THE RING $D + XD_S[X]$ AND t-SPLITTING SETS

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الخلاصـة :

نسمّي فئة S، في حلقة تامة D، فئة قاسمة بواسطة عملية نجمية محدودة النوع ★ إذا كانت S مغلقة تحت الضرب ولكل $d \neq 0$ في D تجد مثاليتين A و B في D بالصفات التالية:

١ – $(d) = (AB)^\star$

٢ – $A^\star \cap sD = sA^\star$ لكل s في S

٣ – $B^\star \cap S \neq \emptyset$.

نثبت في هذا البحث أنّ الحلقة $D + XDs[X]$ – من التركيبات الخلفية – تكون من نوع PVMD (أو GGCD) إذا وإذا فقط كانت D من نوع PVMD (أو GGCD) وكانت S فئة قاسمة بواسطة العملية النجمية "t" (أو العملية النجمية "d"). يضم البحث أيضاً نتائج عديدة تتمحور حول مفهوم الفئة القاسمة بواسطة عملية نجمية محدودة النوع ★.

ABSTRACT

Let D be an integral domain, S a multiplicatively closed subset of D, and ★ a finite character star-operation on D. We say that S is a ★-splitting set if for each $0 \neq d \in D$, there exist integral ideals A and B of D with $(d) = (AB)^\star$, where $A^\star \cap sD = sA^\star$ for all $s \in S$ and $B^\star \cap S \neq \emptyset$. We show that $D^{(S)} = D + XD_S[X]$ is a PVMD (resp., GGCD domain) if and only if D is a PVMD (resp., GGCD domain) and S is a t-splitting (resp., d-splitting) subset of D. Let S be a t-splitting set of D and let $\mathcal{T} = \{A_1 \cdots A_n |$ each $A_i = d_i D_S \cap D$ for some nonzero $d_i \in D\}$. Then $D = D_S \cap D_{\mathcal{T}}$. We relate the t-operation on D to the t-operation on D_S and $D_{\mathcal{T}}$.

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1. INTRODUCTION

Throughout this paper, D denotes an integral domain with quotient field K, $D^* = D - \{0\}$, and $U(D)$ is the group of units of D. As usual, $X^{(1)} = X^{(1)}(D)$ is the set of height-one prime ideals of D. For a multiplicatively closed subset S of D, let $D^{(S)} = D + XD_S[X] \subseteq K[X]$. The $D^{(S)}$ construction has proved useful in constructing examples, see [1].

A saturated multiplicatively closed subset S of D is said to be a *splitting set* if for each $d \in D^*$ we can write $d = sa$ for some $s \in S$ and $a \in D$ with $s'D \cap aD = s'aD$ for all $s' \in S$. The set $T = \{t \in D^* \mid sD \cap tD = stD$ for all $s \in S\}$ is also a splitting set, called the *m-complement of* S. Each $d \in D^*$ has a unique representation (up to unit factors) $d = st$, where $s \in S$ and $t \in T$. If $d = st$ $(s \in S, t \in T)$, then $dD_S \cap D = tD$. In fact, a saturated multiplicatively closed subset S of D is a splitting set if and only if $dD_S \cap D$ is principal for each $d \in D^*$. For these, and other, results on splitting sets, see [2]. Splitting sets are investigated further in [3].

In [1, Theorem 1.1], it was shown that $D^{(S)}$ is a GCD domain if and only if D is a GCD domain and GCD(d, X) exists in $D^{(S)}$ (equivalently, $(d, X)_t$ is principal in $D^{(S)}$) for each $d \in D^*$; while in [4, Corollary1.5], it was shown that $D^{(S)}$ is a GCD domain if and only if D is a GCD domain and (the saturation of) S is a splitting set.

In Section 2 of this paper, we introduce the notion of a t-splitting set. We say that a multiplicatively closed subset S of D is a *t-splitting set* if for each $d \in D^*$, $(d) = (AB)_t$ for some integral ideals A and B of D, where $A_t \cap sD = sA_t$ (or equivalently, $(A, s)_t = D$) for all $s \in S$ and $B_t \cap S \neq \varnothing$. Here, as usual, the *$t$-operation* is the star-operation given by $A \to A_t = \bigcup \{(a_1, \ldots, a_n)_v \mid a_1, \ldots, a_n \in A - \{0\}\}$ and $(a_1, \ldots, a_n)_v = \left((a_1, \ldots, a_n)^{-1}\right)^{-1}$. Also, recall that a nonzero fractional ideal I of D is a *t-ideal* if $I = I_t$ and that I is *t-invertible* if there exists a fractional ideal J with $(IJ)_t = D$. Note that $dD_S \cap D$ is a t-ideal for any $d \in D^*$ and multiplicatively closed subset S of D. Clearly a splitting set is a t-splitting set.

