit) is expressible as the product of a finite number of prime quanta.

Let $x = p_1 p_2 \ldots p_n$, where $p_i$ are prime quanta. Then if (say) $p_1, p_2$ are not distinct then by (3) of Lemma 1, $p_1 p_2$ is a prime quantum similar to $p_1$ and $p_2$, and after a finite number of steps we are able to express x as a prime quantum or as the product of a finite number of distinct prime quanta.

Cor 7. An atomic HCF domain is a UFD.

Now we have enough material to be able to prove

Theorem 12. An integral domain R is a GUFD iff it is an HCF-GKD.

Proof. Let R be a GUFD then

(1) every non zero non unit of R is contained in a finite number of minimal prime ideals (Cor. 3 and the fact that every non zero non unit of R is the product of a finite number of prime quanta)

(2) for every minimal prime P, $R_P$ is a valuation domain( Prop. 4 and Cor. 6)

(3) $R = \cap R_P$, where P ranges over all minimal prime ideals of R.

Proof of(3). Obviously $R \subseteq \cap R_P$ where P ranges over minimal primes. Let $x \in \cap R_P$, then since R is an HCF domain, we can write $x = r/s$ where $(r,s) = 1$ . Now $r/s \in R_P$ for every minimal prime P implies that s is a unit in each $R_P$, consequently s is in no minimal prime ideal and so has no prime quantum as a factor which in a GUFD is possible only if s is a unit and hence $x \in R$.

The properties (1),(2) and (3) as we have mentioned at the beginning of this section, show that R is a GKD and with the help of Prop. 4 we have proved that a GUFD is an HCF-GKD.

Conversely let R be an HCF-GKD. Let x be a non zero non unit element of R, then by the definition of a GKD, x is contained in a finite number of minimal prime ideals $P_1, P_2, . P_n$ say. We may assume that there is no other minimal prime which contains x. Now since $P_i$ are distinct there exists an element $y \in P_1$ such that $y \notin P_2$. We claim that $(x,y) \neq 1$, for otherwise $(x,y) = 1$ in R implies that $xR \cap yR = xyR$ in R and so $xR_{P_1} \cap yR_{P_1} = xyR_{P_1}$ in $R_{P_1}$ (cf Proof of Lemma 9) which further implies that $(x,y) = 1$ in $R_{P_1}$. But $R_{P_1}$ being a valuation domain either x or y is a unit in $R_{P_1}$ i.e. either x or y is not in $P_1$ a contradiction.

Let$(x,y) = d$ , and so $x = x_1 d$, $y = y_1 d$ where $(x_1, y_1) = 1$ and by the previous argument, $x_1$ and $y_1$ cannot both belong to $P_1$ . Let $x_1$ be such that $x_1 \notin P_1$ , then $x_1 d \in P_1$ implies that $d \in P_1$. Obviously since d is a factor of y, $d \in P_1$ and being a factor of x, d belongs at most to $P_1, P_3, \ldots, P_n$. Further let $y_2 \in P_1$ such that $y_2 \notin P_3$, and repeating the above argument we get $d_1 = (d, y_2)$ where $d_1$ is a non unit factor of x which can belong at most to $P_1, P_4, \ldots, P_n$. And it needs a finite number of steps to reach the conclusion that x has a non unit factor q say, which is contained in $P_1$ and is contained in no other minimal prime ideal.

Now as $q \in P_1$ and belongs to no other minimal prime ideal, $q^n$ is also in no minimal prime ideal other than $P_1$, because if we suppose on the contrary that $q^n \in P$ a minimal prime other than $P_1$ then $q \in P$ a contradiction.

Further let a non unit $h|q$ then since a GKD is completely integrally closed, there exists a positive integer n such that $h^n \nmid q$. But R being an HCF domain $h^n$ and q have a highest common factor d say, then $h^n = rd$ , $q = q'd$ where $(r,q') = 1$.

Since $h^n \nmid q$, r is not a unit, and if we assume that q' is also a non unit then either r or q' is not in P, a contradiction and hence q' is a unit. In other words, for every non unit factor h of q there exists an n such that $q|h^n$ i.e. q is a quantum and so by Lemma 8, q is a prime quantum.

Now the prime ideal $Q_q$ associated to q is obviously contained in P, but P being minimal $Q_q = P$ (cf (2) of Prop. 5)

Finally we know that for every minimal prime P of R, $x \in P$ implies that x is in a finite number of minimal primes

$P, P_1', \ldots, P_m'$ and so by the above process we can show that $P = Q_{q'}$ where $q'$ is the prime quantum emerging from the above process.

It is well known that in a GKD every non zro prime ideal contains a minimal prime ideal and so we have proved that in an HCF-GKD every prime ideal contains a minimal prime ideal associated to a prime quantum which by Proposition 7 is equivalent to say that R is a GUFD.

Remark 3. The above proof does not demonstrate as to how we can write a non zero non unit x in an HCF-GKD R. This end may be achieved as follows:

Let $\{ P_1, P_2, \ldots, P_r \}$ be the set of all non zero minimal prime ideals containing x. We have shown that $x \in P_1$ implies that there exists a prime quantum $q_1$ in P such that $q_1 | x$. Suppose that $q_1$ does not divide x completely (cf Def. 2), then R being a GKD, is completely integrally closed and so there is an n such that $q^n \nmid x$. Now by the HCF property $q_1' = (q_1^n, x)$ divides x completely. Similarly proceeding for $P_2, \ldots P_r$ we conclude that $x = q_1' q_2' \ldots q_r'$ and this factorization is obviously unique.

It is well known that if R is a GKD and S is a multiplicative set in R then $R_S$ is a GKD (cf[9] p 513). Further by Lemma 9, if R is an HCF domain and S in R is multiplicative then $R_S$ is an HCF domain and so using the above theorem we can prove the

Proposition 13. If R is a GUFD and S is a multiplicative set in R then $R_S$ is a GUFD.

Further if R is a GKD and x is an indeterminate over R then R[x] is a GKD ([9] p. 517)and it is well known that if R is an HCF domain then so is R[x]. Hence follows the

Proposition 14. If R is a GUFD and x is an indeterminate over R then $R[x]$ is a GUFD.

We end this section with an application of the theory developed in the previous sections and state the

Proposition 15. Let R be an integral domain such that for every non zero non unit x in R

$$x = Q_1 \cap Q_2 \cap \dots \cap Q_n$$

where $Q_i$ are primary ideals such that $\sqrt{Q_i}$ is a minimal prime ideal, then R is a GUFD if it is an HCF domain.

Proof. (1) from the hypothesis it follows that every non zero non unit of R is contained in a finite number of minimal prime ideals of R.

(2) R being an HCF ring $R_P$ is a valuation domain for every non zero minimal prime ideal P of R.

(3) The proof that $R = \cap R_P$ follows the same lines as the proof of ~~Prop.~~ Theo. 12.

From (1),(2) and (3) above it follows that R is an HCF GKD and hence is a GUFD.

5. Ideal Theory.

This section includes a brief account of the behaviour of minimal prime ideals of a GUFD. We then pass on to the ideal theory of GKD's which are Prüfer(Bezout), the primary decomposition being our main concern. We shall find that the primary decomposition of every non zero ideal in a Prüfer GKD is unique, in other words a Prüfer GKD is a W-ring. At the end of the section we show that the necessary and sufficient condition for a Prüfer domain to be a Prüfer GKD is that its non zero ideals have primary decompositions.

For the sake of reference we quote the definition and some properties of W-rings from [10].

Definition . A ring R is a W-ring if each ideal of R may be uniquely represented as an intersection of finitely many primary ideals.

A W-ring R is called a <u>W*-ring</u> if each ideal of R contains a power of its radical.

Theorem A( [10] Th. 1). A ring is a W-ring iff it is a finite direct sum of primary rings and one dimensional integral domains in which every non zero ideal is contained in only finitely many maximal ideals.

Theorem B ([10] Th.2). A W-ring is a W*-ring iff each non zero ideal of R contains a product of non zero prime ideals.

Theorem C ([10] Th. 4). If a W*-domain is strongly( completely) integrally closed then it is a Dedekind domain.

First we take up the behaviour of minimal prime ideals in GUFD's. We note that in the case of UFD's it is well known that an integral domain R is a UFD iff every non zero prime ideal of R contains a principal(non zero) prime, and that an analogue of this result appears in this chapter as Prop. 7. And to clarify the structure of minimal prime ideals of GUFD's still further we prove the

Theorem 16. If P is a minimal prime ideal in a GUFD R, then P is either principal or idempotent.

Proof. Let P be a minimal prime ideal in a GUFD R then by (2) of Proposition 5, $P = Q_q$ for a prime quantum q.

Suppose that $P^2 \neq P$ and let $x \in P - P^2$. Since $P = Q_q$ $(x,q) \neq 1$ , obviously $q_1 = (x,q)$ is contained in P and no other minimal prime ideal. We claim that $q_1$ is an atom. For supposing on the contrary that $q_1 = q_2 q_3$, where $q_2, q_3$ are both non units. Since $q_1 \in P$ and is in no other minimal prime ideal every non unit factor of $q_1$ is in P. This implies that $q_2, q_3 \in P$ and so $q_1 = q_2 q_3 \in P^2$ i.e. $x = x_1 q_1 \in P^2$, a contradiction and hence $q_1$ is an atom.

Now since a GUFD is an HCF domain and it is well known that an atom in an HCF domain is a prime(cf.e.g. [5]), $q_1 R$ is a prime ideal contained in P, that is $q_1 R = P$ ((2) of Prop. 5)

To study another feature of GUFD's, let q be a prime quantum and let $ab \in qR$, that is $q|ab$. By Definition 3, $q = q_1 q_2$ such that $q_1 | a$ and $q_2 | b$, that is $a = a_1 q_1, b = b_1 q_2$ say. Obviously if $b \notin qR$, $q_1$ is a non unit and so there is a positive integer m(say) such that $q | q_1^m$ i.e. $q | a_1^m q_1^m = a^m$, that is if $b \notin qR$, $a^m \in qR$. In other words qR is primary. Further we note that

$$\sqrt{qR} = \{ x \mid (x,q) \neq 1 \} = Q_q$$ , which in a GUFD, is the minimal prime ideal associated to q.

Now let x be a non zero non unit in a GUFD R then

$x = q_1 q_2 \ldots q_n$, where $q_i$ are distinct prime quanta can be written as $xR = q_1 q_2 \cdots q_n R = q_1 R \cap q_2 R \cap \ldots \cap q_n R$ and a consideration of $\sqrt{q_i R}$ shows that xR has a unique primary decomposition. And so we have proved the

Theorem 17. In a GUFD, every non zero principal ideal has a primary decomposition $xR = P_1 \cap P_2 \cap \ldots \cap P_n$ where each $P_i$ is primary to a minimal non zero prime ideal and is principal.

It may be pointed out that the above theorem is closely related to Prop. 15. In connection to these and specially as a corollary to Prop. 15, we state

Corollary 7. If in an HCF domain R every principal ideal is primary then R is a rank one valuation ring.

Proof. Let x,y be any two non zero non units of R. According to the hypothesis, xR, yR and xyR are primary. Obviously since x and y are non units, $x,y \notin xyR$ and consequently there exist m and n such that $x^m, y^n \in xyR$ i.e. $xy | x^m, y^n$. Now

$xy|x^m, y^n$ implies that $x|y^{n-1}$ and $y|x^{m-1}$ i.e. every non zero non unit of R is a quantum and hence a prime quantum because of the HCF property and hence R is a rank one valuation domain.

To proceed further we need some more definitions.

An integral domain R in which every finitely generated ideal is principal(invertible) is called a Bezout(Prüfer) domain. It is well known that a Prüfer domain which is also an HCF domain is a Bezout domain and equally well known is the fact that an integral domain R is Prüfer iff $R_P$ is a valuation domain for each prime ideal P (cf e.g. [5]). A generalized Krull domain which is also Prüfer(Bezout) will be called a Prüfer(Bezout) GKD.

As no convenient and to the point reference is available we include

Lemma 18. A GKD R is a Prüfer GKD iff every non zero prime ideal of R is maximal.

Proof. Let R be a Prüfer GKD and let P be a non zero prime ideal in R, then $R_P$ is a GKD ([9] p. 513). But the Prüfer condition implies that $R_P$ is a valuation domain. If P is not minimal then $R_P$ is a valuation domain of rank greater than 1, which implies that there exist non units in $R_P$ which are contained in no minimal prime ideals, a contradiction to the fact that $R_P$ is a GKD and hence implying that every non zero prime ideal of R is minimal. The converse is obvious.

Now a GUFD is an HCF-GKD and so for a GUFD to be Bezout all we need to state is

Corollary 8. A GUFD R is a Bezout GUFD(Bezout GKD) iff every non zero prime ideal of R is maximal.

Gilmer and Ohm in [18] prove that a UFD is a PID iff it has the Q.R-property, where an integral domain R is said to have the Q.R-property if every over-ring(ring between R and its quotient field K) is a quotient ring. In a similar way it can be proved that a GUFD is a Bezout GKD iff it has the Q.R-property, but a more general result is in order and we state

Proposition 19. A Schreier domain R is a Bezout domain iff it has the Q.R-property.

Proof. If R has the Q.R property, it is a Prüfer domain( [9] p. 319) and R being Schreier also is Bezout(cf [5]). Conversely it is well known that a Bezout ring has the Q.R property (cf e.g. [5]).

It is obvious that a Bezout GKD(Prüfer GKD) is a W- domain and so every non zero ideal of a Prüfer GKD has a unique primary decomposition. The above stated fact makes a Prüfer( Bezout)GKD very similar to a Dedekind(Principal ideal) domain. In fact the only point of difference is that Prüfer(Bezout) GKD's admit idempotent ideals while Dedekind domains(PID's) do not. To establish this fact we prove

Proposition 20. A Prüfer GKD R is a Dedekind domain iff each non zero prime ideal of R is non idempotent.

Proof. If R is a Dedekind domain the result is obvious. Conversely let R be a Prüfer GKD such that every non zero prime ideal of R is non idempotent. Then if P is a non zero prime ideal of R every P-primary ideal contains a power of P(cf [28]) and so every non zero ideal of R contains a product of a finite number of maximal ideals, that is R is a W*- domain (cf Th. B) but since R is a GKD and hence completely integrally closed it is a Dedekind domain by Theorem C.

A Bezout GKD being a GUFD, we can state as a corollary to Theorem 16 the

Proposition 21. In a Bezout GKD a prime ideal is either principal or idempotent.

Finally to study the primary decomposition in Prüfer domains we proceed as follows.

Let R be an integral domain, an ideal P is said to be an S-ideal in R if (1) P is prime (2) the set of P-primary ideals is linearly ordered (3) the intersection of all the P-primary ideals is a prime ideal M (4) M contains each prime ideal properly contained in P. An integral domain R is said to be an S-domain if every prime ideal of R is an S-ideal (cf [13] pp. 249-250 ).

According to Cor. 2.5 of [13],( $\subsetneq$ = proper containment) " If D is an S-domain and $Q_1, Q_2$ are primary ideals for $P_1, P_2$ respectively, where $P_1 \subsetneq P_2$, then $Q_1 \subseteq Q_2$ "------------(S)

It is easy to establish that a Prüfer domain is an S-domain and that(S) can be proved for a Prüfer domain. But for the convenience of reference we adopt (S) for Prüfer domains and use it to prove

Theorem 22. If a non zero ideal A in a Prüfer domain R has a reduced primary decomposition

$$A = P_1 \cap P_2 \cap \dots \cap P_n \quad \text{-------------------(a)}$$

then (a) is unique.

Proof. Let Rad $P_i = Q_i$ ( i = 1,2,...,n), we claim that if (a) is reduced then

(1) $Q_i$ are incomparable under inclusion (i = 1,...,n)

(2) no two $P_i, P_j$ $i \neq j$ are contained in the same prime ideal Q.

First let $Q_i \subseteq Q_j$ for some $i \neq j$, then if $Q_i = Q_j$; $P_i \subseteq P_j$ or $P_j \subseteq P_i$ because each of the $Q_i$ is an S-ideal and this contradicts the assumption that (a) is reduced. Further if $Q_i \subsetneq Q_j$ then by (S) above $P_i \subseteq P_j$ which again contradicts the assumption that (a) is reduced and hence establishes (1). For (2) let $P_i, P_j \subseteq Q$ a prime ideal then Rad $P_i$, Rad $P_j \subseteq Q$. But since $R$ is a Prüfer domain $R_Q$ is a valuation domain and so Rad $P_i \subsetneq$ Rad $P_j$ or Rad $P_j \subsetneq$ Rad $P_i$, this contradicts (1) and hence establishes (2). Now let

$$A = P_1' \cap P_2' \cap \ldots \cap P_m' \quad \text{-------------------(b)}$$

be another primary decomposition of A and since every primary decomposition can be reduced, suppose that (b) is reduced and let Rad $P_j' = Q_j'$ $(j = 1,2,\ldots m)$

We note that the above claim holds for (b) as well and that $(P_1 \cap P_2 \cap \ldots \cap P_n)R_{Q_i} = (P_1' \cap P_2' \cap \ldots \cap P_m')R_{Q_i}$, $(i=1,\ldots n)$ can be written as

$$P_1R_{Q_i} \cap P_2R_{Q_i} \ldots \cap P_nR_{Q_i} = P_1'R_{Q_i} \cap \ldots \cap P_m'R_{Q_i} \quad \text{-----------(c)}$$

(cf [9] p 34 )

In view of the above claim there exists only one primary ideal $P_i \subseteq Q_i$ in the decomposition (a) and so (c) can be written as

$$\bigcap_{k=1}^{n} P_kR_{Q_i} = P_1'R_{Q_i} \cap P_2'R_{Q_i} \cap \ldots \cap P_m'R_{Q_i} \quad \text{------(d)}$$

Now on the right hand side of (d), no two of $P_j'$ are in $Q_i$ and since the left hand side is a proper ideal of $R_{Q_i}$ there must at least one of $P_j'$ be contained in $Q_i$ and thus

$P_iR_{Q_i} = P_j'R_{Q_i}$, but since $P_i$ is $Q_i$-primary and $R$ is a Prüfer domain (cf [28])

$$P_i = P_iR_{Q_i} \cap R = P_j'R_{Q_i} \cap R \quad \text{----------------------(e)}$$

we have $P_j' \subseteq P_i$ - --------------------(f)

Similarly considering

$$(P_1' \cap P_2' \cap \ldots \cap P_m')R_{Q_j'} = (P_1 \cap P_2 \cap \ldots \cap P_n)R_{Q_j'}$$

where $Q_j' = $ Rad $P_j'$, we find that there exists some primary ideal $P_k$ in the decomposition (a) such that

$$P_k \subseteq P_j' \quad \text{------------------------ --(g)}$$

Combining (f) and (g), $P_k \subseteq P'_j \subseteq P_i$ and recalling that (a) is reduced $P_k = P_i = P'_j$ . Hence m = n and the primary decomposition is unique.

And all that interests us at present may be stated as

Corollary 9. A Prüfer(Bezout) domain R is a Prüfer(Bezout) GKD iff every ideal of R has a primary decomposition.