The integral domain D is called a *Prüfer v-multiplication domain* (PVMD) if each nonzero finitely generated ideal of D is *t-invertible*, or equivalently, if for each maximal prime t-ideal P of D, D_P is a valuation domain. We show that S is a t-splitting set if and only if $dD_S \cap D$ is a t-invertible t-ideal for each $d \in D^*$, or equivalently, (d, X) is a t-invertible ideal of $D^{(S)}$ for each $d \in D^*$. The main result of this paper, Theorem 2.5, is that $D^{(S)}$ is a PVMD if and only if D is a PVMD and (d, X) is t-invertible in $D^{(S)}$ for each $d \in D^*$, if and only if D is a PVMD and S is a t-splitting set of D.

Let D be an integral domain. The set of t-invertible fractional t-ideals forms an abelian group under the t-product $A * B = (AB)_t$. The *t-class group of* D is $\mathrm{Cl}_t(D)$, the abelian group of t-invertible fractional t-ideals of D modulo its subgroup of principal fractional ideals. For D a Krull domain, $\mathrm{Cl}_t(D) = \mathrm{Cl}(D)$, the divisor class group of D; while for D a Prüfer domain, $\mathrm{Cl}_t(D) = \mathrm{Pic}(D)$, the ideal class group of D. Recall that D is a *generalized GCD domain* (*GGCD domain*) if the intersection of two nonzero principal ideals of D is invertible or, equivalently, if each finite type v-ideal of D is invertible [5]. If D is a PVMD, then $\mathrm{Cl}_t(D) = 0$ (resp., $\mathrm{Cl}_t(D) = \mathrm{Pic}(D)$) if and only if D is a GCD (resp., GGCD) domain [6, Corollary 1.5] (resp., [6, Corollary 2.3]). For more on the t-class group, see [7].

Let $\star$ be a finite character star-operation on D. In Section 3, we generalize the notion of a t-splitting set to a $\star$ -splitting set. A multiplicatively closed subset S of D is a *$\star$-splitting set* if for each $d \in D^*$, $(d) = (AB)^\star$, where $A^\star \cap sD = sA^\star$ for each $s \in S$ and $B^\star \cap S \neq \varnothing$. In the case where $\star$ is the d-operation ($A_d = A$), S is a d-splitting set if and only if $dD_S \cap D$ is invertible for each $d \in D^*$, or equivalently, $(d, X)_t$ is an invertible ideal of $D^{(S)}$ for each $d \in D^*$. We show (Theorem 3.3) that $D^{(S)}$ is a GGCD domain if and only if D is a GGCD domain and $(d, X)_t$ is invertible in $D^{(S)}$ for each $d \in D^*$, if and only if D is a GGCD domain and S is a d-splitting set.

Thus $D^{(S)}$ is a PVMD (resp., GCD domain, GGCD domain) if and only if D is a PVMD (resp., GCD domain, GGCD domain) and $(d, X)_t$ is t-invertible (resp., principal, invertible) in $D^{(S)}$ for all $d \in D^*$. In Theorem 3.6, we show that $D^{(S)}$ is a Prüfer (resp., Bezout) domain if and only if D is a Prüfer (resp., Bezout) domain and (d, X) is invertible (resp., principal) in $D^{(S)}$ for all $d \in D^*$.

If S is a t-splitting set of D, then the set t-Max(D) of maximal t-ideals of D is the union of two disjoint subsets, $F = \{P \in t\text{- Max}(D) \mid P \cap S = \varnothing\}$ and $G = \{P \in t\text{-Max}(D) \mid P \cap S \neq \varnothing\}$, and D is "split" as $D = D_1 \cap D_2$, where $D_1 = \bigcap_{P \in F} D_P$ and $D_2 = \bigcap_{P \in G} D_P$. These decompositions are studied in Section 4. We show that $D_1 = D_S$ and $D_2 = D_{\mathcal{T}}$ is a generalized quotient ring of D, where $\mathcal{T} = \{A_1 \cdots A_n \mid \text{each } A_i = d_i D_S \cap D \text{ for some } d_i \in D^*\}$. We also show that if A is a nonzero integral ideal of D, then $A_t = \big((AD_S)_t \cap D\big) \cap \big((AD_{\mathcal{T}})_t \cap D\big) = \big(((AD_S)_t \cap D)((AD_{\mathcal{T}})_t \cap D)\big)_t$ and $A_t D_S = (AD_S)_t$.

The notation and terminology used in this paper are standard and may be found in [8]. Also see [8] for an introduction to star-operations. For results on PVMD's, see [9] and [10], and for results on GGCD domains, see [5]. For results on t-ideals and t-invertibility, and for star operations in general, the reader is referred to [7], [11], [12], or [13].

2. t-SPLITTING SETS AND PVMD's

Let D be an integral domain with quotient field K and let S be a multiplicatively closed subset of D. We say that $d \in D^*$ is *t-split by S* if $(d) = (AB)_t$ for integral ideals A and B of D, where $A_t \cap sD = sA_t$ (or equivalently, $(A, s)_t = D$, *cf.* Theorem 4.4) for all $s \in S$ and $B_t \cap S \neq \varnothing$. Note that A and B are both t-invertible. So by replacing A by A_t and B by B_t, we can assume that A and B are t-invertible t-ideals and $B \cap S \neq \varnothing$. We say that S is a *t-splitting set* if each $d \in D^*$ is t-split by S. Note that S is a t-splitting set if and only if the saturation $\bar{S}$ of S is a t-splitting set.

Lemma 2.1. *Suppose that D is an integral domain and S is a multiplicatively closed subset of D. Suppose that $d \in D^*$ is t-split by S. Thus $(d) = (AB)_t$, where A and B are integral ideals of D with $A_t \cap sD = sA_t$ for all $s \in S$ and $B_t \cap S \neq \varnothing$. Then $A_t = dD_S \cap D$, and hence $dD_S \cap D$ is a t-invertible t-ideal. Also, $B_t = dA^{-1}$.*

Proof. Since $A_t B_t \subseteq (AB)_t \subseteq (d)$, $A_t \subseteq A_t D_S \cap D = A_t B_{tS} D_S \cap D = A_t B_t D_S \cap D \subseteq dD_S \cap D$. Let $x \in dD_S \cap D$. Then there exists $s \in S$ with $sx \in (d)$. Thus $sx \in A_t \cap sD = sA_t$, and hence $x \in A_t$. Thus $dD_S \cap D = A_t$ is a t-invertible t-ideal. Also, $dA^{-1} = (dA^{-1})_t = (ABA^{-1})_t = (AA^{-1}B)_t = B_t$. □

Lemma 2.2. *Suppose that D is an integral domain and S is a multiplicatively closed subset of D. Let $d \in D^*$ such that $dD_S \cap D$ is t-invertible. Then d is t-split by S.*

Proof. Let $A = dD_S \cap D$. Hence A is a t-ideal. Since $(d) \subseteq A$, $B = dA^{-1}$ is an integral t-invertible t-ideal of D and $(d) = (AB)_t$. Now $B_S = (dA^{-1})_S = dD_S(A^{-1})_S = dD_S(AD_S)^{-1} = dD_S(dD_S)^{-1} = D_S$. (Here $(A^{-1})_S = (AD_S)^{-1}$ follows from [14, Lemma 4].) Hence $B \cap S \neq \varnothing$. We next show that $A \cap sD = sA$. Clearly $sA \subseteq A \cap sD$. Let $z \in A \cap sD$. Then $z = sb$ for some $b \in D$. Thus $b = z/s \in A_S \cap D = dD_S \cap D = A$. So $z \in sA$, and thus $A \cap sD = sA$. □

Corollary 2.3. *Suppose that D is an integral domain and S is a multiplicatively closed subset of D. Then $d \in D^*$ is t-split by S if and only if $dD_S \cap D$ is t-invertible. Moreover, if $(d) = (AB)_t$, where $A_t \cap sD = sA_t$ for all $s \in S$ and $B_t \cap S \neq \varnothing$, then $A_t = dD_S \cap D$ and $B_t = dA^{-1}$. Hence S is a t-splitting set if and only if $dD_S \cap D$ is t-invertible for all $d \in D^*$.*

Lemma 2.4. *Let D be an integral domain and S a multiplicatively closed subset of D. Then $d \in D^*$ is t-split by S if and only if (d, X) is t-invertible in $D^{(S)} = D + XD_S[X]$.*

Proof.