Proof. If R is a Prüfer domain and every ideal of R has a primary decomposition then these decompositions being unique by the above theorem show that R is a W-domain and a W-domain which is Prüfer is a Prüfer GKD.

Conversely in a Prüfer GKD every non zero prime ideal is maximal and every ideal is contained in a finite number of maximal ideals, and this is a condition for a domain to be a W-domain.

# CHAPTER 2

## SEMIRIGID DOMAINS

### 0. Introduction.

In the theory of Unique Factorization the concept of a prime element is basic. Similarly it is clear that a discrete rank one valuation domain is the simplest UFD ( in the sense that it has only one prime and its associates). In the previous chapter we replaced the concept of prime element by a more general concept ,prime quantum which resulted in the replacement of a discrete rank one valuation domain by a rank one valuation domain as the simplest GUFD(every non zero non unit in a rank one valuation domain is a prime quantum similar to any other). But the generalization of Unique Factorization in the above mentioned fashion gives rise to the following

Question . Is it possible to work out a theory of Unique Factorization in which a general valuation domain replaces a rank one valuation domain ?

We note that in a general valuation domain R; no non zero non unit x can be expressed as a product of two co-prime non units. Moreover for all v,u|x in R, u|v or v|u. In other words the lattice L(xR,R) is a chain for each non zero element x in a valuation domain R. According to [6] p. 129 an element x in an integral domain R is called rigid if L(xR,R) is a chain, and an integral domain R with all non zero elements rigid is called a rigid domain (cf [6] p 129). It can be easily seen that a commutative valuation domain is a rigid domain.

An obvious programme is, that we should consider an integral domain in which every non zero non unit element is expressible as the product of a finite number of mutually co-prime rigid non units. For a clearer picture of factorization into rigid non units we consider the following

Example 1. Let V be a valuation domain, x an indeterminate over V and let $R = V[x]$.

Pick a general non zero non unit element

$$y = \sum_{i=0}^{n} v_i x^i ; \quad v_i \in V.$$

Since V is an HCF domain, we can calculate the HCF, d of $v_0, v_1, \ldots, v_n$ and so $y = d(\sum_0^n v_i' x^i)$; where $\{ v_i' \}$ have no non unit common factor( in fact one of them is a unit).

In the factorization of $y' = \sum_0^n v_i' x^i$ , every non unit element has positive degree in x and hence $\sum_0^n v_i' x^i$ is a product of atoms. Moreover since, V is an HCF domain and so is $V[x]$, every atom in $V[x]$ is a prime (cf [5]) and thus

$\sum_0^n v' x^i = p_1^{a_1} p_2^{a_2} \ldots p_s^{a_s}$ ; $(p_i, p_j) = 1$ for $i \neq j$ . That is

$$y = dp_1^{a_1} p_2^{a_2} \ldots p_s^{a_s} \; ; \; (d, p_i) = 1 \; (i = 1, 2, \ldots, s)$$

$$(p_i, p_j) = 1 \text{ for } i \neq j \text{ -----(A)}$$

Obviously each prime power is a rigid non unit and d being a member of V is rigid and so if y is non unit, it is the product of a finite number of mutually co-prime rigid non units. It is also obvious that the factorization in the expression (A) is unique up to associates of the rigid non units. And since, y is arbitrary we conclude that every non zero non unit element in $R = V[x]$ is uniquely expressible as the product of a finite number of mutually co-prime rigid elements.

Here we note that while an atom is rigid, a quantum according to its definition, need not be . For example, in a one dimensional quasi-local domain every non zero non unit element is a quantum but a one dimensional quasi-local domain need not be a valuation domain and to show that there does exist at least one, one dimensional quasi-local domain which is not a valuation domain we take up the following

Example 2.(cf [5] p. 262). Let G be the additive semigroup of all rationals $\geqslant 0$ and reals $\geqslant 1$, form the semigroup algebra F[G] and let F(G) be the ring obtained by adjoining inverses of all elements with non zero constant term. We can write

$$F(G) = \{ \Sigma u_i x^{\alpha_i} \mid \alpha_i \geqslant 0 \text{ if rational and } \alpha_i \geqslant 1 \text{ if real and } u_i \text{ are units} \}$$

No two elements of F(G) are co-prime and it can be verified that one divides a power of the other and that F(G) is a one dimensional quasi-local domain, because if $(\alpha),(\beta) \in G$ where $\beta > \alpha$ then there exists a positive integer n such that $n\alpha > \beta + 1$ ($\alpha, \beta$ being real numbers). But F(G) is not a valuation domain, since $x^{1/2} \nmid x^{1+\gamma}$, where $\gamma$ is an irrational number less than 1/2.

Further it can be verified that a prime quantum is a rigid non unit while a rigid non unit may not be a prime quantum, for example every non zero non unit in a rank two valuation domain R is rigid, while if P is the maximal ideal of R and Q is the minimal non zero prime ideal then every integral power of $x \in P - Q$ will divide every element of Q, that is elements of Q do not satisfy the condition of being a quantum and hence are not prime quanta.

In the case of prime quanta it was easy to develop a theory of factorization on classical lines, as we did in the previous chapter, but in the case of rigid elements it looks not only difficult but also unnecessary to go through all those details. So we shall consider the properties of rigid non units in HCF domains and will investigate the structure of those HCF domains in which every non zero non unit element is expressible as the product of a finite number of mutually co-prime rigid non units and these domains we shall call Semirigid Domains.

This chapter consists of only two sections. In the first section we formally define a rigid element and discuss its properties in an HCF domain, while in the second section we introduce the concept of a semirigid element - the product of a finite number of mutually co-prime rigid non units and prove that if in an HCF domain an element can be expressed as the product of a finite number of mutually co-prime rigid non units then this factorization is unique up to associates of the rigid non units and up to order. And from this we derive the definition of a Semirigid Domain. Moreover in the same section we give, what may be called the local characterization of a Semirigid Domain, in the form of Theorem 2, which eventually induces the definition of another generalization of Krull domains.

## 1. Preliminary Definitions and Basic Results.

Definition 1. A non zero element $r$ in a commutative integral domain $R$ is said to be rigid if for every $u,v \mid r$ ; $u \mid v$ or $v \mid u$.

From the definition it follows immediately that every factor of a rigid non unit is also rigid . We proceed to investigate the properties of rigid non units in an HCF domain and prove the

Lemma 1. In an HCF domain $R$ the following are valid.

(1) Let $r,s$ be any two non co-prime rigid non units of $R$, then $r \mid s$ or $s \mid r$.

(2) Let r,s be any two non co-prime rigid non units of R their product rs is again a rigid non unit ( obviously non co-prime to both r and s).

(3) To each rigid non unit $r \in R$, there is associated a prime ideal $P(r) = \{ x \in R \mid x \text{ is non co-prime to } r \}$.

(4) Let r,s be two rigid non units in R then $P(r) = P(s)$ iff r,s are non co-prime .

(5) If r is a rigid non unit in R and P(r) is the prime ideal associated to r then the localization $R_{P(r)}$ is a valuation domain.

Proof. Let $(r,s) = d$ ( $\neq 1$); $r = r_1 d$ , $s = s_1 d$ where $(r_1, s_1) = 1$. If either of $r_1, s_1$ is a unit, (1) holds and we have nothing to prove. So we suppose on the contrary that $r_1, s_1$ are both non units . By the definition of a rigid element $r_1 | d$ or $d | r_1$ ---------------------------------- -(a)

and $s_1 | d$ or $d | s_1$ ----------------------------------------(b)

Now if $r_1 | d$ and $d | s_1$ ; $r_1 | s_1$ a contradiction -----(i)

and if $r_1$ and $s_1$ divide d which being a factor of a rigid element is it self rigid and hence $r_1 | s_1$ or $s_1 | r_1$

a contradiction ---------------(ii).

Further if $d | r_1$ and $s_1 | d$ then $s_1 | r_1$ a contradiction ---(iii).

Finally if $d | r_1$ and $d | s_1$ then again $(r_1, s_1) \neq 1$

a contradiction ---------------(iv).

To sum up we get contradiction as a result in all the four cases which arise from the assumption that $r \nmid s$ and $s \nmid r$ and this confirms the truth of (1).

(2) Let $z = rs$, where r,s are non co-prime rigid elements. Let x,y be any pair of factors of z and suppose that $x \nmid y$ and $y \nmid x$ ( in other words we suppose that z is not a rigid element). Now let $(x,y) = d$, where $x = x_1 d$, $y = y_1 d$ and

$(x_1, y_1) = 1$ and obviously $x_1, y_1$ are both non units. We note that $x_1 | x$ and $x | z = rs$, therefore $x_1 | rs$, and by the HCF property of R,

$x = x'x''$ where $x'|r$ and $x''|s$ ----------------------(c)

Similarly $y = y'y''$, where $y'|r$ and $y''|s$ -------------(d)

Further $y'|y_1, x'|x_1$ and $(x_1, y_1) = 1$ implies that $(x', y') = 1$. But since r is a rigid element $x'|y'$ or $y'|x'$ which is possible only if one of $x', y'$ is a unit --------------(e). Similarly we conclude that either of $x'', y''$ is a unit--(f).

Let $x'$ be a unit, then since $x_1$ is a non unit and $x_1 = x'x''$, $x''$ is a non unit and is an associate of $x_1$ but then $y''$ is a unit (by (f)). Again since $y_1$ is a non unit $y'$ is a unit and so we conclude that

$y'|r$ where $y'$ is an associate of $y_1$ and

$x''|s$ where $x''$ is an associate of $x_1$.

I.e. there exist two co-prime elements $x_1, y_1$ such that $y_1|r$ and $x_1|s$. But since r and s are non co-prime rigid elements $r|s$ or $s|r$ by (1) above. And in both cases $x_1$ and $y_1$ become factors of a rigid non unit (e.g. $x_1, y_1$ are factors of s if $r|s$ because $y_1|r$ and $r|s$ i.e. $y_1|s$ while $x_1|s$ is assumed) but this being in contradiction with $(x_1, y_1) = 1$ implies that the assumption $x \nmid y$ and $y \nmid x$ is wrong and z is a rigid non unit.

(3) Let $P(r) = \{ x \in R \quad (x,r) \neq 1 \}$.

Because of (1) above, if x and y are non co-prime to r and if $(x,r) = d$, $(y,r) = d_1$ then, being factors of a rigid non unit $d|d_1$ or $d_1|d$. Consequently if $d_1|d$ then $d_1|x,y$ and so $d_1|x+y$, similarly if $d|d_1$, $d|x+y$. In other words if $x,y \in P(r)$ then $x+y \in P(r)$. Moreover if $x \in P(r)$ then $ax \in P(r)$ for all $a \in R$, that is $P(r)$ is an ideal. Finally because of the HCF property $(xy,r) \neq 1$ iff $(x,r) \neq 1$ or

$(\gamma, \delta) \neq 1$ i.e. if $xy \in P(r)$ then $x \in P(r)$ or $y \in P(r)$ and this establishes (3).

(4) If $P(r) = P(s)$ then since $r \in P(r)$ , $(r,s) \neq 1$.

Conversely let $(r,s) \neq 1$ then by (1), $r|s$ or $s|r$ . If $r|s$ then $(x,r) \neq 1$ implies that $(x,s) \neq 1$, that is $P(r) \subseteq P(s)$. If on the other hand $(x,s) \neq 1$ then by the HCF property $x = x_1 s_1$ and $s = s_1 s_2$ , where $(x_1 , s_2) = 1$. Since $s_1 | s$, $s_1$ is a rigid element which is non co-prime to r(since we have assumed that $r|s$) that is $(x,s) \neq 1$ implies that $(x,r) \neq 1$ i.e. $P(s) \subseteq P(r)$ and combining the two inclusion relations the result follows.

(5) Since R is an HCF domain, $R_{P(r)}$ is an HCF domain (cf Lemma 9, Ch. 1). To prove that a quasi-local HCF domain ( $R_{P(r)}$ in this case) is a valuation domain, all we have to show is that no two non units of this domain ( $R_{P(r)}$) are co-prime. Suppose on the contrary that there exist x,y in $P(r)R_{P(r)}$, such that $(x,y) = 1$ and let

$$x = u_1/v_1 \ ; \ y = u_2/v_2 \ \text{( we can assume that } (u_i , v_i) = 1 \text{ ).}$$

Now since $v_1 , v_2$ are units in $R_{P(r)}$ we get $(u_1 , u_2) = 1$ in $R_{P(r)}$, that is $(u_1 , u_2) \notin P(r)R_{P(r)}$. But since we assumed that x,y are non units in $R_{P(r)}$, $u_1 , u_2 \in P(r)$ and so $(u_i , r) = r_i$ $(i = 1,2.)$ are such that $r_i \neq 1$ that is $d = (u_1 , u_2)$ is a multiple of $r_1$ or of $r_2$ in R (since $r_i$ are factors of a rigid element r ) and thus $(u_1 , u_2) = d \in P(r)$ i.e. $u_1 , u_2$ are non co-prime in $R_{P(r)}$ a contradiction establishing that no two non units in $R_{P(r)}$ are co-prime which implies the result.

## 2. Semirigid Domains.

Using Lemma 1, we first prove the

Theorem 1. Let R be an HCF domain and suppose that an element $x \in R$, can be expressed as the product of a finite number of mutually co-prime rigid non units then this factorization is unique up to associates of the rigid non units and up to their order.

Proof. Let R be an HCF domain and let $x \in R$ be such that

$x = r_1 r_2 \ldots r_m$ ; $r_i$ rigid , $(r_i, r_j) = 1$ for $i \neq j$.

Further suppose that

$x = s_1 s_2 \ldots s_n$ ; $s_i$ rigid(non unit) $(s_i, s_j) = 1$, for $i \neq j$.

Since $s_1 | x$, by the HCF property

$s_1 = s_{11} s_{12} \ldots s_{1m}$ ; where $s_{1i} | r_i$ and since $\{ r_i \}_{i=1}^{i=m}$ are co-prime, at most one of $s_{1i}$ say $s_{1k}$ is a non unit and so $s_1 | r_k$ for some $k$ ( $=1,2,\ldots,m.$ ).

Reversing the process we take $r_k | x$ and so

$r_k = r_{k1} r_{k2} \ldots r_{kn}$ where $r_{ki} | s_i$ ( $i = 1,2,\ldots,n$).

By the above argument there exists an $s_j$ such that $r_k | s_j$ and obviously $s_j$ is an associate of $s_1$, for if not so $(s_1, s_j) = 1$ while $s_1 | r_k$ and $r_k | s_j$ that is $s_1 | s_j$ a contradiction establishing the fact that $s_1$ is an associate of $r_k$.

Repeating the above process for $s_2, s_3, \ldots, s_n$ we get $m = n$ and each $s_i$ associate of some $r_j$. In other words the factorization $x = r_1 r_2 \ldots r_m$ is unique up to associates of $r_i$ and up to a suitable permutation of the rigid non units.

We can call the non unit of Theorem 1, a <u>Semirigid element</u> and based on this notion we make the following

Definition 2. An HCF domain in which every non zero non unit is semirigid will be called a <u>Semirigid Domain</u>.

We note that in an HCF domain a rigid non unit generalizes a prime quantum ( since a prime quantum satisfies the properties of a rigid non unit) and it is easy to see that a

Semirigid Domain is a generalization of a GUFD. And to display another feature of Semirigid Domains we prove the following

Theorem 2. Let R be a Semirigid Domain, then there exists a family $\Phi = \{ P_\alpha \}$ ($\alpha \in I$ an index set) of prime ideals of R such that

(1) $R_{P\alpha}$ is a valuation domain for each $\alpha \in I$

(2) each non zero non unit $x \in R$ is contained in only a finite number of members of $\Phi$

(3) $P_{\alpha_1} \cap P_{\alpha_2}$ does not contain a non zero prime ideal if $\alpha_1 \neq \alpha_2$, $\alpha_i \in I$

(4) $R = \cap R_{P_\alpha}$, $\alpha \in I$.

Proof. By part(3) of Lemma 1, in an HCF domain R, corresponding to each rigid non unit r, there exists a prime ideal $P(r) = \{ x \in R \mid (x,r) \neq 1 \}$ associated to r, and by (4) of Lemma 1, $P(r) = P(s)$ iff s is a rigid non unit non co-prime to r.

Now let $\Gamma$ be a set of mutually co-prime rigid non units $r_\alpha$ of the given Semirigid domain R, where $\alpha \in I$ an index set. According to the above observation we have a family of prime ideals $\Phi = \{ P(r_\alpha)( = P_\alpha ) \mid r_\alpha \in \Gamma ; \alpha \in I \}$, and by part (5) of Lemma 1, $R_{P_\alpha} = R_{P(r_\alpha)}$ is a valuation domain for each $\alpha \in I$, that is (1) holds for the selected family $\Phi$.

Since R is a Semirigid Domain, each non zero non unit being a product of a finite number of mutually co-prime rigid non units is a member of at most a finite number of elements of $\Phi$, that is (2) also holds for $\Phi$.

Now let Q be a non zero prime ideal contained in the intersection $P_{\alpha_1} \cap P_{\alpha_2} \equiv P(r_{\alpha_1}) \cap P(r_{\alpha_2})$, ($P_{\alpha_1} \neq P_{\alpha_2}$) and let $x \in Q$. Then since x is semirigid

$x = x_1 x_2 \ldots x_s$, where $x_i$ are mutually co-prime rigid non units. Since $x \in P(r_{\alpha_1})$; one of the $x_i$ $(i = 1,2,\ldots s)$ say $x_1$ is non co-prime to $r_{\alpha_1}$. Also since $x \in P(r_{\alpha_2})$ one of the $x_i$ $(i = 2,3,\ldots,s)$ say $x_2$ is non co-prime to $r_{\alpha_2}$ so that

$x = x_1 x_2 a$ ; where $a \notin P(r_{\alpha_i})$ $i = 1,2$ (because $(a,x_i)=1$ which is equivalent to saying that $(a,r_{\alpha_i}) = 1$).

Since we assume that Q is prime and since $a \notin P(r_{\alpha_i})$ $a \notin Q$, and so $x_1 x_2 a = x \in Q$ implies that $x_1 x_2 \in Q$, that is $x_1 \in Q$ or $x_2 \in Q$. In other words $x_1 \in P(r_{\alpha_1}) \cap P(r_{\alpha_2})$ or $x_2 \in P(r_{\alpha_1}) \cap P(r_{\alpha_2})$ that is $x_1$ or $x_2$ is a rigid non unit non co-prime to two co-prime rigid non units (since $\alpha_1 \neq \alpha_2$) a contradiction that confirms that (3) holds for $\Phi$.

To prove (4) for $\Phi$ let $R' = \cap R_{P_\alpha}$, $\alpha \in I$, and suppose that $x = u/v \in R'$, then since R is an HCF domain we can assume that $(u,v) = 1$, but this implies that v is a unit in each $R_{P_\alpha}$, that is v cannot be expressed as a product of rigid non units and we are forced to conclude that v is a unit and $x \in R$ which confirms that

$$R = \cap R_{P_\alpha} \;;\alpha \in I.$$

in view of Cor 1 p61,

The above theorem, ~~obviously~~ is a local characterization of Semirigid Domains, and, gives us another generalization of Krull domains. Being short of a suitable name for these integral domains, we call them <u>*GKD's</u>.