($\Leftarrow$) Suppose that (d,X) is t-invertible in $D^{(S)}$. Then $(d,X)_tD_S[X] = \big((d,X)_tD^{(S)}\big)_S = \big(((d,X)D^{(S)})_S\big)_t = \big((d,X)D_S[X]\big)_t$, where the second equality follows from the fact that $C_tD_S = (CD_S)_t$ for any t-invertible ideal C of D. This may be seen by combining [6, Lemma 2.9], which says that $C_tD_S = (C_tD_S)_t$ for any t-invertible t-ideal C_t, with [15, Lemma 3.9], which states that $(C_tD_S)_t = (CD_S)_t$ for any nonzero ideal C of D. But $\big((d,X)D_S[X]\big)_t = D_S[X]$, so $(d,X)_tD_S[X] = D_S[X]$, and hence $(d,X)_t \cap S \neq \varnothing$. Thus $(d,X)_t \supseteq XD_S[X]$, and hence $(d,X)_t = B + XD_S[X]$ for some ideal B of D (with $B \cap S \neq \varnothing$). Now $B + XD_S[X] = BD^{(S)}$; so B is a t-invertible t-ideal of D by [16, Proposition 3.9]. We show that $dB^{-1} = dD_S \cap D$, which gives that $dD_S \cap D$ is a t-invertible t-ideal. Thus by Lemma 2.2, d is t-split by S. Let $0 \neq x \in dD_S \cap D$; so $x = d\frac{r}{s}$, where $r \in D$ and $s \in S$. Then $\frac{r}{s}(B+XD_S[X]) = \frac{r}{s}(d,X)_t = (d\frac{r}{s}, \frac{r}{s}X)_t \subseteq D^{(S)}$. Thus $\frac{r}{s}B \subseteq D$, and hence $\frac{r}{s} \in B^{-1}$. So $x \in dB^{-1}$, and hence $dD_S \cap D \subseteq dB^{-1}$. Now $d \in B$, so $dB^{-1} \subseteq D$. Also, $D^{(S)} \supseteq B^{-1}BD^{(S)} = B^{-1}(B + XD_S[X]) = B^{-1}B + XB^{-1}D_S[X]$; so $B^{-1} \subseteq B^{-1}D_S \subseteq D_S$ (or note that $dB^{-1} \subseteq dB^{-1}D_S = d(BD_S)^{-1} = dD_S$, with the first equality following from [14, Lemma 4]). Thus $dB^{-1} \subseteq dD_S$; so $dB^{-1} \subseteq dD_S \cap D$.

($\Rightarrow$) Suppose that d is t-split by S. Then $(d) = \big((dD_S \cap D)B\big)_t$, where B is a t-invertible t-ideal of D with $B \cap S \neq \varnothing$. Again by [16, Proposition 3.9], $BD^{(S)} = B + XD_S[X]$ is a t-invertible t-ideal of $D^{(S)}$. Now $(d,X) \subseteq BD^{(S)}$, so $(d,X)_t \subseteq BD^{(S)}$. We need to show that $BD^{(S)} \subseteq (X,d)_t$, or $(X,d)^{-1} \subseteq (BD^{(S)})^{-1} = B^{-1}D^{(S)} = B^{-1} + XD_S[X]$ (the first equality follows from [17, Lemma 3.2] and the second since $B \cap S \neq \varnothing$). Let $\frac{f}{h} \in (X,d)^{-1}$, where $f,h \in D[X]$ with f and h having no common factor in $D[X]$ of degree ≥ 1. First suppose that $h(0) = 0$, so $h = Xh'$. Then $\frac{f}{Xh'}d \in D^{(S)}$ implies $f(0) = 0$. But then $X|f$, contrary to our assumption that f and h have no common factor in $D[X]$ of degree ≥ 1. Thus we can assume $h(0) \neq 0$. Now $\frac{f}{h}X = a_0 + \frac{a_1}{s}X + \cdots + \frac{a_n}{s}X^n$, where each $a_i \in D$ and $s \in S$. Hence $fX = \big(a_0 + \frac{a_1}{s}X + \cdots + \frac{a_n}{s}X^n\big)h$. Since $h(0) \neq 0$, $a_0 = 0$. So $\frac{f}{h}X = \frac{a_1}{s}X + \cdots + \frac{a_n}{s}X^n$, and hence $\frac{f}{h} = \frac{a_1}{s} + \frac{a_2}{s}X + \cdots + \frac{a_n}{s}X^{n-1} := g(x) \in D_S[X]$. Now $dg(x) \in D^{(S)}$, so $d\frac{a_1}{s} = dg(0) \in dD_S \cap D$, and hence $\frac{a_1}{s} \in d^{-1}(dD_S \cap D) = B^{-1}$. Thus $\frac{f}{h} = g \in B^{-1}D^{(S)}$. □

Theorem 2.5. *Let D be an integral domain and S a multiplicatively closed subset of D. Then the following statements are equivalent.*

1. *$D^{(S)}$ is a PVMD.*
2. *D is a PVMD and (d,X) is t-invertible in $D^{(S)}$ for each $d \in D^*$.*
3. *D is a PVMD and S is a t-splitting set.*
4. *D is a PVMD and, for each prime t-ideal P of D with $P \cap S = \varnothing$, there is a t-invertible t-ideal $A \subseteq P$ with $A \cap sD = sA$ for all $s \in S$.*

Proof.

(1) $\Rightarrow$ (2) Suppose that $D^{(S)}$ is a PVMD. Let B be a nonzero finitely generated ideal of D. Then $(BD^{(S)})^{-1}$ is a t-invertible t-ideal of $D^{(S)}$. Since $B^{-1}D^{(S)} = (BD^{(S)})^{-1}$ [17, Lemma 3.1], $B^{-1}D^{(S)}$ is a t-invertible t-ideal of $D^{(S)}$, and hence B^{-1} is a t-invertible t-ideal of D by [16, Proposition 3.9]. Thus B is t-invertible. Hence D is a PVMD. The second statement is clear.

(2) $\Rightarrow$ (3) Lemma 2.4.

(3) $\Rightarrow$ (1) Let P be a prime t-ideal of $D^{(S)}$. To show that $D^{(S)}$ is a PVMD, it suffices to show that $D^{(S)}{}_P$ is a valuation domain. If $P \cap D = 0$, then $D^{(S)}{}_P$ is a quotient ring of $K[X]$, and hence is a DVR. Thus we may assume $P \cap D \neq 0$.

We first note that $p = P \cap D$ is a prime t-ideal of D. Let $d_1, \ldots, d_n$ be nonzero elements of p. Since D is a PVMD, $(d_1, \ldots, d_n)^{-1}$ is a finite type v-ideal of D, and hence by [17, Lemma 3.2], $\big(((d_1, \ldots, d_n)^{-1})D^{(S)}\big)^{-1} = (d_1, \ldots, d_n)_v D^{(S)}$. But $(d_1, \ldots, d_n)^{-1} D^{(S)} = \big((d_1, \ldots, d_n)D^{(S)}\big)^{-1}$ by [17, Lemma 3.1]. Thus:

$$(d_1, \ldots, d_n)_v D^{(S)} = \big((d_1, \ldots, d_n)D^{(S)}\big)_v \subseteq P.$$

Hence $(d_1, \ldots, d_n)_v \subseteq P \cap D = p$. So p is a prime t-ideal of D, and hence D_p is a valuation domain. Alternatively, by [16, Lemma 3.6] $D \subset D^{(S)}$ is a flat extension. Then $p \neq 0$ gives $p_t \neq D$. But since D is a PVMD and in a PVMD a nonzero prime ideal contained in a maximal t-ideal is again a prime t-ideal, p is a prime t-ideal.

Suppose $P \cap S \neq \varnothing$. Then ${D^{(S)}}_{D-p} = D_p + XD_{S(D-p)}[X] = D_p + XK[X]$ is a Bezout domain by [1, Corollary 4.13]. Here the fact that $D_{S(D-p)} = K$ follows from Lemma 4.2. Thus ${D^{(S)}}_P$ is a localization of the Bezout domain $D_p + XK[X]$, and hence is a valuation domain.