Definition 3. An integral domain R will be called a *GKD if there exists a family $\Phi = \{ P_\alpha \}_{\alpha \in I}$ of prime ideals of R such that

*1- every non zero non unit element of R is contained in only a finite number of members of $\Phi$.

* 2- for each $P_\alpha$ ;$\alpha \in I$, $R_{P_\alpha}$ is a valuation domain

*3- for each pair $P_\alpha, P_\beta \in \Phi, P_\alpha \cap P_\beta$ contains a non zero prime ideal iff $P_\alpha = P_\beta$.

*4- $R = \bigcap_{\alpha \in I} R_{P_\alpha}$ .

It is not very difficult to prove that an HCF- *GKD is a Semirigid Domain, but since there does exist yet another generalization of Krull domains, namely Rings of Krull Type( cf [21]), which also generalizes a *GKD, we postpone the proof till we are able to consider the factorization of an arbitrary non zero non unit in an HCF Ring of Krull Type. Briefly a ring of Krull type is an integral domain with a family $\Phi = \{ P_\alpha \}_{\alpha \in I}$ of prime ideals, for which *1,*2 and *4 hold. But since the rings of Krull type are not much known we need to give an introduction to the theory of rings of Krull type, while it seems difficult to inject it into the discussion of Semirigid Domains, and so we close this chapter with the remark that *3 of Definition 3, holds automatically in the case of Krull domains and of Generalized Krull domains, because of the fact that the families of prime ideals in these cases consist only of minimal primes and in this sense a *GKD is one of the nearest generalizations of Krull domains.

# CHAPTER 3

# UNIQUE REPRESENTATION DOMAINS

## 0. Introduction.

We concluded our previous chapter with the local characterization of Semirigid Domains (cf Th. 2 Ch. 2) which shows that a Semirigid Domain is a generalization of a Krull domain ( is a *GKD). The fact that the two generalizations of UFD's we have worked out are also generalizations of Krull domains leads us to think that if there exists yet another generalization R of Krull domains, which is also an HCF domain, then it is possible that the factorization of non zero non units of R should exhibit some interesting pattern . But we have to be selective in choosing a particular generalization of Krull domains for an examination ; because arbitrary generalizations of Krull domains can range over an uncontrolably large family of integral domains, which may be irrelevant too. For example an integrally closed domain generalizes a Krull domain in the sense that a Krull domain is integrally closed, but choosing an HCF integrally closed domain is absurd, because an HCF domain is already integrally closed (cf [23] p.33 ). We did mention at the end of the last chapter that a ring of Krull type satisfies *1,*2 and *4 of Def. 3 , in view of this, a ring of Krull type seems to be very near to the generalizations of Krull domains we could achieve through a generalization of the concept of Unique Factorization.

Thus it looks worth while to consider the factorization of a non zero non unit in an HCF ring of Krull type and to set up a more general theory if some pattern shows up. And our first step towards this end should be to give an introduction to the theory of the rings of Krull type because these rings are not very widely known. Section 1, of this chapter includes an introduction to the theory of rings of Krull type. Briefly for the sake of completeness of the present section we note that

(1) if R is an integral domain, K its field of fractions and S an integral domain such that $R \subseteq S \subseteq K$ then S is called an <u>overring</u> of R,

(2) if R is an integral domain and S a valuation overring of R then S is called an <u>essential valuation overring</u> of R if $S = R_P$ for some prime ideal P in R,

(3) an integral domain R is called <u>essential</u> if it can be expressed as an intersection of essential valuation domains

(4) an essential integral domain $R = \cap R_{P_\alpha}$ ; $\alpha \in I$ is a ring of Krull type, if for each non zero non unit x in R, x is a non unit in only a finite number of $R_{P_\alpha}$ ;$\alpha \in I$.

If P is a prime ideal such that $R_P$ is a valuation domain, we shall call P, a <u>valued prime</u>, and every prime ideal Q such that $0 \neq Q \subseteq P$, will be called a <u>subvalued prime</u> in P. In section 2, we show that if P is a valued prime and $0 \neq x \in P$ then there exists a unique minimal subvalued prime which is minimal with respect to containing x such that $x \in Q \subseteq P$, and this we shall call the <u>minimal subvalued prime of x in P</u>. In the same section we show that if an element p in an HCF ring of Krull type has only one minimal subvalued prime w.r.t. [illegible] all the valued primes containing x then p is such that if $p = p_1 p_2$; $p_i$ non units then $(p_1, p_2) \neq 1$ and there exists a positive integer n such that $p_1 | p_2^n$ or $p_2 | p_1^n$. Such an element will be called a <u>packet</u>. Finally we shall prove in the same section that a non zero non unit in an HCF ring of Krull type is expressible as the product of a finite number of mutually co-prime packets.

In section 3, we show with the help of a counter example that an HCF domain in which every non zero non unit can be expressed as the product of a finite number of mutually co-prime packets may not be a ring of Krull type. We shall call the above mentioned integral domains, Unique Representation Domains (URD's). After the counter example we proceed to investigate the conditions under which an HCF domain should become a URD. This gives rise to the concept of *-essential domains which can be explained as follows.

Let R be an essential domain and let $\{P_\alpha\}_{\alpha \in I}$ be the family of valued primes of R such that $R = \bigcap R_{P_\alpha}$ ; $\alpha \in I$, and that no two members of $\{P_\alpha\}_{\alpha \in I}$ are comparable w.r.t. inclusion, then R is a *-essential domain if every non zero non unit of R has a finite number of minimal subvalued primes which are contained in the members of $\{P_\alpha\}$. Finally we shall prove that a * -essential domain is a URD iff it is an HCF domain.

In section 4, we consider the stability properties of URD's under the operations of adjoining indeterminates and localization. We shall also prove that an integral domain R is a URD iff $R + xK[x]$ is a URD, where K is the field of fractions of R and x is an indeterminate over R. At the end of section 4, we establish that the concepts of GUFD, Semi-rigid Domain , HCF ring of Krull type and URD signify distinct classes of integral domains, out of a pair of which, one generalizes the other.

Our procedure of going from one generalization to a further generalization may look repetitive especially the distinct treatment of HCF rings of Krull type and of URD's. But we have adopted this approach because it is easier going from HCF rings of Krull type to URD's in the sense that we get the concept of a packet using the strict definition of the rings of Krull type, which it would have been difficult to visualize in the general case.

1. Rings of Krull Type.

Griffin in [21] introduced the notion of a ring of Krull type as a special case of the rings of finite character. The basic notion in the theory of rings of finite character is that of a <u>valuation v over a field K.</u> And for the sake of completeness we include the

Definition 1. Let G be a totally ordered group under addition and let $G^* = G \cup \{\infty\}$ be the group including the symbol $\infty$ with the properties

$$g + \infty = \infty + g = \infty + \infty = \infty \quad ; \quad g \in G$$

then the function $v : K \longrightarrow G^*$ such that

(1) $v(a) = \infty$ iff $a = 0$

(2) $v(xy) = v(x) + v(y)$

(3) $v(x + y) \geq \min(v(x), v(y))$

is called a valuation of K (or over K).

If v is a valuation of a field K, then the set

$R_v = \{ x \in K \mid v(x) \geq 0 \}$ is a valuation domain and is called the <u>valuation ring of v.</u>

Let $\Omega$ be a family of valuations of a field K and let $R = \cap R_v$ ; $v \in \Omega$ then R is called the ring determined by the family $\Omega$. Moreover the family $\Omega$ of valuations of K is said to be of <u>finite character</u> if for each $x \in K$ the set $\{ \omega \in \Omega \mid \omega(x) \neq 0 \}$ is finite.

Definition 2. Let $\Omega$ be a family of valuations of a field K and let $\Omega$ be of finite character then the ring determined by $\Omega$ is called a ring of <u>finite character</u>.

Now let R be a ring determined by a family $\Omega$ of valuations, let $R_v$ be a valuation ring of $v \in \Omega$, and let $M_v$ be the maximal ideal of $R_v$, then the prime ideal $R \cap M_v = Z(v)$ is called the <u>centre of v on R</u>. If the localization $R_{Z(v)}$ is equal to $R_v$ we call v an <u>essential valuation</u>. And according to Griffin, a ring R of finite character is called a <u>ring of Krull type</u> if it has a defining family of valuations consisting of essential valuations only.

Equivalently we can define a ring of Krull type as follows

Definition 3. An integral domain R is said to be a ring of Krull type if, there exists a family of prime ideals $\{P_\alpha\}_{\alpha \in I}$ such that

(1) $R_{P_\alpha}$ is a valuation domain for each $\alpha \in I$

(2) every non zero non unit element of R is contained in only a finite number of members of $\{P_\alpha\}_{\alpha \in I}$

(3) $R = \cap R_{P_\alpha}$ ; $\alpha \in I$

We shall adopt Definition 3, as the standard definition of a ring of Krull type. The family $\{P_\alpha\}_{\alpha \in I}$ can be assumed to be such that $P_\alpha, P_\beta$ are incomparable w.r.t. inclusion for each $\alpha \neq \beta \in I$. Because if $P_\alpha \subset P_\beta$; $R_{P_\beta} \supset R_{P_\alpha}$ and so $R_{P_\alpha} \cap R_{P_\beta} = R_{P_\alpha}$ i.e. $P_\beta$ can be dropped from the family. Moreover if there exists a chain of prime ideals $\{P_{\alpha'}\} = C$ in $\{P_\alpha\}$ i.e. $P_\gamma \subset P_\delta$ or $P_\delta \subset P_\gamma$ for each pair $P_\gamma, P_\delta \in C$ then since the unions and intersections of all the elements of C exist we can replace the elements of C by $P = \cup Q$ , $Q \in C$. In other words we can assume that $\{P_\alpha\}$ consists of the largest possible prime ideals for which, $R_{P_\alpha}$ is a valuation domain for each $\alpha \in I$. Thus by the <u>family of valued primes defining a ring R of Krull type</u> we shall in future mean the family$\{P_\alpha\}$ consisting of the largest valued primes of R. We recall that

Definition 4. An integral domain R is called a Krull domain if

(1) every non zero non unit element of R is contained in only a finite number of minimal prime ideals of R

(2) $R_P$ is a discrete rank one valuation ring for each minimal (non zero ) prime ideal P of R

(3) $R = \cap R_P$ where P ranges over all the minimal prime ideals of R.

Comparing the Definitions 3 and 4, we infer that a Krull domain is a ring of Krull type with the difference that the defining family of prime ideals of a Krull domain consists only of minimal non zero prime ideals, and of course that $R_P$ is a discrete for each P in the defining family. Similarly recalling Def. 3 of Chapter 2, we infer that a *GKD is also a ring of Krull type. Thus if $\leqslant$ denotes ," Form a special case of " then

Krull domains $\leqslant$ GKD's $\leqslant$ *GKD's $\leqslant$ Rings of Krull type.

The examples given or mentioned at the end of section 4 of this chapter ensure that the above is a chain of distinct classes of integral domains.

There may be many further generalizations of a ring of Krull type but we shall restrict our attention to essential domains and their special case to which we have given the name *-essential domains.

2. Factorization in an HCF ring of Krull type.

In this section we first take up a non zero non unit element in an HCF ring of Krull type and prove a sequence of lemmas to establish the notions in terms of which we can describe its factorization. In brief we shall first derive the notion of a packet as we mentioned before and then prove that in an HCF ring of Krull type a non zero non unit is expressible as the product of a finite number of mutually coprime packets.

Let R be a ring of Krull type and let $\Phi = \{ P_\alpha \}_{\alpha \in I}$ be the family of valued primes defining R. We start by showing that if $0 \neq x \in P$ ( $\in \Phi$ ) then there exists a unique prime ideal Q, minimal subject to the property $x \in Q \subseteq P$.

To achieve the above mentioned result we proceed a bit more generally as follows.

Let P be a prime ideal in an integral domain R and denote the set $\{ Q \mid Q$ is a prime ideal contained in P $\}$ by $\mathcal{E}(P)$. We note that if P is a valued prime then $\mathcal{E}(P)$ is totally ordered under inclusion and keeping in view the fact that every prime ideal contains a minimal (rank zero) prime ideal we state the

Lemma 1. Let P be a prime ideal in an integral domain R such that $\mathcal{E}(P)$ is totally ordered under inclusion, then for each non zero $x \in P$, there exists a unique prime ideal Q in P which is minimal subject to the property $x \in Q \subseteq P$.

Proof. P/xR is a prime ideal in R/xR and so contains a minimal prime ideal Q' = Q/xR for some $Q \subseteq P$, but since $\mathcal{E}(P)$ is totally ordered, Q is unique and hence the lemma.

And as a result of the above lemma we can state that, "If x is a non zero non unit in a ring of Krull type R, then each valued prime P of R with $x \in P$; contains a unique minimal subvalued prime satisfying $x \in Q \subseteq P$." We shall call Q, the <u>minimal subvalued prime of x in P</u>.

Now let x be a non zero non unit in an HCF ring of Krull type and let $P_1, P_2, \ldots, P_n$ be the only valued primes containing x. By the above lemma, each valued prime $P_i$ contains a unique minimal subvalued prime $Q_i$ containing x ( $i = 1,2,\ldots,n$). Here we note that unlike a *GKD, a ring of Krull type admits valued primes $P_\alpha, P_\beta \in \{ P_\alpha \}$ ( the family defining the ring of Krull type) such that $P_\alpha \cap P_\beta$ contains non zero prime ideals. And so the minimal subvalued primes $Q_i$ ( $\subseteq P_i$ ) of x may not all be distinct. The case where $Q_i \subseteq Q_j$; $i \neq j$ does not arise, because then $Q_i$ becomes the minimal subvalued prime of x in $P_i$ and $P_j$ both.

Striking repetitions out of $\{ Q_i \}_{i=1}^{n}$ and denoting the set of distinct minimal subvalued primes of x by $\{ q_j \}_{j=1}^{r}$ we can regroup $\{P_i\}_{i=1}^{n}$ after a suitable permutation of $\{ P_i \}$ as

$$\{ P_i \}_{i=1}^{n} = \bigcup_{j=1}^{r} \Pi_j \quad \text{where} \quad \Pi_j = \{ P_k \in \{ P_i \}_{i=1}^{n} | q_j \subset P_k \}$$

We shall call the set $\Pi_j$ , the bunch of valued primes of x containing $q_j$ only( among all $q_j$ of course).

Now let y be such that $y \in q_1$ but $y \notin q_2$ ( since $q_1, q_2$ are distinct we can have such a y ), then since R is an HCF domain and $R_{q_1}$ is a valuation domain, $(y,x) = d_1 \in q_1 - q_2$, because $y = y'd_1$ , $x = x'd_1$ , $(x',y') = 1$ ( since $d_1$ is the HCF ) and because of the HCF property $(x',y') = 1$ in $R_{q_1}$ that is at least one of $x',y'$ is not in $q_1$ but since $x,y \in q_1$ $d_1 \in q_1$, further since $y \notin q_2$ and $d_1 | y$ , $d_1 \notin q_2$. Further let $y_1 \in q_1 - q_3$, then as before $(y_1, d_1) = d_2 \in q_1 - q_3$( and also

$d_2 \in q_1 - q_2$ ).

Replacing $d_2$ by $d_3 = (y_2, d_2)$, where $y_2 \in q_1 - q_4$, and repeating the process, we conclude that there exists a factor d of x such that $d \in q_1$ and $d \notin q_j$ ( $j = 2,3,\ldots,r$). In other words, with a suitable permutation of $\{ q_j \}_{j=1}^{r}$ we have proved the

Lemma 2. Let x be a non zero non unit of an HCF ring R of Krull type with the family $\{ P_\alpha \}_{\alpha\in I}$ of valued primes defining R, $\{ P_1, P_2, \ldots, P_n \}$ be the set of all the valued primes containing x and let $\{ q_j \}_{j=1}^{r}$ be the set of all the distinct minimal subvalued primes of x, then corresponding to each $q_j$ there exists a $p_j | x$ such that $p_j \in q_j$ and $p_j \notin q_k$ for all $k \neq j$ ( $k, j = 1,2,\ldots,r$).

Lemma 2 leads to the notion of an element( in an HCF ring of Krull type at present) with a single minimal subvalued prime and to study the properties of such elements we state the

Lemma 3. Let d be a non zero non unit element in an HCF ring of Krull type R. Let $P_1, P_2, \ldots, P_r$ be the only valued primes( in the family $\{ P_\alpha \}_{\alpha\in I}$) of R containing d and suppose that d has only one minimal subvalued prime q then

(1) If $d = d_1 d_2$ , then $(d_1, d_2) = 1$ only if either of $d_i$ is a unit ( $i = 1,2$).

(2) If $x \notin q$ but the set of all the valued primes containing x is a subset of $\{ P_1, P_2, \ldots, P_r \}$ then $x^n | d$ for all positive integers n.

(3) If there exists another element d' such that d' has q as the only minimal subvalued prime containing it, then d' belongs to $P_1, P_2, \ldots P_r$ and to no other valued prime in the defining family and there exists a positive integer n such

that $d|d'^n$ and $d'|d^n$.

(4) If $x$ has $q$ as one of its minimal subvalued primes and $d|x$, then there exists a positive integer n such that $d^n \nmid x$. Moreover $x = x_1 x_2$ such that $(x_1, x_2) = 1$ and $x_1$ has q as its only minimal subvalued prime.

(5) If $d = d_1 d_2$ ; $d_i$ non units ( $i = 1,2$) then there exists a positive integer n such that $d_1 | d_2^n$ or $d_2 | d_1^n$.

Proof. (1) Suppose that $(d_1, d_2) = 1$ and that both $d_i$ are non units. Obviously $(d_1, d_2) = 1$ in any localization of R ( since R is an HCF domain).

Since q is a prime $d_1 d_2 = d \in q$, implies that $d_1 \in q$ or $d_2 \in q$. We note that both of $d_i$ cannot belong to q, because if$(d_1, d_2) = 1$ in R, $(d_1, d_2) = 1$ in $R_q$ and since $R_q$ is a valuation domain( q is a subvalued prime ) at least one of $d_i$ is a unit in $R_q$, in other words at least one of $d_i$ is not in q.

Let $d_2 \notin q$ then since $d_2 | d$ and since we have assumed that $d_2$ is a non unit the set $\{ P_\beta \in \{ P_\alpha \}_{\alpha \in I} \mid d \in P_\beta \}$ is a subset of $\{ P_1, P_2, \dots, P_r \}$ ( for if not so $\{P_1, \dots, P_r\}$ is not the set of all the valued primes containing d).

Select a member $P_j$ of $\{P_1, \dots, P_r\}$ such that $d_1, d_2 \in P_j$ but since $(d_1, d_2) = 1$ in R and$(d_1, d_2) = 1$ in $R_{P_j}$ and thus $d_2$ does not belong to $P_j$ i.e. if $(d_1, d_2) = 1$ and $d_2 \notin q$ then there exists no valued prime in the defining family of R which should contain $d_2$, a contradiction to the definition of a ring of Krull type and hence $d_2$ is a unit. Similarly if we had assumed $d_1 \notin q$ we would conclude that $d_1$ is a unit. thus if $(d_1, d_2) = 1$ then either of $d_i$ is a unit( but of course not both).