Thus we may assume $P \cap S = \varnothing$. Suppose $P \cap D[X]$ is not a t-ideal of $D[X]$. Since $D[X]$ is a PVMD, $(P \cap D[X])_t = D[X]$. Hence there exists $f(X) \in P \cap D[X]$ with $(A_f)_v = D$ [18, Lemma 10]. (Here A_f is the content of f.) Let $0 \neq d \in p$. Since S is a t-splitting set, $(d) = (AB)_t$, where we can take A to be finitely generated and $B \cap S \neq \varnothing$. Then $AB \subseteq p$ and $B \not\subseteq p$ imply $A \subseteq p$, and hence $A \subseteq P$. Thus $(f(X), A)D^{(S)} \subseteq P$. Since P is a t-ideal, $\big((f(X), A)D^{(S)}\big)_v \subseteq P$. Hence there exist $\alpha, \beta \in D[X]$ with $\alpha \nmid \beta$ in $D^{(S)}$ such that $(f(X), A)D^{(S)} \subseteq \frac{\alpha}{\beta} D^{(S)}$. Now $AD^{(S)} \subseteq \frac{\alpha}{\beta} D^{(S)}$, so we can take $\alpha \in D$. Since $\beta f(X) D^{(S)} \subseteq \alpha D^{(S)}$, there exists $s \in S$ with $\beta s f(X) \in \alpha D[X]$. Thus $sA_{\beta f} \subseteq \alpha D$, and hence $sA_\beta \subseteq s(A_\beta)_t = s(A_\beta A_f)_t = s(A_{\beta f})_t \subseteq \alpha D$. Thus $s\beta \in \alpha D[X]$, and hence $s\beta \in \alpha D^{(S)}$. Now $(A, s)_t = D$, so $\big((A, s)D^{(S)}\big)_t = D^{(S)}$. Hence $\beta(A, s)D^{(S)} \subseteq \alpha D^{(S)}$ gives $\beta \in \beta\big((A, s)D^{(S)}\big)_t \subseteq \alpha D^{(S)}$; so $\alpha | \beta$ in $D^{(S)}$, a contradiction. Thus $P \cap D[X]$ is a prime t-ideal of $D[X]$. Since $D[X]$ is a PVMD, $(P \cap D[X])D_S[X]$ is a prime t-ideal of the PVMD $D_S[X]$. By [1, Lemma 2.3], $(P \cap D[X])D_S[X] = PD_S[X]$. Hence ${D^{(S)}}_P = ({D^{(S)}}_S)_{P_S} = D_S[X]_{PD_S[X]}$ is a valuation domain.

(3) $\Rightarrow$ (4) Suppose that P is a prime t-ideal of D with $P \cap S = \varnothing$. Let $0 \neq d \in P$. Thus $(d) = (AB)_t$, where $(A, s)_t = D$ for every $s \in S$ and $B_t \cap S \neq \varnothing$. Then $A_t B_t \subseteq (d) \subseteq P$ and $B_t \cap S \neq \varnothing$ gives $B_t \not\subseteq P$, and hence $A_t \subseteq P$. Thus A_t is the desired t-invertible t-ideal.

(4) $\Rightarrow$ (1) If A is the ideal given by (4), then $A = F_t$ for some finitely generated ideal $F \subseteq A$. Now just replace the ideal A in the proof of (3) $\Rightarrow$ (1) by the ideal F. The proof of (3) $\Rightarrow$ (1) carries through verbatim. □

Theorem 2.5 allows us to recover the two characterizations of when $D^{(S)}$ is a GCD domain mentioned in the Introduction.

Corollary 2.6. *Let D be an integral domain and S a multiplicatively closed subset of D with saturation $\bar{S}$. Then the following statements are equivalent.*

(1) *$D^{(S)}$ is a GCD domain.*
(2) *D is a GCD domain and $(d, X)_t$ is principal for each $d \in D^*$.*
(3) *D is a GCD domain and $\bar{S}$ is a splitting set.*

Proof. The equivalence of statements (1) and (2) follows directly from Theorem 2.5 since $\mathrm{Cl}_t(D^{(S)})$ is isomorphic to $\mathrm{Cl}_t(D)$ when D is integrally closed [16, Corollary 4.5] and a PVMD is a GCD domain if and only if it has trivial t-class group [6, Corollary 1.5]. The equivalence of (1) and (3) follows as above since in a GCD domain a saturated multiplicatively closed set is a splitting set if and only if it is a t-splitting set. □

An integral domain D is said to be *weakly factorial* [19] if every nonzero nonunit of D is a finite product of primary elements. In [20], it is shown that the following conditions are equivalent for an integral domain D: (1) D is weakly factorial; (2) $D - P$ is a splitting set for each prime ideal P of D; and (3) each saturated multiplicatively closed subset S of D is a splitting set. Since a Krull domain is weakly factorial if and only if it is factorial [19, Theorem 15], a Krull domain D has every saturated multiplicatively closed subset of D a splitting set if and only if D is factorial (or, it is easily checked that for a height-one prime ideal P of a Krull domain D, $S = D - P$ is a splitting set if and only if P is principal). Also, it follows from [19, Theorems 18 and 19] that the following conditions on an integral domain D are equivalent: (1) D is a PVMD and every saturated multiplicatively closed subset of D is a splitting set (*i.e.*, D is weakly factorial); (2) D is a weakly factorial GCD domain; (3) D is a weakly factorial generalized Krull domain; and (4) D is a generalized Krull domain and a GCD domain. (Recall that an integral domain D is a *generalized Krull domain* if $D = \bigcap_{P \in X^{(1)}} D_P$, where the intersection is locally finite and each D_P is a valuation domain. It is easily checked that a generalized Krull domain is a PVMD.) The integral domains satisfying the equivalent conditions (1)–(4) are the focus of study of [21], where they are called *generalized unique factorization domains* (GUFD's) (see [21, page 402] for the formal definition of a GUFD). A number of other characterizations of GUFD's may be found in [21, Theorem 10].

An integral domain D is *weakly Krull* if $D = \bigcap_{P \in X^{(1)}} D_P$, where the intersection is locally finite, or equivalently, if every nonzero proper principal ideal of D is a t-product of primary ideals (necessarily of height-one) [22, Theorem 3.1]. Let D be weakly Krull and let S be a multiplicatively closed subset of D. Let d be a nonzero nonunit of D. Then $(d) = (Q_1 \cdots Q_n)_t$, where each Q_i is P_i-primary with $P_i \in X^{(1)}$. Let $A = \Pi\{Q_i \mid Q_i \cap S = \varnothing\}$ and $B = \Pi\{Q_i \mid Q_i \cap S \neq \varnothing\}$. Hence $(d) = (AB)_t$, where $(A, s)_t = D$ for each $s \in S$ (for each P_i is a maximal t-ideal of D) and $B_t \cap S \neq \varnothing$. Hence S is a t-splitting set. Thus, *via* [22, Theorem 3.1] and the above remarks, we have that the following conditions are equivalent for an integral domain D: (1) D is weakly Krull; (2) $S = D - P$ is a t-splitting set for each prime ideal P of D minimal over a principal ideal; and (3) each (saturated) multiplicatively closed subset of D is a t-splitting set. It is easily checked that the following two conditions are equivalent for an integral domain D: (1) D is a PVMD and weakly Krull; and (2) D is a generalized Krull domain. Hence we have the following corollary to Theorem 2.5.

Corollary 2.7. *Let D be a generalized Krull domain and S a multiplicatively closed subset of D. Then $D^{(S)} = D + XD_S[X]$ is a PVMD. Moreover, $\mathrm{Cl}_t(D^{(S)}) \cong \mathrm{Cl}_t(D)$.*

Proof. The second statement follows from [16, Corollary 4.5] since a generalized Krull domain is integrally closed. □

3. ⋆-SPLITTING SETS

Let D be an integral domain, S a multiplicatively closed subset of D, and $\star$ a finite character star-operation on D. We say that $d \in D^*$ is *$\star$-split by S* if $(d) = (AB)^\star$, where A and B are integral ideals of D with $A^\star \cap sD = sA^\star$ for all $s \in S$ and $B^\star \cap S \neq \varnothing$. Note that A and B are both $\star$-invertible. By replacing A by $A^\star$ and B by $B^\star$, we can assume that A and B are $\star$-invertible $\star$-ideals of D and that $B \cap S \neq \varnothing$. If A is a $\star$-invertible ideal with $(A, s)^\star = D$, then using Theorem 4.4 it is easy to show that $A^\star \cap sD = sA^\star$. If A is t-invertible with $A_t \cap sD = sA_t$, then $(A, s)_t = D$. However, for an arbitrary finite character star-operation $\star$, we may have $A^\star \cap sD = sA^\star$, but $(A, s)^\star \neq D$. (For example, let $\star$ be the d-operation on $D = K[X, Y]$, K a field, and take $A = (X)$ and $s = Y$.) We say that S is a *$\star$-splitting set* if each $d \in D^*$ is $\star$-split by S. It is easy to check that S is a $\star$-splitting set if and only if the saturation $\bar{S}$ of S is a $\star$-splitting set. When $\star = t$, we have the t-splitting sets considered in Section 2. Also, S is a d-splitting set if for each $d \in D^*, (d) = AB$, where A and B are integral ideals of D, necessarily invertible, with $A \cap sD = sA$ for all $s \in S$ and $B \cap S \neq \varnothing$. Certainly a splitting set is a d-splitting set; in fact, a splitting set is a $\star$-splitting set for any finite character star-operation $\star$. Now each multiplicative subset of a Krull domain (resp., Dedekind domain) is a t-splitting (resp., d -splitting) set. Also, each saturated multiplicative subset of a Krull domain (resp., Dedekind domain) D is a splitting set if and only if D is a UFD (resp., PID). Thus, if P is a non-principal prime ideal of a Dedekind domain D, then $S = D - P$ is a d-splitting set that is not a splitting set. It is easily checked that Lemmas 2.1 and 2.2 (and hence Corollary 2.3) hold for any finite character star-operation. We state this as the first result of this section.