(2) Let x and d be as in the hypothesis and let

$(x,d) = h$ i.e. $x = x_1 h$ , $d = d_1 h$ where $(x_1, d_1) = 1$. Since $x \notin q$, $h \notin q$ ($\because h|x$), further since q is a prime and $d_1 h = d \in q$ ; $d_1 \in q$. Now $(x_1, d_1) = 1$ and we claim that $x_1$ is a unit, for if not $x_1$ is a member of at least one of $P_1, P_2, \ldots, P_r$. Suppose that $x_1 \in P_s$, then since $q \subset P_s$ ; $x_1, d_1 \in P_s$. Further since R is an HCF domain and $R_{P_s}$ is a valuation domain $x_1, d_1$ are non units in $R_{P_s}$ and so $(x_1, d_1) \neq 1$ in $R_P$ a contradiction implying that $x_1$ is a unit i.e. $x|d$ and obviously the same procedure holds for each integral power of x.

(3) Let d,d' be as in the hypothesis and let $(d,d') = h$ i.e. $d = d_1 h$ , $d' = d_1' h$ such that $(d_1, d_1') = 1$. Obviously $h \in q$ and this leaves us with two possibilities to consider

(a) $d_1, d_1' \notin q$

(b) one of $d_1, d_1'$ is in q.

In case (a) holds $d_1, d_1' | h$ by (2) above and so $d|d'^2$ and $d'|d^2$. And in case (b) holds ; if $d_1'$ is in q then $d|d'$. To show that there exists a positive integer r such that $d'|d^r$ we first prove that there exists an m such that $d^m \nmid d'$. Suppose on the contrary that $d^m | d'$ for each m, then for all m, $d^m | d'$ in $R_q$. But then $R_q$ being a valuation domain $Q = \bigcap_{n=1}^{\infty} d^n R_q$ is a prime ideal properly contained in $dR_q$ (cf Theorem 17.1 (3) page 187 [11]) that is $d'R_q \subsetneq Q \subsetneq qR_q$ i.e. $Q' = Q \cap R$ contains the minimal prime of d', but since we assumed that q is the minimal prime of d' and this result contradicts our assumption we infer that there exists a positive integer m such that $d^m \nmid d'$. Now if we let $(d^n, d') = d''$ (n greater than m) such that $d^n = ad''$ ; $d' = bd''$, then $(a,b) = 1$ and $b \notin q$ ( for if $b \in q$, $a \notin q$

and so by (2) $a|b$ i.e. $d^n|d'$, a contradiction to the fact that $d^m \nmid d'$ for an $m \leqslant n$.) and so $d'|d^{2n}$ which is the required result.

(4) Let x and d be as in the hypothesis. Using a method similar to the one used in the proof of (3) above, we can prove that there exists an n such that $d^n \nmid x$. Suppose that $d^n \nmid x$ and consider $(d^n,x) = h$, that is $d^n = ah$, $x = bh$ and $(a,b) = 1$ i.e. at least one of a,b is not in q. If $b \in q$ then $a \notin q$ and so $h \in q$ ($\because ah \in q$). Now b has a factor contained in q such that q is the only minimal subvalued prime of this factor (cf Lemma 2) and thus by (2) above $a^m|b$ for each m, and so $d^n = ah|bh = x$ ($\because a|b$ and $h|h$) a contradiction and hence $b \notin q$. If we assume that $a \in q$ then since $(a,b) = 1$ and q is the minimal prime of d and hence of a and h we have $(h,b) = 1$ (since if $(a,b) = 1$ then $(a^n,b^n) = 1$ and by (3) above there exists an n such that $h|a^n$) i.e.

$x = bh$ where $(b,h) = 1$. ------------------------(A)

Similarly if $a \notin q$ we can consider $(d^{n+1},x) = x'$ and then $d^{n+1} = x'k$, $x = x'' x'$, $(k,x'') = 1$ and if $k \notin q$ then $d^n|x'$ and so $d^n|x$ a contradiction establishing that k must be in q. As in (A) above $(k,x'') = 1$ implies that $(x'',x') = 1$ i.e. $x = x''x'$ where $x'$ has q as its only minimal subvalued prime and $(x'',x') = 1$. ---------------------(B)

Now combining (A) and (B) we get the result.

(5) the proof follows as an application of (2) and (3).

The properties (1) and (5) of d in Lemma 3 give rise to the following

Definition 4. A non zero non unit element d in an integral domain R, will be called a <u>packet</u> if every factorization of d into two non units, $d = d_1d_2$ (if it exists) is such that

($p_1$) $(d_1, d_2) \neq 1$

($p_2$) there exists a positive integer n such that

$$d_1 | d_2^n \text{ or } d_2 | d_1^n.$$

Finally we state the

Theorem 1. In an HCF domain of Krull type R, a non zero non unit x, is expressible as the product of a finite number of mutually co-prime packets and this factorization is unique up to associates of the respective packets and up to their order.

Proof. Let x be a non zero non unit in R, let $P_1, P_2, \ldots, P_n$ be the set of all the valued primes containing x and let $q_1, q_2, \ldots, q_m$ be the set of all the distinct minimal subvalued primes of x. By Lemma 2, corresponding to each $q_i$ there exists a $p_i | x$ such that $p_i \in q_i$ and $p_i \notin q_j$ for each $i \neq j$.

We first take up $q_1$; there exists a $p_1$ such that

$$x = p_1 x' \text{ where } p_1 \in q_1 \text{ and } p_1 \notin q_j \quad j = 2, \ldots, m.$$

And by (4) of Lemma 3 we can write

$$x = x_1 x_2' \text{ where } (x_1, x_2') = 1 \text{ and } x_1 \text{ has } q_1 \text{ as its only}$$

minimal subvalued prime i.e. $x_1 \notin q_j$ ( $j = 2, \ldots, m$).

Similarly corresponding to $q_2$, there exists $p_2 \in q_2$ such that $p_2 | x$ and $p_2 \notin q_j$ $j \neq 2$. Being in $q_2$, $p_2$ is not in the bunch of valued primes of x containing $q_1$ we conclude that $x = x_1 p_2 x_2''$ and by an application of (4) of Lemma 3 again $\quad x = x_1 x_2 x_3' \quad ; (x_1 x_2, x_3') = 1.$

Repeating the above process we get

$$x = x_1 x_2 \ldots x_m \text{ ; where each } x_i \text{ is a packet}$$

$$\text{and } (x_i, x_j) = 1 \text{ whenever } i \neq j.$$

Moreover if $x = y_1 y_2 \ldots y_s$ where $y_j$ are mutually co-prime packets then s = m, because the set of the valued primes(and hence of the minimal subvalued primes) remains the same. Suppose that $y_i$ are permuted such that, $x_k, y_k$ are in the same minimal subvalued prime q , then $x_k \mid (y_k^r, x) = (y_k, y_1 y_2 \dots y_m) = y_k$ that is $x_k | y_k$, and similarly we can show that $y_k | x_k$. I.e. for each packet $x_k | x = x_1 x_2 \dots x_n$ there exists its associate $y_k | x = y_1 y_2 \dots y_m$ which is the required result.

Corollary 1. In an HCF *-GKD a packet is rigid and hence an HCF *-GKD is a Semirigid Domain.

Proof. We recall that a *GKD R is a ring of Krull type with the family $\{ P_\alpha \}_{\alpha \in I}$ of primes defining it, such that for $\alpha \neq \beta \in I$ , $P_\alpha \cap P_\beta$ contains no non zero prime ideal(cf Def.3 Ch. 2, and Def.3 of this chapter).

Let q be a packet in the HCF *-GKD R and let $Q$ be the minimal subvalued prime containing q( it can be easily deduced from Lemmas 2 and 3 that in an HCF ring of Krull type an element x is a packet iff it has a single minimal subvalued prime), then q is contained in a single valued prime P of R(because of *3 of Def.3, Ch. 2). And obviously every non unit factor of q is in P (since otherwise q will not be in a single minimal subvalued prime e.g. if $q \in P \not\subseteq P'$ with no containment relation between P and P'; P' contains a minimal subvalued prime Q' of q such that $Q \neq Q'$).

Now let $q_1, q_2$ be two non unit factors of q then $q_1, q_2 \in P$. We claim that $(q_1, q_2) \neq 1$ for if $(q_1, q_2) = 1$ in R then since R is an HCF domain $(q_1, q_2) = 1$ in $R_P$ i.e. at least one of $q_1, q_2$ is a unit in $R_P$ which in other words means that at least one of $q_1, q_2$ is not in P a contradiction implying that no two non unit factors of q are co-prime. We now take any two non unit factors q', q" of q and suppose that $(q', q'') = d$ then $q' = xd, q'' = yd$ where $(x,y) = 1$. But since x,y also are factors of q, both of x,y cannot be non units and hence $q'|q''$ or $q''|q'$. That is q is a rigid non unit (cf Def. 1, Ch.2). Once we have shown that every packet in R is a rigid non unit it becomes obvious in the light of Theorem 1, that R is a Semirigid domain.

## 3. Unique Representation Domains.

In the previous section, we were able to show that every non zero non unit in an HCF ring of Krull type is the product of a finite number of mutually co-prime packets. But from the definition of a packet follows the

Proposition 2. Let R be an HCF domain and suppose that a non zero non unit x in R is expressible as the product of a finite number of mutually co-prime packets, then the factorization of x in this manner is unique up to associates of the packets and up to order.

And this Proposition gives us the concept of a <u>Unique Representation Domain (URD)</u>, as an HCF domain in which every non zero non unit is expressible as the product of a finite number of mutually co-prime packets.

In this section after formally proving the Proposition 2 we show with the help of an example that a URD is not necessarily a ring of Krull type. We show that an HCF domain is an essential domain and prove that the necessary and sufficient condition for an HCF domain R to be a URD is that every non zero non unit in R has only a finite number of minimal subvalued primes, and this gives rise to the definition of a <u>*-essential domain</u> as an essential domain in which every non zero non unit has a finite number of minimal subvalued primes.

Proof of Proposition 2. Let x be a non zero non unit in an HCF domain R and suppose that x is expressible as

$x = x_1 x_2 \ldots x_n$ ; $x_i$ are packets and $(x_i , x_j) = 1, i \neq j$ ---(A).

Further suppose that x is also expressible as

$x = y_1 y_2 \ldots y_m$ ; $y_j$ are packets and $(y_j , y_k) = 1$, $j \neq k$ ---(B).

Now $x_1 | y_1 y_2 \ldots y_m$, implies that $x_1 = x_{11} x_{12}$ such that $x_{11} | y_1$ and $x_{12} | y_2 y_3 \ldots y_m$ . But since $(y_1, y_2 y_3 \ldots y_m) = 1$, either $x_{11}$ is a unit or $x_{12}$ is (cf Def.4 ). In other words $x_1 | y_1$ or $x_1 | y_2 y_3 \ldots y_m$, and proceeding in this manner we can show that there exists only one $y_j$ such that $x_1 | y_j$. Reversing the process and considering $y_j | x_1 x_2 \ldots x_n$ and using the definition of a packet as above, we conclude that there exists an $x_k$ such that $y_j | x_k$. Moreover $x_1 | y_j$ and $y_j | x_k$ implies that $x_1 | x_k$ i.e. k = 1 ( since if $k \neq 1$ then $(x_k , x_1) = 1$ a contradiction ) and obviously for each $x_i | x$ in (A) there exists a $y_j | x$ in expression (B) such that $x_i$ is an associate of $y_j$. And consequently n = m and the factorizations (A)and (B) are unique up to associates and a suitable permutation of the packets.

Definition 5. An HCF domain R will be called a <u>Unique Representation Domain</u> if every non zero non unit of R is expressible as the product of a finite number of mutually co-prime packets.

Now to show that a URD is not necessarily an HCF ring of Krull type we put forward the following

Example 1. Let R be a PID, K its field of fractions and let x be an indeterminate over R. The integral domain

$S = R + xK[x]$ ; called the <u>almost integral closure</u> of R (cf [24] page 9) is a Bezout domain.

Consider a general non zero non unit element y in S, that is $y = r_0 + \sum_{i=1}^{n} a_i x^i$ ; $r_0 \in R$ , $a_i \in K$.

Now y can be of two possible types i.e. such that (1) $r_0 = 0$, (2) $r_0 \neq 0$.

In the first case $y = bx^s(1 + \sum_{j=1}^{n-s} a_j' x^j)$ ; $a_j', b \in K$. We see that $bx^s$ is a packet, because if

$bx^s = d_1 d_2$ ; $d_i$ non units and $(d_1, d_2) = 1$, then at least one of $d_i$ say $d_1$ is of degree zero in x and thus belongs to R, but then $d_1^n | d_2$ for each n and $d_2$ is of degree $s > 0$ in x; $d_1 | d_2$; a contradiction establishing that$(p_1)$ of Def. 4 holds for $bx^s$. Further if $bx^s = d_1 d_2$, $s > 0, d_i$ non units either $d_1 \in R$ or $d_1 = b_1 x^{s_1}$. If $d_1 \in R$ obviously $d_1 | d_2$ and if $d_1 = b_1 x^{s_1}, s_1 > 0$ then $d_2 = b_2 x^{s_2}$, where $b_1 b_2 = b$ , we note that if $s_2 = 0$ then $d_2 | d_1$ and so we take up $s_2 > 0$ and in this particular case $d_1$ divides a power of $d_2$ and vice versa. And to sum up $(p_2)$ of Def. 4 holds for $bx^s$, that is $bx^s$ is a packet . It is obvious that $( 1 + \sum_{j=1}^{n-s} a_j' x^j)$ is a product of atoms . But since, an atom in a Bezout domain is a prime, $( 1 + \sum_{j=1}^{n-s} a_j' x^j)$ is a product of powers of primes and can be written as the product of a finite number of mutually coprime powers of primes and thus is a product of a finite number of mutually co-prime packets because a prime power satisfies the requirements of a packet. Moreover since $(bx^s, 1 + \sum_{j=1}^{n-s} a_j' x^j) = 1$ , $y = bx^s( 1 + \sum_{j=1}^{n-s} a_j' x^j)$ is the product of a finite number of mutually co-prime packets.

In the second case, $y = r_0(1 + \sum_{i=1}^{n} a_i' x^i)$, where $r_0 \in R$

and so is a product of powers of primes and similarly as in the first case $(1 + \Sigma a_i' x^i)$ is a product of powers of primes and combining these observations $r\ (1 + \sum_{i=1}^{n} a_i' x^i)$ is the product of a finite number of mutually co-prime packets. And thus we have established that $S = R + xK[x]$ is a URD. But S is not necessarily a ring of Krull type, follows from the fact that $x \in pS$ for each prime p in R and if the number of prime ideals in R is infinite, S is not a ring of Krull type.

The above example gives rise to the question of characterization of a URD. We note that a URD by definition is an HCF domain and so, part of our task would be done if we explain the structure of an HCF domain in terms of its valuation overrings. For this purpose we prove that an HCF domain is an essential domain. To achieve this result we need to introduce some concepts which are to serve as tools.

Let R be an integral domain and K be its field of fractions and let F(R) be the set of non zero fractional ideals of R. If $A \in F(R)$, by $A^{-1}$ we mean the set

$\{ x \in K \mid xA \subseteq R \}$ and this again is a fractional ideal. We denote by $A_v$ the fractional ideal $(A^{-1})^{-1}$. The operation of associating $A_v$ with each fractional ideal $A \in F(R)$ is called the v-operation (cf [11] page 416]

It is well known (cf 32.1 [11]) that if $a \in K$ and $A, B \in F(R)$

(1) $(a)_v = (a)$ ; $(aA)_v = aA_v$

(2) $A \subseteq A_v$ ; if $A \subseteq B$ then $A_v \subseteq B_v$

(3) $(A_v)_v = A_v$

It is also known (cf (c) 32.2 [11]) that

$(AB)_v = (AB_v)_v = (A_vB_v)_v$ ----------------------------(VM)

A fractional ideal A is a v-ideal if $A = A_v$, and a v-ideal F is said to be of finite type if there exists a finitely generated ideal A such that $A_v = F$.

Definition . An integral domain R is called a <u>Prüfer v-multiplication domain</u> if the v-ideals of finite type in F(R) form a group under v-multiplication as (VM) above.

Note . Griffin [19] and [20] calls these integral domains, "<u>v-multiplication rings</u>" while in the present literature, a v-multiplication ring is an integral domain in which $(AB)_v \subset (AC)_v$ implies that $B_v \subset C_v$.

Turning our attention towards HCF domains we see that it is well known (cf e.g.[8]page 584) that each v-ideal of finite type of an HCF domain is principal. And to prove that an HCF domain is a Prüfer v-multiplication domain we only need to verify that the principal fractional ideals in F(R) form a group under multiplication which is evident. Thus an HCF domain is a Prüfer v-multiplication domain and hence according to Griffin [19] an essential domain.

We recall that an integral domain R is an essential domain if there exists a family $\Phi = \{P_\alpha\}_{\alpha \in I}$ of prime ideals such that $E_1$. $R_{P_\alpha}$ is a valuation domain for each $\alpha \in I$

$E_2$. $R = \cap R_{P_\alpha}, \alpha \in I$.

~~We can assume that no two members of $\Phi$ are comparable w.r.t. inclusion and~~ We shall call $\{P_\alpha\}_{\alpha \in I}$ the family of valued primes defining R. Clearly by $E_2$ above, a non zero non unit x in R is contained in at least one valued prime in

$\{P_\alpha\}$, for if not: x is a unit in each $R_{P_\alpha}$ $\alpha \in I$

i.e. $x^{-1} \in R_{P_\alpha}$ $\alpha \in I$

i.e. $x^{-1} \in \cap R_{P_\alpha} = R$, that is x is a unit in R, a contradiction establishing the result.

In what follows, the family of valued primes defining an HCF domain R will be denoted by $\{P_\alpha\}_{\alpha\in I}$ and by a valued prime we shall mean a valued prime in $\{P_\alpha\}_{\alpha\in I}$ and by a subvalued prime we shall mean a prime contained in a valued prime in $\{P_\alpha\}$.

Lemma 4. A non zero non unit x in an HCF domain R is a packet iff x has a single minimal subvalued prime .

Proof. Let x be a non zero non unit in an HCF domain R and let x have a single minimal subvalued prime q. We have to show that x is a packet i.e.

($p_1$) if $x = x_1x_2$, where $x_i$ are non units then $(x_1,x_2) \neq 1$

($p_2$) if $x = x_1x_2$, with $x_i$ non units then there exists a positive integer n such that $x_1 | x_2^n$ or $x_2 | x_1^n$.

We first show that ($p_1$) holds for x, for if we assume on the contrary that $x = x_1x_2$, $x_i$ non units and $(x_1,x_2) = 1$ then $x_1$ and $x_2$ cannot both belong to the same valued prime P because then $(x_1,x_2) = 1$ in R implies that $(x_1,x_2) = 1$ in $R_P$ which in turn implies that at least one of $x_1, x_2$ is not contained in a given valued prime.

Let $P_1$ be one of the valued primes containing $x_1$ and $P_2$ be one of those containing $x_2$ then the minimal subvalued primes $q_1, q_2$ of $x_1$ and $x_2$ respectively are distinct and obviously these are minimal subvalued primes of x, a contradiction establishing that $(x_1,x_2) \neq 1$.

Before establishing that ($p_2$) holds for x, we prove the following lemma to make our task easier.

Lemma 5. Let x be a non zero non unit with a single minimal subvalued prime q in an HCF domain and let $\{P_\beta\} \subset \{P_\alpha\}$ be the family of valued primes containing x, then for every element y which is contained only in the intersection of a subfamily of $\{P_\beta\}$ such that $y \notin q$ then $y^n | x$ for all n.

Proof. Let x and y be as in the hypothesis, then for each n, $xy^n \in q$ and $xy^n$ has q as its minimal subvalued prime ( any minimal subvalued prime of y is some subvalued prime containing q).

Now suppose that $y \nmid x$ and let $d = (x,y)$ where $x = x_1 d$, $y = y_1 d$ and $(x_1, y_1) = 1$, then since $y \notin q$, $d \notin q$ and so $xy/d^2 \in q$ and q is the single minimal subvalued prime of $xy/d^2$. But $xy/d^2 = x_1 y_1$ where $(x_1, y_1) = 1$. In other words $xy/d^2$ has a single minimal prime and is expressible as a product of two co-prime non units, a contradiction of $(p_1)$ unless $y_1$ is a unit i.e. $y|x$. Similarly we can proceed with $y^n$ and can show that $y^n | x$ for each n.

To show that $(p_2)$ also holds for x of Lemma 4, we first note that q being a prime ideal, $x_1 \in q$ or $x_2 \in q$, and we have two cases to consider:

(a) $x_1 \in q$, and $x_2 \notin q$ ( or $x_2 \in q$ and $x_1 \notin q$ )

(b) $x_1, x_2 \in q$.

If (a) holds, $x_2$ belongs to a subfamily of the valued primes containing x and by Lemma 5, $x_2^n | x$ for each n, i.e. $x_2 | x_1$. And in case (b) holds, $x_1, x_2 \in q$ implies that $x_1, x_2$ both have q as their minimal subvalued prime and that $(x_1, x_2) = d \in q$ (   R is an HCF domain and $R_q$ is a valuation domain). Now if $(x_1, x_2) = d$ then $x_1 = x_1' d$, $x_2 = x_2' d$ where $(x_1', x_2') = 1$ i.e. at least one of $x_1', x_2'$ is not in q. This in turn gives rise to the following two cases:

(i) $x_1' \notin q$ and $x_2' \in q$ (or $x_1' \in q$ and $x_2' \notin q$ )

(ii) $x_1', x_2' \notin q$ .

In the first case if $x_1'$ is not a unit, $x_1'$ belongs to a subfamily of the family of valued primes containing x(and hence $x_2'$ ) and so $x_1' | x_2'$ , that is $x_1 | x_2$. And in the second case $x_i'^n | d$ ( i = 1,2.) for each n and so $x_1 | x_2^2$ and $x_2 | x_1^2$ .

Combining all the above cases we conclude that ($p_2$) holds for x. In other words x is a packet.

Conversely let x be a packet in an HCF domain R and let $\{P_\beta\}$ be the family of all the valued primes containing x, further let P,Q be two distinct minimal subvalued primes of x and consider $y \in P - Q$, then $(x,y) = d \in P - Q$ ( can be verified easily by using the fact that R is an HCF domain and $R_P$ is a valuation domain), and d has P as one of its minimal subvalued primes. We claim that there exists a positive integer n such that $d^n \nmid x$. For if not let $d^n | x$ for each n, then $d^n | x$ for each n in $R_P$ and so $x \in \cap\ d^n R_P = P_1 R_P$ where $P_1 R_P$ is a prime ideal properly contained in $PR_P$ i.e. $x \in P_1 R_P \cap R$ , and by the one-one correspondence between primes in $R_P$ and those contained in P, x has $P_1 R_P \cap R$ as its minimal subvalued prime a contradiction to the assumption that P is one of the minimal subvalued primes of x, and hence there exists a positive integer n such that $d^n \nmid x$.

Now consider $h = (x, d^n)$ where $d^n \nmid x$ in $R_P$ then $d^n = ah$ $x = bh$ and $(a,b) = 1$. We claim that $b \notin P$ for if $b \in P$, then $a \notin P$ and so $a | b$ in $R_P$ and consequently $ah | bh$ in $R_P$ that is $d^n | x$ in $R_P$ , a contradiction establishing the claim.

Further $h | d^n \notin Q$ and so $h \notin Q$ but since $bh \in Q; b \in Q$ (Q being a prime) that is we have $x = bh$ where

$b \in Q$ and $b \notin P$ ------------------(1)

$h \in P$ and $h \notin Q$ ------------------(2)

but since x is a packet there exists an n such that $b|h^n$ or $h|b^n$. Now if $b|h^n$ then $h^n \in Q$ i.e. $h \in Q$ which contradicts(2) and if $h|b^n$ ; $b \in P$ in contradiction to (1) and this establishes that a packet x in an HCF domain R cannot have more than one minimal subvalued primes.

Now going from packets to products of mutually co-prime packets, we prove the following

Theorem 3. An HCF domain R is a URD iff every non zero non unit x in R has a finite number of minimal primes.

Proof. Let R be a URD and let x be a non zero non unit in R. We can write

$$x = x_1 x_2 \ldots x_n \; ; \; (x_i, x_j) = 1 \text{ if } i \neq j$$

where each of the $x_i$ is a packet. Being mutually co-prime, no two of the $x_i$ have a valued prime common to them and hence no subvalued prime, while each of the $x_i$ has a single minimal subvalued prime(being a packet) and consequently x has a finite number of minimal subvalued primes.

Conversely let x be a non zero non unit in an HCF domain R and let $q_1, q_2, \ldots, q_n$ be all the minimal subvalued primes containing x then following exactly the same lines as in the proof of Theorem 1, of this chapter we can show that $x = x_1 x_2 \ldots x_n$ ; where each of the $x_i$ is a packet such that $(x_i, x_j) = 1$ if $i \neq j$. And to conclude the proof we mention that a minimal prime P of a principal ideal is a minimal subvalued prime. For if not let $R_P$ be not a valuation domain. Then since $R_P$ is an HCF domain and thus is essential there exists a valued prime $Q$ ( $\neq PR_P$) containing x. But then $x \in Q \cap R \subsetneq P$ a contradiction.

Theorem 3, gives rise to the following

Definition 5. An essential domain R with the defining family $\{ P_\alpha \}_{\alpha \in I}$ of primes will be called *-essential if every non zero non unit x in R has a finite number of minimal subvalued primes.

Finally in view of Theorem 3, and the earlier work we can state that a non zero non unit x in an HCF domain R is the product of a finite number of mutually co-prime packets iff x has a finite number of minimal primes.

4. Stability Properties of URD's.

We begin this section with results about the behaviour of Unique Representation under the operations of adjoining indeterminates and localization. We then go on to establish a property of URD's which is not shared by UFD's that is if R is a URD x an indeterminate over R and K the field of fractions of R then the almost integral closure

$$S = R + xK[x]$$

is a URD. Finally with the help of examples we show that the integral domains we have considered under distinct names are in fact distinct.

Like Unique Factorization, the concept of Unique Representation remains stable under adjoining indeterminates and this we prove with

Proposition 4. Let R be a URD and x an indeterminate over R then R[x] is a URD.

Proof. Since a URD is an HCF domain, and we have mentioned before that an atom in an HCF domain is a prime. Moreover if R is an HCF domain then so is R[x].

Now consider an arbitrary non zero non unit

$$y = \sum_{i=0}^{n} r_i x^i \quad ; \; r_i \in R.$$

Let d be the highest common factor of $r_0, r_1, r_2, \ldots r_n$ then

$y = d(\sum_{i=0}^{n} r_i' x^i)$; the expression in braces is a primitive polynomial in x, and since every non unit factor of the expression in braces is of degree greater than zero in x, it has only a finite number of factors. I.e. $\Sigma r' x^i$ is a product of atoms and hence of primes and since a prime power is a packet; $\Sigma r_i' x^i$ is a product of a finite number of mutually co-prime packets.

Finally it can be verified that $(d, \Sigma r_i' x^i) = 1$. And since d is in R ( and so is a product of mutually co-prime packets if it is a non unit)

$y = d(\Sigma r_i' x^i) = \sum_0^n r_i x^i$ is a product of a finite number of mutually co-prime packets. Since y is arbitrary the result follows.

Since a prime power is a rigid element we can state the

Corollary 3. If R is a Semirigid domain and x is an indeterminate over R, then R[x] is a Semirigid domain.

Further let R be a URD, S a multiplicative and saturated set of R and let x be a packet in R then we claim that if x is not a unit in $R_S$ then it is a packet in $R_S$. For if not let $x = x_1 x_2$; where $x_i$ are non units in $R_S$ such that $(x_1, x_2) = 1$ in $R_S$. Now if $x_1 = u_1/v_1$, $x_2 = u_2/v_2$;(since R is an HCF domain we can take $(u_i, v_i) = 1$, $i = 1,2.$) then $x = (u_1/v_1)(u_2/v_2)$ implies that $v_2 | u_1$ and $v_1 | u_2$ i.e. $u_1 = u_1' v_2$, $u_2 = u_2' v_1$ and $x = u_1' u_2'$ where $u_1', u_2' \in R$ and

$(u_1', u_2') = 1$ in $R_S$. Since we are approaching from a localization to the original ring, it is possible that $(u_1', u_2') \neq 1$ in R (moreover $u_i'$ being non units in $R_S$ are non units in R) and thus there exists a positive integer n such that $u_1' | u_2'^n$ or $u_2' | u_1'^n$(x being a packet ). If we have $u_1' | u_2'^n$ then obviously $u_1' | u_2'^n$ in $R_S$ , but since $(u_1', u_2') = 1$ in $R_S$ which is an HCF domain, $(u_1', u_2'^n) = 1$ in $R_S$, which implies that $u_1'$ is a unit in $R_S$ a contradiction to the assumption that $x_1$ , $x_2$ are both non units in $R_S$ and hence x is a packet in $R_S$.

Now according to the definition

$$R_S = \{ r/s \quad r \in R \; ; \; s \in S \}.$$

If r/s is a non unit in $R_S$ and if $r = p_1 p_2 \dots p_n$, $p_i$ packets and $(p_i , p_j) = 1$ if $i \neq j$ then

$$r/s = (p_1/s_1)(p_2/s_2)\dots(p_n/s_n); \text{ where } s = s_1 s_2 \dots s_n$$

$(p_i/s_i)$ are packets if non unit and because of the HCF property $((p_i/s_i), (p_j/s_j)) = 1$ if $i \neq j$, that is if R is a URD then so is $R_S$ and so we state the

Proposition 5. Let R be a URD and S be a multiplicative and saturated set in R then $R_S$ is a URD.

The concept of a rigid non unit being simpler than that of a packet we can easily prove the

Corollary 4. If R is a Semirigid domain and S is a multiplicative and saturated set in R then $R_S$ is again a Semirigid domain.

In Example 1, we showed that the almost integral closure of a PID is a URD, we now extend this result and state the

Theorem 6. Let R be an integral domain, K its field of fractions and x an indeterminate over R, then R is a URD iff its almost integral closure $S = R + xK[x]$ is a URD.

Proof. If S is a URD, all the non units of R are non units of S and hence products of mutually co-prime packets and R is an HCF domain as well.

To prove the converse we first prove the

Lemma 6. Let R be an HCF domain, K its field of fractions and x an indeterminate over R then $R + xK[x]$ is an HCF domain.

Proof. A general element $y \in S$ can be written as

$$y = r_o + \sum_{i=1}^{n} a_i x^i \quad ; r_o \in R \text{ and } a_i \in K.$$

As we observed in Example 1, y can be of two types corresponding to $r_o = 0$ or $r_o \neq 0$, that is

$(\alpha)$ $(r_o = 0)$ ; $y = bx^s(1 + \sum_{j=1}^{n-s} a'_j x^j)$; $b \in K$

$(\beta)$ $(r_o \neq 0)$ ; $y = r_o(1 + \sum_{i=1}^{n} (a_i/r_o)x^i)$.

The case where one of the elements of S is zero or is a unit, is obvious and so we consider a pair $y_1, y_2$ of arbitrary non zero non units of S. Let

$y_1 = r_{o1} + \sum_{i_1=1}^{n_1} a_{i_1} x^{i_1}$, $y_2 = r_{o2} + \sum_{i_2=1}^{n_2} a_{i_2} x^{i_2}$, the following cases are possible:

(a) both $y_1, y_2$ are of type $(\alpha)$

(b) $y_1$ is of type $(\alpha)$ and y is of type $(\beta)$ (~~or otherwise~~) an vice versa

(c) $y_1, y_2$ are both of type $(\beta)$.

In case (a) holds, let

$$y_1 = b_1 x^{s_1}(1 + \sum_{j_1=1}^{n_1-s_1} a'_{j_1} x^{j_1}) , \; y = b_2 x^{s_2}(1 + \sum_{j_2=1}^{n_2-s_2} a'_{j_2} x^{j_2})$$

the expressions in braces being elements of $K[x]$ are products of primes and so the HCF

$$d = ( (1 + \sum_{j_1=1}^{n_1-s_1} a'_{j_1} x^{j_1}),(1 + \sum_{j_2=1}^{n_2-s_2} a'_{j_2} x^{j_2}) )$$

can be calculated.

Now if $s_1 < s_2$ it is easy to see that $b_1 x^{s_1} d$ is the highest common factor of $y_1, y_2$. Further if $s_1 = s_2 = s$, the highest common factor of $b_1 x^{s_1}$ and $b_2 x^{s_2}$ ( if it exists ) must be of degree s in x. If $b_1 = c_1/d_1$ and $b_2 = c_2/d_2$ ( we can assume $(c_i, d_i) = 1$ because of R being HCF) it can be verified that $((c_1, c_2)/[d_1, d_2])x^s d$ is the highest common factor of $y_1, y_2$, where $[d_1, d_2]$ denotes the least common multiple of $d_1$ and $d_2$.

If the case (b) holds let $y_1$ be of type ($\alpha$) and $y_2$ be of type ($\beta$), that is $y_1 = b_1 x^{s_1}(1 + \sum_{j_1=1}^{n_1-s_1} a'_{j_1} x^{j_1})$

$y_2 = r_{02}(1 + \sum_{i_2=1}^{n_2} a'_{i_2} x^{i_2})$ and if d is the HCF of the elements in the braces then $r_{02}d$ is the HCF of $y_1$ and $y_2$.

Finally if (c) holds let $y_1 = r_{01}(1 + \sum_{i_1=1}^{n_1} a'_{i_1} x^{i_1})$

$y_2 = r_{02}(1 + \sum_{i_2=1}^{n_2} a'_{i_2} x^{i_2})$

and if d is the HCF of the elements in the braces then $(r_{01}, r_{02})d$ is the HCF of $y_1, y_2$.

To sum up, each pair of non units in S has the highest common factor and this establishes the lemma.

Now let y be a general non zero non unit element in S then $y = r_0 + \sum_{i=1}^{n} a_i x^i$ ; $r_0 \in R$, $a_i \in K$, and y can be of two types; ($\alpha$) $y = bx^s(1 + \sum_{j=1}^{n-s} a'_j x^j)$ ; $b \in K$, or

($\beta$) $y = r_0(1 + \sum_{i=1}^{n} a'_i x^i)$.

We note that the expressions within the braces in both cases being elements of K[x] are products of powers of primes and hence of mutually co-prime packets.

In case ($\beta$) $0 \neq r_0 \in R$ is the product of a finite number of mutually co-prime packets (provided it is a non unit) and $(r_0, 1 + \sum_{i=1}^{n} a_i' x^i) = 1$ that is y is a product of a finite number of mutually co-prime packets . And in case ($\alpha$) obviously $( bx^s, 1 + \sum_{j}^{n} a_j' x^j) = 1$ ; $b \in K$, and $bx^s$ is a packet itself (cf Example 1, this chapter). Consequently y is a product of a finite number of mutually co-prime packets in case ($\alpha$) as well, and this completes the proof.

Remark 1 . Thoerem 6, marks the basic difference of the concepts of Unique Factorization and Unique Representation, because the almost integral closure of a UFD is not completely integrally closed and hence cannot be a UFD.

We have hitherto mentioned different classes of integral domains, one generalizing the other; that is if we take $\triangleright$ to mean <u>generalize</u> we have

URD's $\triangleright$ HCF rings of Krull type $\triangleright$ Semirigid Domains $\triangleright$ GUFD's $\triangleright$ ~~URD's~~ UFD's.

We have shown by Example 7, of Chapter 1, that there exists a GUFD which is not a UFD. Similarly Example 1 of Chapte 2, ensures the existence of a Semirigid Domain which is not a GUFD. We have also shown, with Example 1, of this chapter, that there exists a URD which is not an HCF ring of Krull type and finally it remains to show that there exists an HCF ring of Krull type which is not a Semirigid Domain and for this we consider the following

Example 2. Let R be a Semi-local PID, K the quotient field of R and x an indeterminate over R. The almost integral closure

$S = R + xK[x]$ , is a two dimensional Bezout

domain and is a URD ( Example 1, this chapter).

If $P_1 = p_1R$, $P_2 = p_2R$ ,...,$P_n = p_nR$ are all the non zero prime ideals of R then correspondingly $p_iS$ $(i = 1,2,\dots,n)$ are maximal ideals of R of rank 2. Now let

$T = \{ y \in S \mid y \notin p_iS$ for any $i = 1,2,\dots,n\}$,

then it can be shown that T is a multiplicative saturated set. Localizing at T, $S_T$ is a two dimensional Bezout domain with exactly n maximal ideals $p_iS_T$ $(i = 1,2,\dots,n)$. Obviously S is a semi quasi-local Bezout domain and so an HCF ring of Krull type. Finally that $S_T$ is not a Semirigid Domain follows from the fact that $0 \neq \cap p_iS$ is a prime ideal. That is $S_T$ is our example of an HCF ring of Krull type which is not a Semirigid Domain.

Note . S it self is an example of an HCF ring of Krull type. We have avoided S as an example on the basis that its verification becomes very lengthy.

Remark 2. Introduction of the concept of Unique Representation is the result of an effort to study and to single out those HCF domains in which the factorization is rather simple. We cannot at present guess the scope of this concept but it can be remarked that this concept could be of some help in the study of HCF rings of Krull type, semi quasi-local Prufer domains, *-essential Bezout domains etc. At least in these cases we could start with the knowledge that the elements of these integral domains have some factorization properties.

# CHAPTER 4

## IDEAL TRANSFORMS IN GENERALIZED KRULL DOMAINS

### 0. Introduction, Definitions and Basic results.

Let R be a commutative integral domain with unity and let K be the field of fractions of R. If A is an ideal of R then the set

$$T(A) = \{ x \in K \mid xA^n \subseteq R \text{ for some positive integer } n \}$$

is a ring and is called the A-transform of R, ideal transform of R or the transform of A. The notion of an ideal transform was introduced and developed by Nagata in [26] and [27].

Gilmer used the ideal transforms in the study of Prüfer domains in [12]. Later appeared [17] by Gilmer and Heinzer. The efficiency of this tool in studying the Prüfer domains, soon attracted the attention of various mathematicians and the study of properties of the ideal transform began. Brewer in [2] put forward some striking results connecting some integral domains and the transforms of their proper principal ideals, while Arnold and Brewer in [3] discussed Generalized transforms. Gilmer and Huckaba [15] discussed some properties of ideal transforms in general and of ideal transforms in Krull domains in particular.

Our interest in the generalization of the concept of Unique Factorization led us to Generalized Krull Domains (cf Ch 1) and the rather easy formulation of Generalized Unique Factorization led to the observation that, with some modifications the GKD's can be studied on the same lines as Krull domains . The realization of Theorems 1, and 2, confirmed our observation as far as the ideal transform is concerned, Theorem 2, in fact has motivated much of the work included in this chapter.

In the first section we improve Lemma 2.12 of [14] to Theorem 1, which gives a formula for the transform of an ideal in an integral domain which is a locally finite intersection of a family of overrings, while Theorem 2, provides a neat formula for the ideal transform of an ideal in a GKD.

In section 2, we generalize the property ($\mu$) discussed in [15] page 207 to property ($\nu$) (cf Definition 1 ) and record the consequences of this generalization.

Brewer's Theorem for Krull domains which is included as (4) of Theorem 9, establishes the relationship of an integral domain (which is not quasi-local) and the transforms of its proper principal ideals. In section 3, we provide an analogue of this result for GKD's, and analyse the situation for quasi local domains.

Section 4, includes miscellaneous results, in other words those results which could not find a place in the earlier sections but seem to be interesting enough to be included in this chapter.

The notions and notations used in this chapter are either familiar or properly explained with the exception that by an <u>ideal</u> we mean an <u>integral ideal</u> including (0) and R (the integral domain) as ideals and by an invertible ideal we mean an ideal which has an inverse in the group of fractional ideals.

In the following we include without proof, some basic results already in the literature, and will use them where necessary with little or no reference.

Definition 0 (cf [15]) An integral domain R is called a

(1) $T_1$-domain if $T(AB) = T(A) + T(B)$ for every two ideals

A,B of R.

(2) $T_2$-domain if T(AB) = T(A) + T(B) for every pair of finitely generated ideals A,B of R.

(3) $T_3$-domain if T(AB) = T(A) + T(B) for every pair of principal ideals of R.

Proposition $O_1$ (cf Prop. 1, [15]) Let $A, A_1, A_2, \ldots, A_n$ and B be ideals of an integral domain R

(a) if k is a positive integer such that $A^k \subseteq B$ then $T(A) \supseteq T(B)$ and T(AB) = T(A) = T(A) + T(B)

(b) if $e_i$ and $f_i$ are positive integers for $1 \leqslant i \leqslant n$, then

$$T(A_1^{e_1} \cdot \ldots \cdot A_n^{e_n}) = T(A_1^{f_1} \cdot \ldots \cdot A_n^{f_n})$$

(c) if the hypothesis is as in (b) then

$$T(A_1^{e_1} + A_2^{e_2} + \ldots + A_n^{e_n}) = T(A_1^{f_1} + \ldots + A_n^{f_n}).$$

In particular if $(a_1, \ldots, a_n)$ is an ideal of R then

$$T(a_1^{e_1}, \ldots a_n^{e_n}) = T(a_1^{f_1}, \ldots, a_n^{f_n})$$

(d) $T(AB) \supsetneq T(A) + T(B)$

(e) if A and B are such that there exists an ideal A* such that $A^* \supseteq B$ and $T(A^*) = T(A)$ then

$$T(AB) = T(B) = T(A) + T(B)$$

(f) if T(A) = R or T(B) = K, the field of fractions of R then T(AB) = T(A) + T(B)

(g) $T(A \cap B) = T(AB)$

(h) $T(A) \cap T(B) = T(A + B)$.

Note. (a) and (e) of Prop. 1 of [15] are combined to give (a) while (e) of Prop. $O_1$ is new but easy to verify.

Theorem $O_2$ (Lemma 1 [15]). (i) Suppose that (A, and B are ideals of R such that (A + B) T(A,B) = T(A,B) then for each positive integer k, $(A^k + B^k)$ T(A,B) = T(A,B).*1

(ii) If A and B are comaximal ideals of R and if C is any ideal of R then, T(ABC) = T(AC) + T(BC).

*1 T(A, B) = T(A+B)

(iii) Suppose that A and B are ideals of R and suppose that A is invertible then $T(AB) = T(A)T(B)$.

(iv) If A and B are ideals of R, and C is a finitely generated ideal of R, and if T(A) contains T(B) then T(AC) contains T(BC).

Theorem $O_3$ ( Theo. 2 [15]) If a and b are elements of R then the following are equivalent:

(1) $T(ab) = T(a) + T(b)$

(2) for every positive integer k there exist $\alpha, \beta$ in R such that $(1/ab)^k = \alpha/a^i + \beta/b^j$ ; i, j positive integers.

(3) $(a,b)T(a,b) = T(a,b)$.

Theorem $O_4$ (Theorem 4, [15]). Let A and B be ideals of R

(1) If A + B is an invertible ideal of R and if C is any ideal of R then $T(ABC) = T(AC) + T(BC)$.

(2) If A is an invertible ideal of R and if T(A) + T(B) is a subring of K, then $T(A) + T(B) = T(AB)$.

(3) Suppose that $T(A) + T(B) = T(C)$ where A and C are finitely generated ideals of R, then $T(A) + T(B) = T(AB)$.

Theorem $O_5$ (Theorem 8 [15]) If x and y are non zero elements of an integral domain R such that $(x) : (y) = (x)$, then $T(xy) = T(x) + T(y)$ implies that $xR + yR = R$.

Theorem $O_6$ (Proposition 1.4 [2]) Let A be a finitely generated ideal of an integral domain R, with $A = (a_1, \ldots a_n)$ then $T(A) = \bigcap_{i=1}^{n} T(a_i)$.

Theorem $O_7$ (Theorem 1.5 [3]). Suppose that A is a finitely generated ideal of an integral domain R. Let $\{P_\alpha\}$ be the collection of all prime ideals of R which do not contain A, then

$$T(A) = \bigcap_{\alpha} R_{P_\alpha} .$$

Theorem $O_8$ (Lemma 2.2 [2]). Let x be a non zero element of an integral domain R. Then $T(x) = R(1/x) = R_S$ ; $S = \{x^i\}_{i=0}^{\infty}$

1. A Formula for the Transform of an ideal in a GKD.

In [14], Lemma 2.12 states," Let D be an integral domain with identity having a quotient field K. If $\{D_i\}_{i=1}^{n}$ is a finite family of overrings of D such that $D = \bigcap_{i=1}^{n} D_i$ and if A is an ideal of D, then $T(A) = \bigcap_{i=1}^{n} T(AD_i)$ ".

In this section we generalize this result to the case of an integral domain R which has a family $\{R_\alpha\}$ of overrings such that $R = \cap R_\alpha$ and each non unit element of R is a non unit in only a finite number of $R_\alpha$ . This generalization appears as Theorem 1, and as a consequence of this theorem we prove Theorem 2, which gives a formula for the transform of a non zero ideal of a GKD.

Theorem 1. Let R be an integral domain with identity and let K be its field of fractions. If $\Pi = \{R_\alpha\}$ is a family of overrings of R such that

(a) $R = \cap R_\alpha$ ; $R_\alpha \in \Pi$

(b) for every non zero non unit x of R, x is a non unit in only a finite number of members of $\Pi$

then for every ideal A of R, $T(A) = \cap T(AR_\alpha)$ ; $R_\alpha \in \Pi$ .

Proof. It is clear that for every overring R' of R,

$T(A) \subseteq T(AR')$ and so

$T(A) \subseteq \cap T(AR_\alpha)$ ; $R_\alpha$ ranging over $\Pi$ .

Now let $y \in \bigcap_\alpha T(AR_\alpha)$ , we can write $y = r/s$ where $r,s \in R$ and $s \neq 0$. According to the hypothesis s is a unit in all but a finite number of members of $\Pi$ . Let

$\Sigma = \{R_1, R_2, \ldots, R_n\}$

be the set of all those overrings of R (in $\Pi$ ) in which s is a non unit, so that

$(r/s)(AR\ ) \subseteq R_\alpha$ for all $R_\alpha \in \Pi - \Sigma$ -----(1)

Now, $y = (r/s) \in T(AR_\alpha)$ implies that

$(r/s) \in T(AR_i)$ ; $i = 1,2,\ldots n$ ( $R_i \in \Pi$), that is there exist $m_i$ ($i = 1,2,\ldots,n$) such that

$$(r/s)(AR_i)^{m_i} \subset R_i .$$

Let $m = \max \{ m_i \mid i = 1,2,\ldots,n\}$, then

$$(r/s)(AR_i)^m \subset R_i \ ; \ i = 1,2,\ldots,n \text{ -------------- } (2)$$

Combining (1) and (2)

$(r/s)(AR_\alpha)^m \subset R_\alpha$, for all $R_\alpha$ in $\Pi$ , that is

$(r/s)(A)^m \subset R_\alpha$ , for all $R_\alpha$ in $\Pi$ , that is

$(r/s)(A)^m \subset \cap R_\alpha = R$ , and thus $y \in T(A)$. In other words $T(A) = \cap T(AR_\alpha)$ ; $R_\alpha \in \Pi$.

For the sake of reference we shall call the ring R with a family $\{R_\alpha\}$ of overrings satisfying (a) and (b) of Theo. 1 a $\Delta$-ring . The family $\{R_\alpha\}$ of overrings of R will be called the <u>defining family of R</u> . If every member of the defining family $\{R_\alpha\}$ of a $\Delta$-ring is such that $R_\alpha = R_{P_\alpha}$ for some prime ideal $P_\alpha$ then R will be called an <u>essential $\Delta$-ring</u>. It is easy to see that if R is an integral domain with a family $\{P_\alpha\}$ of prime ideals such that

(1) $R = \cap R_{P_\alpha}$ ; $P_\alpha \in \{P_\alpha\}$

(2) for every non zero non unit x in R, x belongs to only a finite number of members of $\{P_\alpha\}$

then R is an essential $\Delta$-ring .

Moreover we can assume that no two members of $\{P_\alpha\}$ are comparable w.r.t. inclusion. The family $\{P_\alpha\}$ will be called the defining family of the essential $\Delta$-ring.

As may be easily seen, an essential $\Delta$-ring is a generalization of the rings of Krull type (cf Def. 3, Ch. 3)

For the present we restrict our attention to the immediate task of finding a formula for the transform of an ideal in a GKD which is a ring of Krull type restricted still further and state as a preliminary, the following

Theorem 1'. Let R be an essential Δ-ring with a defining family $\{P_\alpha\}$ of primes, then for every ideal A of R

$$T(A) = \cap\, T(AR_{P_\alpha}) \; ; \; P_\alpha \in \{P_\alpha\}.$$

Corollary 1. ( Prop.7, [22]) Let R be a ring of Krull type with the defining family $\{R_{P_\alpha}\}$, then for every ideal A in R

$$T(A) = (\bigcap_{A \subset P} T(AR_P)) \cap (\bigcap_{A \not\subset P} R_P) \quad ; \; R_P \in \{R_{P_\alpha}\}.$$

In the case of a generalized Krull Domain R, we find a somewhat neater formula for the transform of a non zero ideal A. We recall that a GKD is a ring of Krull type in which the defining family $\{P_\alpha\}$ consists of all the minimal non zero prime ideals of R. To bring about the said formula we prove the

Theorem 2. Let A be a non zero ideal in a GKD R, then $T(A) = \cap\, R_P$ where P ranges over all non zero minimal prime ideals of R for which $(AR_P)^2 = AR_P$.

Proof. By Corollary 1, above

$$T(A) = (\bigcap_{A \not\subset P} R_P) \cap (\bigcap_{A \subset P} T(AR_P))$$

and so the task of finding the transform of A has been reduced to that of finding the transform of ideals in a finite number of rank one valuation rings.

Now it is well known that the maximal ideal $PR_P$ of the rank one valuation domain $R_P$ is either (i) principal or (ii) idempotent (cf. e.g. [28])

(i) Let $PR_P$ be principal, then

$AR_P = (PR_P)^n = (pR_P)^n = p^nR_P$ for some n, that is

$T(AR_P) = T(PR_P)^n = T(PR_P) = T(pR_P)$ (cf (d) Cor. 2.4[28])

$= R_P[1/p]$ (cf Theo. 0₈ )

$= K$, the field of fractions of R.

(ii) Let $PR_P$ be idempotent, then $R_P$ being of rank one is completely integrally closed and so $T(PR_P) = R_P$.

Now for $AR_P$, there are two possibilities:

(a) $AR_P = PR_P$ idempotent

(b) $AR_P \subsetneq PR_P$ non idempotent (cf (b) Cor 2.4 [28])

We have seen that in case (a)

$T(AR_P) = T(PR_P) = R_P$, and to deal with the case (b) let $x \in A$, and consider $xR_P \subseteq AR_P \subseteq PR_P$ since $R_P$ is a rank one valuation domain $xR_P$ and $AR_P$ are both $PR_P$-primary and so, there exists a positive integer n such that $(AR_P)^n \subsetneq xR_P \subsetneq AR_P$ (cf(c) Cor. 2,4 [28]) and consequently $T(AR_P) = T(xR_P)$ (cf (a) Prop. 0, )

$= R_P[1/x] = K$, the field of fractions of R.

And in view of case (a) we conclude that if $A \subseteq P$; $T(AR_P) = R_P$ iff $(AR_P)^2 = AR_P$ ( since otherwise $T(AR_P) = K$, as we have shown above ). Moreover if $A \not\subseteq P$ then $AR_P = R_P$ and so $T(AR_P) = R_P$ for minimal primes P such that $(AR_P)^2 = AR_P$.

To conclude our proof, we consider the expression

$$T(A) = \left(\bigcap_{A\not\subset P} R_P\right) \cap \left(\bigcap_{A\subset P} T(AR_P)\right)$$

Obviously $\bigcap_{A\subset P} T(AR_P) = K \cap \left(\bigcap_{A\subset P} R_P\right)$; where $(AR_P)^2 = AR_P$

therefore $T(A) = \left(\bigcap_{A\not\subset P} R_P\right) \cap \left(\bigcap_{A\subset P} R_P\right)$ and $(AR_P)^2 = AR_P$

$$= \bigcap_{(AR_P)^2 = AR_P} R_P$$

It may be noted that in a Krull domain R, for every minimal prime ideal P, $R_P$ is a discrete rank one valuation domain and thus $(AR_P)^2 = AR_P$ implies that $A \not\subset P$ and so this result proves to be a generalization of Nagata's Theorem (cf Theo. 10, [15]) which we include as

Theorem 3. If A is a non zero ideal in a Krull domain R then $T(A) = \cap R_P$ where P ranges over minimal prime ideals $P \not\supset A$ of R.

Corollary 2. If A is an ideal in R such that A is contained in no minimal prime ideal of R(a GKD) then $T(A) = R$.

Proof. Since for every minimal prime ideal P of R, $A \not\subset P$, therefore $(AR_P)^2 = AR_P$ and so

$T(A) = \cap R_P = R$ ; because P ranges over all the minimal primes of R.

Corollary 3. If A is a finitely generated ideal in a GKD,R then $T(A) = \cap R_P$ where P ranges over minimal primes $P \not\supset A$.

Proof. Immediate from the fact that if A is finitely generated then so is $AR_P$ and so $T(AR_P) = K$ for every prime $P \not\supset A$.

2. The Property ($\nu$)

According to Gilmer and Huckaba [15] page 207, an integral domain R is said to have property ($\mu$) if for every ideal A in R there exists a finitely generated ideal $A^* \subset A$ such that $T(A) = T(A^*)$ moreover $T(AB) = T(A^*B^*)$ for any pair of ideals A,B of R. Connected with this property they state the following three results:

(1) (Cor. 16, [15]). If A, B and C are ideals of an integral domain D satisfying ($\mu$) ( having the property ($\mu$)) and if $T(A) \supset T(B)$ then $T(AC) \supset T(BC)$.

(2) (COr. 17 [15]). If A and B are ideals of a domain D with Property ($\mu$) and if $T(A) + T(B)$ is the transform of an ideal of D, then $T(AB) = T(A) + T(B)$ .

(3) (COr. 18 [15]) If D is an integral domain with property ($\mu$) then the property $T_2$ holds iff $T_1$ holds.

These results in fact are the tools with the help of which the behaviour of ideals in an integral domain with property ($\mu$) can be examined. As may be verified easily, Theorem 10, and Theorem 12, in [15] imply that a Krull domain has the property ($\mu$). Theorem 12 of [15] being of interest to us is included as

Theorem 4. If A is an ideal of a generalized Krull domain $D$ then there exist $x,y \in A$ such that the ideals A and (x,y) are contained in exactly the same prime ideals of D. If D is a Krull domain then $T(A) = T(x,y)$.

The last statement in Theorem 4, is exactly where we get interested, and start questioning the necessity of the condition that $x,y \in A$, as is imposed upon $x,y$ in Theorem 4. Our reasons for this behaviour being:

(1) We have generalized Theorem 10 of [15] to Theorem 2, for generalized Krull domains, that gives a formula for the transform of a non zero ideal, and the formula is remarkably similar to that provided by Theo. 10 of [15], for Krull domains.

(2) While proving Cors. 16 - 18 in [15], no use has been made of the condition that $A^* \subset A$.

And in view of these reasons and observations we put forward the

Definition 1. An integral domain R will be said to have the property $(\nu)$ if for every ideal A of R there exists a finitely generated ideal $A^*$ such that $T(A) = T(A^*)$, moreover $T(AB) = T(A^*B^*)$ for every pair of ideals A,B of R.

To show that this definition is consistent with the tools used by Gilmer and Huckaba in [15], we prove the

Proposition 5. Let R be an integral domain with property $(\nu)$

(1) If A,B and C are ideals of R and $T(A) \supset T(B)$ then $T(AC) \supset T(BC)$.

(2) If A and B are ideals of R and if $T(A) + T(B)$ is the transform of an ideal then $T(AB) = T(A) + T(B)$.

(3) The property $T_2$ holds in R iff $T_1$ holds.

Proof. (1) Let $A^*$, $B^*$ and $C^*$ be finitely generated ideals of R such that $T(A) = T(A^*)$, $T(B) = T(B^*)$ and $T(C) = T(C^*)$, then by the hypothesis $T(A^*) \supset T(B^*)$ and since $C^*$ is finitely generated $T(A^*C^*) \supset T(B^*C^*)$ (cf (iv) Theo. $O_2$ ) but according to the definition of property $(\nu)$

$$T(AC) = T(A^*C^*) \quad T(B^*C^*) = T(BC).$$

(2) Let $T(A) + T(B) = T(C)$, because of the property ($\nu$) there exist finitely generated $A^*, B^*$ and $C^*$ such that $T(A) = T(A^*)$ etc. and so $T(A^*) + T(B^*) = T(C^*)$. By part 3, of Theo. $O_4$ $T(A) + T(B) = T(A^*) + T(B^*) = T(A^*B^*) = T(AB)$.

(3) follows from (2).

Although Proposition 5, is a mere reproduction of Cors. 16,17,18 of [15] it proves that the property ($\nu$) is a generalization of the property ($\mu$) and provides room for conjecturing analogues of results about domains with property ($\mu$) in case of those with property ($\nu$).

The generalized Krull domains being our immediate concern we state the

Proposition 6. A generalized Krull domain has property($\nu$).

Proof. Let A be an ideal in a GKD R . If $A = (0)$ , it is finitely generated and so we may assume $A \neq (0)$, for general considerations.

Let A be a non zero ideal in R such that $T(A) = R$, then we make a convention that $A^* = R = (1)$ (cf explanation at the end of this section).

Now let A be a non zero ideal in R such that $T(A) \neq R$, then $T(A) = \cap R_P$ where P ranges over all the minimal primes of R for which $(AR_P)^2 = AR_P$ (cf Theorem 2).

Let $S = \{P_1, P_2, \ldots, P_n\}$ be the set of all the prime ideals of R which contain A, and let

$S_1 = \{P_1', P_2', \ldots P_m'\}$ be the set of all those prime ideals for which $(AR_{P_i'})^2 \neq AR_{P_i'}$. Obviously $T(A) = \cap R_P$ where P ranges over all the minimal primes not in $S_1$.

Now consider $B = P_1' \cap P_2' \cap \ldots \cap P_m'$, by Theo. 4, there exist $x, y \in B$, such that $(x,y)$ is contained exactly in $P_i'$ $(i = 1, 2, \ldots, m)$. So that

$$T(x,y) = \bigcap_{P \not\supset (x,y)} R_P = \bigcap_{P \not\in S_1} R_P = T(A)$$

and we can take A* to be (x,y).

Further let A,B be any two ideals in the GKD R, and let A*,B* be the finitely generated ideals such that T(A) = T(A*) and T(B) = T(B*). To show that T(AB) = T(A*B*) we proceed as follows:

T(AB) = $\cap R_P$ where P ranges over all those minimal primes for which $(ABR_P)^2 = ABR_P$. But since $(ABR_P)^2 = ABR_P$ implies that $(AR_P)^2 = AR_P$ and $(BR_P)^2 = BR_P$; P ranges over minimal primes of R for which P $\not\supset$ A* and B* i.e. P $\not\supset$ A*B* while $\cap R_P$ (where P ranges over P $\not\supset$ A*B*) is the trasform of A*B* and to sum up T(AB) = T(A*B*), and a GKD has the property ($\nu$).

Corollary 4. A Prüfer GKD is a $T_1$ domain.

Proof. Let R be a Prüfer GKD, by the above Proposition, R has property ($\nu$) and being a Prüfer domain, R is a $T_2$ domain (cf (ii) Cor. 5 [15]). Thus applying (3) of Proposition 5, the result follows.

Compared to Corollary 13 of [15], we state

Corollary 5. In a GKD R, the following are equivalent:

(1) R is a $T_1$ domain

(2) R is a $T_2$ domain

(3) R is a $T_3$ domain

(4) R is a Prufer GKD.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) follow from the definition of $T_i$ domains (3) $\Rightarrow$ (4) follows from Theorem 11 of [15], while (4) $\Rightarrow$ (1) follows from Cor. 4, above.

In a similar fashion Cor. 14, of [15] can be restated for GUFD's, replacing PID by GUFD Bezout, but a more general result can be brought about with the help of the

Lemma 1. An HCF domain is a Bezout domain iff it is a $T_3$ domain.

Proof. Let R be a Bezout domain, then R is a Prüfer domain which is also an HCF domain, but a Prüfer domain is a $T_2$ and hence a $T_3$ domain. Conversely let R be an HCF $T_3$ domain. The strategy of our proof is to show that R is a Pre-Bezout ring, we recall that an integral domain in which $(x,y) = 1$ implies that $xR + yR = R$ is a Pre-Bezout domain( cf [5]). Once we prove that R is Pre-Bezout ,the result will follow from Proposition 3.2 of [5], which states," A ring R is a Bezout ring iff it is a Pre-Bezout ring and an HCF ring".

So to show that R is a Bezout ring we have to show that any two co-prime elements in R are co-maximal.

Let x,y be two co-prime elements in R, then obviously $(x) : (y) = (x)$ ( ∵ R is an HCF domain ) and since R is a $T_3$ domain also, $T(xy) = T(x) + T(y)$ , which by Theo. $O_5$ is possible only if $xR + yR = R$. Now x,y being arbitrary, the result follows.

The above Lemma enables us to state the

Corollary 6. In an HCF domain R with property $(\nu)$, the following are equivalent:

(1) R is a $T_1$ domain.

(2) R is a $T_2$ domain.

(3) R is a $T_3$ domain.

(4) R is a Bezout domain.

Proof. (1) ⇒ (2) ⇒ (3) obvious, (3) ⇒ (4) by Lemma 1, above and (4) ⇒ (1) follows from the fact that R has property $(\nu)$ and is a $T_2$ domain (being Bezout).

We know that in a GUFD R, each non zero minimal prime ideal P is associated to a prime quantum (cf Ch. 1). In other words each minimal prime P contains an element which belongs to no other minimal prime ideal. So that if $P_1, P_2, \ldots, P_n$ are minimal primes associated to the prime quanta $q_1, q_2, \ldots q_n$, then $q_1 q_2 \ldots q_n$ is an element contained precisely in $P_1, P_2, \ldots, P_n$. This property of the GUFD's gives rise to the

Corollary 7. The transform of every non zero ideal A in a GUFD R, is a localization $R_S$ of R w.r.t. a set S generated by a single element of R.

Proof. Let A be a non zero ideal in a GUFD R, and suppose that $\mathcal{P} = \{\Pi_1, \Pi_2, \ldots, \Pi_n\}$ is the set of all those minimal primes for which $(AR_{\Pi_i})^2 \neq AR_{\Pi_i}$ $(i = 1,2,\ldots,n)$ and let $q_1, q_2, \ldots, q_n$ be the prime quanta contained in $\Pi_1, \Pi_2, \ldots, \Pi_n$ respectively. Then $x = q_1 q_2 \ldots q_n$ is precisely the element for which $(xR_{\Pi_i})^2 \neq xR_{\Pi_i}$ $(i = 1,2,\ldots n)$ and thus $T(A) = T(x)$ where $T(x) = R_S$ ; $S = \{x^i\}_{i=0}^{\infty}$ (cf Theo. $O_8$ ). If on the other hand A is contained in no minimal prime ideal, $T(A) = R$ and so we can choose $x = 1$.

The property $(\nu)$ being at hand we can go still further to state the

Proposition 7. If A is a non zero ideal in an HCF domain with property $(\nu)$ then there exists an element $x \in R$ such that $T(A) = T(x) = R_S$ where $S = \{x^i\}_{i=0}^{\infty}$.

Mainly for our convenience we first state the

Lemma 2. Let R be an HCF domain and let B be an ideal of R generated by $x_1, x_2, \ldots, x_n$, such that $x_i$ have a unit as their highest common factor then $T(B) = R$.

Proof. $T(B) = T(x_1, x_2, \ldots x_n) = \bigcap_{i=1}^{n} T(x_i)$ (cf Theo. $O_6$).

Suppose that $T(B) \neq R$ and let $y = r/s \in T(B)$ , since R is an HCF domain we can assume that $(r,s) = 1$, moreover we can assume that s is not a unit (since $T(B) \neq R$). Now $r/s \in T(B)$ implies that $r/s \in T(x_i)$ for each $i = 1,2,\ldots,n$, that is$(r/s)x_i^{n_i} \in R$ for some $n_i$ (cf definition of the ideal transform) but as $(r,s) = 1$ and R is an HCF domain $s | x_i^{n_i}$ $(i = 1,2,\ldots,n)$ for some $n_i$ (in each case)a contradiction to the fact that $x_1, x_2, \ldots, x_n$ have 1, as their highest common factor and thus the lemma follows.

Proof of the Proposition. Let A be a non zero ideal in R then, by the property ($\nu$) there exists a finitely generated ideal $A^* = (y_1, y_2, \ldots, y_n)$ say such that $T(A) = T(A^*)$. Let d be the highest common factor of $y_1, y_2, \ldots, y_n$, then

$$T(A) = T(A^*) = T(d(y_1', y_2', \ldots, y_n'))$$ where

$y_1', y_2', \ldots, y_n'$ have 1 as their highest common factor, and so $T(y_1', y_2', \ldots y_n') = R$ by the above lemma and

$$T(A) = T(y_1, \ldots y_n) = T(d(y_1', \ldots, y_n')) = T(d) + T(y_1', \ldots, y_n')$$
$$= T(d) = R[1/d] = R_S.$$

Remark 1. Obviously in the presence of Proposition 7, Corollary 7, becomes redundant, we have included it because (i) it shows the extent to which we could go without the property ($\nu$) (ii) it serves as a step towards the more general result i.e. the Proposition 7, and (iii) an analogous result for UFD's is known (cf [3]).

Another explanation that is due is to cover the convention that if $T(A) = R$ then we can assume that $A^* = (1) = R$. Our first reason for this convention is that there is no clash between the convention and the requirements of the definition; that is $T(AB) = T(A^*B^*)$ . Because if $T(A) = R$ , $T(AB) = T(A) + T(B) = T(B) = T(B^*) = T(A^*B^*)$. Thus even if there does exist $A^* \neq R$, there is no harm in replacing $A^*$ by R. Secondly, we need this convention because, by defining the property ($\nu$) we have dropped the condition that $A^* \subset A$ to cover more general domains and as a result we come across certain ideals A for which we cannot choose $A^*$ other than (1) = R. For example: Let P be a minimal prime ideal in a generalized Krull domain R such that $PR_P$ is idempotent,, then $T(P) = \cap R_P$ where P ranges over all the minimal prime ideals of R, that is $T(P) = R$. And obviously there exists no finitely generated $P^* \neq R$ such that $T(P) = T(P^*) = R$.

## 3. Rings and their Principal ideal Transforms.

An important result about the ideal transforms appears in Brewer [2], as Theorem 2.1. For the sake of completeness we include it here as

Theorem 8. Let R be a non quasi local integral domain (domain with more than one maximal ideals) and let $\{x_\alpha\}$ be the collection of non units of R then $R = \bigcap_\alpha T(x_\alpha)$.

Using this theorem as a tool Brewer proved results which can be summed up as the

Theorem 9. Let R be a non quasi-local domain and let U be the set of units of R then

(1) (Cor. 2.3 [2]) R is integrally closed iff T(x) is integrally closed for each $x \in R - U$.

(2) ( Proposition 2.4 [2]) R is a Prüfer ring iff T(x) is Prüfer for each $x \in R - U$.

(3) ( Proposition 2.5 [2]) R is almost Dedekind iff T(x) is almost Dedekind for each $x \in R - U$.

(4) ( Proposition 2.6, [2]) R is a Krull domain iff T(x) is a Krull domain for each $x \in R - U$.

Note . "Non quasi-local domain" sounds awkward but we adopt it as an economic equivalent of, " An integral domain which has more than one maximal ideal" or, " An integral domain which is not quasi local".

It may be observed that the proofs of parts (2), (3) and (4) of Theorem 9, depend upon the selection of maximal ideals or of certain prime ideals which have some property in common (e.g. the property of being minimal in part(4)). So it is possible to push the results stated in Theorem 9, to a greater generality. To illustrate our observation we state

Proposition 10. Let R be a non quasi local domain then R has Krull dimension 1 iff T(x) has Krull dimension less than or equal to 1, for each $x \in R - U$.

Proof. If R has Krull dimension 1 then every localization of R has Krull dimension $\leq 1$, and T(x) being a localization of R, dimension of $T(x) \leq 1$. Conversely let P be a maximal ideal of R. Since R is non quasi local, there exists a non unit $x \in R - P$. Now $T(x) = R_S$ where $S = \{x^i\}_{i=0}^{\infty}$. Clearly $P \cap S = \phi$ and so $PR_S$ is a maximal ideal of $R_S$. But as $R_S$ is of Krull dimension 1, $PR_S$ is also minimal in $R_S$, while by the one-one correspondence between primes in $R_S$ and those primes in R which are disjoint from S, P is minimal in R as well. Thus every maximal ideal in R is minimal also; implying that R has Krull dimension 1.

We recall that an integral domain R is a W-domain if

(1) Every non zero prime ideal of R is maximal.

(2) Every ideal (equivalently every principal ideal) is contained in a finite number of maximal ideals of R(cf [10])

Corollary 8. A non quasi local domain R is a W- domain iff T(x) is a W-domain for every $x \in R - U$.

Proof. Clearly if R is a W-domain, every localization of R is a W-domain and T(x) being a localization of R the necessity follows. Conversely assume that T(x) is a W-domain for each $x \in R-U$, that every prime ideal of R is maximal follows from Proposition 10 above. And so to prove that R is a W-domain, it remains only to show that every element of R is contained in only a finite number of maximal ideals of R.

Let $\{P_\alpha\}$ $\alpha \in I$, be the family of all the maximal ideals of R. Let x be a non unit in R and let $\{P_\beta\}$ be the family of all the maximal ideals of R which contain x. Considering $R - \bigcup_\beta P_\beta$, two possibilities arise:

(a) $R - \bigcup_\beta P_\beta$ contains a non unit

(b) $R - \bigcup_\beta P_\beta$ contains no non unit.

In case (a) holds, let y be a non unit in $R - \bigcup_\beta P_\beta$ and consider $T(y) = R_S$ ; $S = \{y^i\}$ . Clearly $P_\beta \cap S = \phi$ for each $P_\beta \in \{P_\beta\}$. And so $\{P_\beta R_S\}$ is the family of maximal ideals of $R_S$ which contain x, but $T(y) = R_S$ being a W-domain $\{P_\beta R_S\}$ is finite and consequently $\{P_\beta\}$ is finite.

In case (b) it is easy to verify that $\{P_\beta\}$ is the set of all the maximal ideals of R. Now select a maximal ideal P of R and consider R - P. Since R is not quasi-local there exists a non unit z in R - P. And obviously z being not in all the maximal ideals comes under the case (a) and hence is contained in only a finite number of maximal ideals of R. Let $\{P_i\}_{i=1}^n$ be the collection of all the maximal ideals containing z and consider $T(x) = R_S$. Only those maximal ideals P' are lost in approaching from R to $R_S$ for which $P' \cap S \neq \phi$ i.e. of which z is a member. Now x is a non unit in $R_S = T(z)$ and T(z) being a W-domain, x is contained in only a finite number of maximal ideals of $R_S$. Let $\{\Pi_i'\}_{i=1}^m$ be the set of all those maximal ideals of $R_S$ which contain x, then the set of all the maximal ideals of R which contain x is a subset of $\{P_i\}_{i=1}^{n} \cup \{\Pi_j\}_{j=1}^{m}$ where $\Pi_j = \Pi_j' \cap R$.

Generalized Krull domains being our immediate concern, we abstain from probing into the matter too generally and state an analogue for generalized Krull domains of part (4) of Theorem 9, as the

Theorem 11. A non quasi local domain R is a GKD iff T(x) is a GKD for each non unit x of R.

Note . Our proof of this theorem is essentially the same as that of Proposition 2.6 of [2], but we treat it in detail since some changes in the proof are needed.

Proof. Since for every x in R, $T(x) = R_S$ the necessity is obvious. Conversely, let for every non unit x in R, T(x) be a GKD. To show that R is a GKD we have to prove that

(1) $R_P$ is a rank one valuation domain for every non zero minimal prime ideal P in R.

(2) $R = \cap R_P$ where P ranges over all minimal prime ideals of R.

(3) Each non zero non unit of R is contained in only a finite number of minimal prime ideals of R.

We first show that every proper prime ideal of R contains a non zero minimal prime ideal and for every minimal prime (1) holds.

Let P be a non zero prime ideal of R. Since R is non quasi-local, there exists at least one non unit a in R - P. Now $T(a) = R_S$ ; $S = \{x^i\}_{i=0}^{\infty}$ and $P \cap S = \phi$. So PT(a) is a prime ideal in T(a), which is a GKD and hence PT(a) contains a minimal prime ideal of T(a) which implies that $P = PT(a) \cap R$ contains a minimal prime ideal of R.

Now let $\{P_\alpha\}$ be the collection of all the minimal prime ideals of R. Select an arbitrary $P \in \{P_\alpha\}$ and let x be a non unit in R - P. Since $T(x) = R_{S_1}$ ; $S_1 = \{x^i\}$ and $P \cap S_1 = \phi$, $PR_{S_1}$ is a minimal prime ideal in $R_{S_1}$ and so $(R_{S_1})_{PR_{S_1}}$ is a rank one valuation domain (because $R_{S_1}$ is a GKD). But

$$(R_{S_1})_{PR_{S_1}} = R_{(PR_{S_1} \cap R)} = R_P \text{ , that is, for every}$$

minimal prime ideal P of R, $R_P$ is a rank one valuation ring.

Further let $\{\Pi_\delta^{(x)}\}$ be the collection of minimal prime ideals of $R_{S_1} = T(x)$; then $R_{S_1} = \cap (R_{S_1})_{\Pi_\delta^{(x)}}$ but $\Pi_\delta^{(x)} \cap R$ is the minimal prime ideal $P_\delta^{(x)}$ of R, which does not contain x, and so $T(x) = R_{S_1} = \bigcap_\delta R_{P_\delta^{(x)}}$ $(P_\delta^{(x)} = \Pi_\delta^{(x)} \cap R)$.

Now R being a non quasi-local domain

$R = \bigcap_{x \in R-U} T(x) = \bigcap_{x \in R-U} (\bigcap_\delta R_{P_\delta^{(x)}}) = \cap R_P$ where P ranges over all the minimal prime ideals of R.

It can be easily verified that every element of R belongs to at least one minimal prime ideal of R and so we proceed to prove that every non zero non unit element of R is contained in only a finite number of minimal prime ideals of R. We first prove that it is sufficient to show that there exists a non unit x in R which is contained in only a finite number of minimal prime ideals, For let x be contained in a finite number of minimal prime ideals $P_1, P_2, \ldots, P_n$ only. We note that $\{P_i\}_{i=1}^n$ is the only set of minimal primes lost in approaching from R to $T(x) = R_{S_1}$ and that T(x) is a GKD. Now let y be a non zero non unit in R, clearly if y is a unit in T(x) then y divides a power of x and hence it cannot belong to a minimal prime other than occurring in the set $\{ P_i \}_1^n$ and hence is contained in a finite number of minimal primes, if on the other hand, y is non unit in T(x) then y belongs to a finite number of minimal primes $\{ \Pi_j \}_{j=1}^m$ of T(x) and consequently y belongs at most to the members of $\{ \Pi_j \cap R \}_{j=1}^m \cup \{ P_i \}_{i=1}^n$; in other words we have established the fact that every element of R is contained in a finite number of minimal primes of R if one is.

Now let x be an arbitrary non zero non unit in R and let $\{ P_{\beta_x} \}$ be the set of all those minimal primes of R which contain x and consider $X = R - \cup P_{\beta_x}$. Two possibilities arise:

(1) X contains a non unit for some element $x \in R - U$

(2) X = U, the set of units of R for each non zero non unit x of R.

If X contains a non unit z for some x then x is a non unit in T(z) and so the family $\{ P_{\beta_x} T(z) \}$ of minimal primes of T(z) (containing x) is finite (since T(z) is a GKD) and we are through in view of the above observation. To complete the proof assume that for each x the family $\{P_{\beta_x}\}$ of minimal primes containing x is such that (2) above holds. But if (2) holds for an element x then x belongs to every maximal ideal of R, because if M is a maximal ideal such that $x \notin M$ then there exists an element d such that $dx + m = 1$ for some $m \in M$, but as $R - \cup P_{\beta_x} = U$, m belongs to some $P_{\beta_x}$, but x also belongs to $P_{\beta_x}$ and so $1 \in P_{\beta_x}$, a contradiction. So if (2) holds for each non unit x in R, each non unit x in R is contained in each maximal ideal of R, which is absurd in a non quasi-local domain.

Corollary 9. Let R be a non quasi-local domain, then R is a Prüfer GKD iff $T(x)$ is a Prüfer GKD for each non unit $x$ in $R$.

Now we consider the case when an integral domain $R$ is quasi-local. One dimensional quasi-local domains turn out to be interesting enough to be treated separately and are the subject of the following

Proposition 12. In an integral domain $R$ with field of fractions $K \neq R$ the following statements are equivalent:

(1) $R$ is a one dimensional quasi-local domain,

(2) for every pair of non zero non units of $R$ there exist $m$ and $n$ such that $x|y^m$ and $y|x^n$.

(3) for every non unit $x$ in $R$, $T(x) = K$

Proof. (1) $\Leftrightarrow$ (2) can be easily established,

(2) $\Rightarrow$ (3) let $x$ be a non zero non unit in $R$, then $T(x) = R_S$ where $S = \{x^i\}_{i=0}^{\infty}$ but by (2) every non zero non unit of $R$ divides a power of $x$, that is $S = R - \{0\}$ for each non zero non unit $x$ in $R$.

(3) $\Rightarrow$ (2) If $T(x) = R_S = R[1/x] = K$ then obviously every element of $R - \{0\}$ divides some power of $x$ and $x$ being arbitrary the result follows.

We note that for a one dimensional quasi-local domain $R$; $R \neq \cap T(x)$ ( $x$ varying over $R - U$ ). And on the other hand for every GKD $R$ which is not a rank one valuation domain , $R = \bigcap T(x)$ ; $x \in R - U$ . This fact can be verified as follows:

Let $R$ be a generalized Krull domain which is not a valuation domain and let $P$ be a minimal prime ideal in $R$ then there exists a non unit $x$ in $R - P$. But $T(x) = \cap R_P$ $P$ minimal and $x \notin P$. And since for each minimal prime $P$, the above expression holds $\bigcap_{x \in R-U} T(x) = \bigcap_{x} ( \bigcap_{x \notin P} R_P) = \cap R_P$ where

P ranges over all the minimal primes of R ( because for each minimal P, $R_P$ appears in the intersection). Moreover if we impose the condition upon R, that $R = \bigcap_{x \in R-U} T(x)$ , the possibility of R being a one dimensional quasi-local domain is automatically ruled out (cf Proposition 12).

Now to be sure of what criteria can be obtained for a quasi-local domain to be a GKD we state the following

Proposition 13. Let R be an integral domain such that

(1) $R = \cap T(x)$ ; $x \in R - U$

(2) T(x) is a GKD for each non unit x in R.

(3) R contains at least one non unit r which is contained in only a finite number of minimal prime ideals of R, then R is a GKD.

Proof. If R is non quasi-local it is sufficient to assume that (2) holds (cf Theorem 8).

Now let R be a quasi-local domain: (1) implies that R is not one dimensional and hence for each minimal prime P of R there exists a non unit z in R - P and so $(T(z))_{PT(z)}$ is a rank one valuation domain(by (2) above) while $(T(z))_{PT(z)} = R_P$ is obvious, in other words, for every minimal prime P of R, $R_P$ is a rank one valuation domain.

Now $T(z) = R_S$ ; $S = \{z^i\}$

$$= \cap R_{(PT(z) \cap R)} = \cap R_P$$ where P ranges over all the minimal primes of R which do not contain z, and since for each minimal prime P of R, there exists a $z \notin P$ $R = \bigcap_{x \in R-U} T(x) = \bigcap_{x \in R-U} ( \bigcap_{P \not\ni x} R_P) = \cap R_P$ (P ranges over all minimal primes of R). Finally as mentioned in the proof of Theorem 11, (3) implies that every non zero non unit in R is contained in only a finite number of minimal primes of R, and thus we have shown that all three requirements for R to

be a GKD are fulfilled.

Corcrollary 10. A Noetherian domain R is a Krull domain iff (1) $R = \cap T(x)$ ; $x \in R - U$

(2) $T(x)$ is a Krull domain for each $x \in R - U$.

Remarks 2.

(1) Condition (3) in Proposition 13, seems to be redundant but we are unable to prove it.

(2) Local Krull domains are not difficult to find but a quasi local GKD does not seem to have appeared in literature before and so we provide an example as follows:

Example A. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{Q}^+$ the set of positive rationals and construct

$$T = \{ \Sigma r_i x^{\alpha_i} \mid r_i \in \mathbb{R} ; \alpha_i \in \mathbb{Q}^+ \} .$$

It can be easily verified that T is a one dimensional Bezout domain. Let y be an indeterminate over T and let $D = T[y]$. Obviously the elements in D are functions of y and of (some positive rational) powers of x. Let

$$S = \{ f(y,x^{\alpha}) \mid f(0,0) \neq 0 \}.$$

It only needs to be pointed out that $D - S$ is a prime ideal and so $D_S$ is a quasi-local domain. Further since T is a Bezout domain, $T[y] = D$ is an HCF domain and consequently the quasi-local domain $D_S$ is an HCF domain(cf Lemma 9 Ch 1).

Now let a be a non zero non unit of $D_S$. We can write $a = ry + sx^{\alpha}$ ; $r,s \in D_S$. And since $ry + sx^{\alpha}$ is a finite sum we can write $a = y^m x (r'y + s'x^{\beta})$ such that the expression in braces is not divisible by y nor by some positive rational power of x. The factorization of $r'y + s'x^{\beta} = z$, depends upon the highest power of y appearing in the reduced expression for z, and so the number of factors of z is finite i.e. $r'y + s'x^{\beta}$ is a product of atoms and hence of primes ( in an HCF domain every atom is a prime). Since a is arbitrary, we conclude that every element a of $D_S$ can be written as $x^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ where $p_i$ are primes. But $x^{\alpha}$ is a quantutum (cf Def. 1) and because of the HCF property is a prime quantum (cf Lemma 8). Now each prime power being a quantum we conclude that every element in $D_S$ is the product of a finite number of distinct prime quanta which means that $D_S$ is a GUFD and hence a GKD (cf Theo. 12,Ch.1)

(3) We feel that it only needs to be pointed out that the construction in the above example is analogous to that of regular local rings. But we do not know to what extent this quasi-local GUFD or any other domain constructed like this one should behave like a regular local ring.

## 4. Miscellaneous Results.

In the first part of this section we shall establish necessary and sufficient conditions for an ideal A in a Prüfer GKD R to be idempotent, using the ideal transform; where an ideal A is called idempotent if $A^2 = A$. Then we go on to consider semi quasi-local Prüfer GKD's which we shall call $\epsilon$-domains for the sake of brevity. Finally we provide a negative answer to a question left open in [15],p. 210.

To start with we prove the following

Lemma 3. Let R be a completely integrally closed integral domain with quotient field K, and let A to be an idempotent ideal in R the $T(A) = R$.

Proof. Suppose that $T(A) = R_1$, then obviously $R_1 \supset R$. Consider an element x in $R_1$, by the definition of the transform $xA^n \subset R$ for some positive integer n. We observe that (a) R being a ring, $x^m \in R$ for all positive integers m

(b) A being idempotent $A^n = A$, for all n, and from these observations we infer that $x^m A \subset R$ for every positive integer m, but since R is completely integrally closed $x \in R$, while x being an arbitrary element of $R_1$ it follows that $R_1 \subset R$, and hence the lemma.

Proposition 14. An ideal A in a Prüfer GKD R is idempotent iff $T(A) = R$.

Proof. If A is idempotent the ~~result follows~~ result follows from the above lemma; because a GKD is completely integrally closed. For the converse we recall that $T(A) = \cap R_P$ where P ranges over all the minimal prime ideals of R for which $(AR_P)^2 = AR_P$ (Theorem 2 ). But $T(A) = R$ implies that $(AR_P)^2 = AR_P$ for each minimal prime P in R, while each minimal prime in a Prüfer GKD is maximal and so $(AR_P)^2 = A^2R_P = AR_P$ for each maximal ideal P of R, and this precisely means that $A^2 = A$ (cf Proposition 3.13 [24]).

Corollary 11. An ideal A in a Prüfer GKD is idempotent iff it is the intersection of idempotent prime ideals.

Proof. If A is expressible as the intersection of idempotent prime ideals, the result is obvious. For the converse we recall that a Prüfer GKD is a W-domain and so

$A = \Pi_1 \cap \Pi_2 \cap \dots \Pi_n$ where $\Pi_i$ are $P_i$-primary$(i = 1,..n)$.

Now $T(A) = \cap R_P = R$, and so $(AR_P)^2 = AR_P$ that is $(AR_{P_i})^2 = (\Pi_i R_{P_i})^2 = AR_{P_i} = \Pi_i R_{P_i}$ which further implies that $\Pi_i^2 R_{P_i} = \Pi_i R_{P_i} = P_i R_{P_i}$. But $P_i$ being $P_i$-primary $\Pi_i = \Pi_i R_{P_i} \cap R = P_i R_{P_i} \cap R = P_i$ and thus follows the result.

$\epsilon$- domains.

To avoid repetition of too long a name we shall call a <u>Semi quasi-local Prüfer GKD</u> an $\epsilon$-domain.

Before we display one or two results about $\epsilon$-domains, we need to mention that, in an integral domain R the intersection J of maximal ideals of R is called the Jacobson radical of R. It is also helpful to keep in view that if an integral domain R is an intersection of a finite number of valuation rings then R is a Bezout ring (cf [23] Theorem 107) in other words a semi quasi-local Prufer domain is a Bezout domain with a finite number of maximal ideals. And from these observations it follows that an $\epsilon$-domain is a Bezout GUFD. Recalling also that the intersection of all the non zero prime ideals of an integral domain is called its Pseudo radical we state the

Lemma 4.

(1) A GKD R with Pseudo radical Q is an $\epsilon$-domain if and only if $Q \neq 0$.

(2) An $\epsilon$-domain R with Pseudo radical Q is a semi-local PID iff $T(Q) = K$ the field of fractions of R.

(3) In an integral domain R with property ($\nu$) (cf Def.1) the folowing are equivalent:

(a) every overring of R is the transform of a finitely generated ideal .

(b) Every overring of R is the transform of an ideal of R.

Proof. (1) can be verified and (3) is just obvious.

(2) If R is a semi-local PID, let $\{p_i R\}_{i=1}^{n}$ be the set of all the maximal ideals of R then $Q = p_1 p_2 \ldots p_n R$ and so

$$T(Q) = T(p_1 p_2 \ldots p_n) = R[1/p_1 p_2 \ldots p_n] = K .$$

To prove the converse we recall that if $\{P_i\}_{i=1}^{n}$ is the set of all the maximal ideals of R then

$$J = Q = \bigcap_{i=1}^{n} P_i \text{ and } T(Q) = \bigcap_{i=1}^{n} T(QR_{P_i})$$

Now suppose that there exists a maximal ideal $P_m$ say which is idempotent i.e. non principal then

$$T(Q) = \bigcap_{j=1}^{n} T(QR_{P_j}) \cap \bigcap_{i \neq m} T(QR_{P_i}) \cap T(QR_{P_m})$$

Obviously $T(QR_{P_m}) = T(P_m R_{P_m}) = R_{P_m} \neq K$ and thus $T(Q) \neq K$, a contradiction implying that every maximal ideal in R is not idempotent and R being a Bezout GUFD the result follows from Theorem 16, Ch. 1.

We recall from [14] that an integral domain R is said to have property (T) if every overring of R is the transform of an ideal of R, and if every overring of R is the transform of a finitely generated ideal of R then R is said to have the property (FT). Moreover a domain with (FT) is a semi quasi-local Prüfer (that is Bezout) domain. And in connection with the GKD's we collect our observations in the form of

Theorem 15. In a GKD R with the field of fractions $K \neq R$ the following are equivalent :

(1) There exists a non zero non unit element x in R such that $T(x) = K$.

(2) The pseudo radical Q of R is non zero.

(3) R is an $\epsilon$-domain.

(4) R has the property (T).

(5) R has the property (FT).

Proof. (1) $\Rightarrow$ (3); $T(x) = K$ implies that there exixts no minimal prime ideal P of R such that $(xR_P)^2 = xR_P$ i.e. x is contained in every minimal prime ideal of R and because R is a GKD R must have a finite number of minimal primes but this makes R an intersection of a finite number of rank one valuations domains and this obviously makes R an $\epsilon$-domain.

(3) $\Leftrightarrow$ (2) follows from Lemma 4, above.

(3) $\Rightarrow$(5) Being an $\epsilon$-domain R is a Bezout GUFD with only a finite number of minimal( also maximal) prime ideals . The Bezout property implies that every overring $R_1$ of R is a localization of R i.e. $R_1 = R_S$ where S intersects only a finite number of minimal primes of R (because R has only a finite number of minimal primes of its own) and the GUFD property implies that there exists an element x which belongs precisely to those minimal primes which intersect S and thus $R_1 = R_S = T(x)$ .

(5)$\Leftrightarrow$ (4) follows from (3) of Lemma 4,. And completing the cycle (5) $\Rightarrow$(1): An integral domain with property (FT) is a semi quasi-local Prüfer and so R being a GKD also has a finite number of maximal ideals which implies that there exists an element x in R which is contained in each maximal ideal of R showing that $T(x) = K$.

Gilmer and Huckaba left a question open in [15] p. 210, which can be stated as follows," If A and B are ideals of a Krull domain D contained in no common minimal prime ideals does $T(AB) = T(A) + T(B)$ imply that $A + B = D$?"

Our answer to this question is,"Not necessarily". For suppose that R is a Krull domain which is not a Dedekind domain and let A,B be two ideals of R such that

$$T(AB) = T(A) + T(B) \text{ and } A + B = R.$$ Since R is not a Dedekind domain there exists a maximal ideal M which is not minimal, further $A + B = R$

$$T(ABM) = T(AM) + T(BM) \quad \text{(cf (2) Theo. O2)}$$

Now obviously AM and BM are contained in no common minimal primes and $T((AM)(BM)) = T(ABM) = T(AM) + T(BM)$

but $(AM) + (BM) \neq R$.

The above explanation of the answer is rather unconventional but it provides us with the

Theorem 16. A GKD R which is not a field is a Prüfer GKD iff for all ideals A,B of R contained in no common minimal primes $T(AB) = T(A) + T(B)$ implies that $A + B = R$.

Proof. Let R be a Prüfer GKD and A,B be two ideals which are contained in no common minimal primes then

(1) $T(AB) = T(A) + T(B)$ follows from the fact that a Prüfer GKD is a $T_1$ domain (cf Cor. 4)

(2) $A + B = R$, follows from the observation that if $A + B$ is contained in a prime ideal P then $A \subset P$ and $B \subset P$, and since A,B are contained in no common minimal prime ideal $A + B$ is contained in no minimal prime ideal . But since every non zero prime ideal in a Prüfer GKD is maximal, $A + B$ is contained in no maximal ideal that is $A + B = R$.

Conversely let R be a GKD in which the given condition holds and let M be a maximal ideal in R which is not minimal. Select a non zero non unit x in R and consider the transform of (xM). Since M is contained in no minimal prime ideal of R, $T(M) = R$ (cf Cor. 2) and the requirement that xR and M should be contained in no common minimal prime ideal is satisfied. Moreover $T(xM) = T(x) + T(M)$ (cf (f) Prop. $O_3$) so that for any non zero non unit x of R, $xR + M = R$, that is if $x \in M$ even then $xR + M = R$, a contradiction, establishing that M is also minimal. Since M is arbitrary, every maximal ideal of the GKD R is minimal i.e. every non zero prime of R is maximal and by Lemma 18 Ch. 1, R is a Prüfer GKD.

Remarks 3. A careful study of [15] reveals that most of the interesting results stem from an effort to study the conditions under which a pair of ideals A,B satisfies the transform formula i.e. T(AB) = T(A) + T(B). Obviously if A,B satisfy the transform formula then T(A) + T(B) is a ring. The conjecture that if T(A) + T(B) is a ring then T(A) + T(B) = T(AB) is not correct, and part (vii) of Corollary 23, [15] ensures the existence of the case where T(A) + T(B) is an overring of the integral domain R but T(AB) ≠ T(A) + T(B). It is natural to ask that if T(A) + T(B) is an overring of R, under what conditions T(A) + T(B) = T(AB) ? The answer is the following simple

Statement A. Let A and B be two ideals in an integral domain R such that T(A) + T(B) is a ring then T(A) + T(B) = T(AB) iff T(AB) = T(A)T(B).

Proof. Since T(A) + T(B) is a ring T(A)T(B) ⊂ T(A) + T(B) so that T(AB) = T(A)T(B) ⊂ T(A) + T(B) ⊂ T(AB) (cf (d) Prop.$O_1$). Conversely T(AB) = T(A) + T(B) implies that T(A) + T(B) is a ring and so T(A) + T(B) = T(A)T(B) and hence T(AB) = T(A)T(B).

According to (iii) Theorem $O_2$, if an ideal A is invertible then T(AB) = T(A)T(B) for any other ideal B, applying this result directly to the Dedekind domains we find that T(AB) = T(A)T(B) for every pair of ideals A,B in a Dedekind domain. And generally

Statement B. For every pair of ideals A,B of a $T_1$ domain R, T(AB) = T(A)T(B).

Proof. It is easy to verify that if T(A) + T(B) is an overring then T(A)T(B) = T(A) + T(B) and since R is a $T_1$ domain, the statement follows.

The above observations lead to the integral domains R in which $T(AB) = T(A)T(B)$ for every pair of ideals A,B of R, we shall call these integral domains, <u>T' domains</u> . The T' property is not very strong as we shall see presently and so we content ourselves with the one or two results worth mentioning:

Statement C. A T' domain R is a $T_1$ domain iff $T(A) + T(B)$ is an overring of R for every pair A,B of ideals of R.

The proof is obvious.

Statement D. An HCF domain R with property $(\nu)$ is a T' domain.

Proof. By the HCF and the $(\nu)$ properties, for every ideal A of R there exists an element $a \in R$ such that $T(A) = T(a)$. So that $T(AB) = T(ab) = T(a)T(b)$ ; because every principal ideal is invertible, and consequently $T(AB) = T(A)T(B)$.

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