Proposition 3.1. *Let D be an integral domain, S a multiplicatively closed subset of D, and $\star$ a finite character star-operation on D. Then $d \in D^*$ is $\star$-split by S if and only if $dD_S \cap D$ is $\star$-invertible. Moreover, if $(d) = (AB)^\star$, where $A^\star \cap sD = sA^\star$ for all $s \in S$ and $B^\star \cap S \neq \varnothing$, then $A^\star = dD_S \cap D$ and $B^\star = dA^{-1}$. Hence S is a $\star$-splitting set if and only if $dD_S \cap D$ is $\star$-invertible for all $d \in D^*$.*

We next give the d-splitting analog of Lemma 2.4.

Proposition 3.2. *Let D be an integral domain and S a multiplicatively closed subset of D. Then $d \in D^*$ is d-split by S if and only if $(d, X)_t$ is an invertible ideal of $D^{(S)} = D + XD_S[X]$.*

Proof.

($\Leftarrow$) Suppose that $(d, X)_t$ is invertible in $D^{(S)}$. As in the proof of Lemma 2.4, we get that $(d, X)_t = B + XD_S[X]$ for some ideal B of D with $B \cap S \neq \varnothing$. Let $0 \neq b \in B$; so $bD + XD_S[X] \subseteq B + XD_S[X]$. Since $B + XD_S[X]$ is invertible, $bD + XD_S[X] = C(B + XD_S[X])$ for some ideal C of $D^{(S)}$. Passing to $D = D^{(S)}/XD_S[X]$, we have $bD = \bar{C}B$. Hence B is a factor of a principal ideal, and thus is invertible. As in the proof of Lemma 2.4, $dD_S \cap D = dB^{-1}$ is an invertible ideal of D.

($\Rightarrow$) This direction is an easy modification of the proof of Lemma 2.4 ($\Rightarrow$). ☐

Our next theorem is the d-splitting analog of Theorem 2.5.

Theorem 3.3. *Let D be an integral domain and S a multiplicatively closed subset of D. Then the following statements are equivalent.*

(1) *$D^{(S)}$ is a GGCD domain.*

(2) *D is a GGCD domain and $(d, X)_t$ is an invertible ideal of $D^{(S)}$ for each $d \in D^*$.*

(3) *D is a GGCD domain and S is a d-splitting set.*

Proof.

(1) $\Rightarrow$ (2) Suppose that $D^{(S)}$ is a GGCD domain. Let B be a nonzero finitely generated ideal of D. Then $B^{-1}D^{(S)} = (BD^{(S)})^{-1}$ is invertible. Choose $0 \neq r \in D$ so that rB^{-1} is an integral ideal of D. Then $(rB^{-1})D^{(S)}$ is an integral invertible ideal of $D^{(S)}$. Hence rB^{-1} is an invertible ideal of D, and so $B_t = r(rB^{-1})^{-1}$ is an invertible ideal of D. Thus D is a GGCD domain. The second statement is clear.

(2) $\Rightarrow$ (3) Proposition 3.2.

(3) $\Rightarrow$ (1) It suffices to show that for all nonzero $a, b \in D^{(S)}$, $aD^{(S)} \cap bD^{(S)}$ is invertible. Since D is also a PVMD and S is a t-splitting set, $D^{(S)}$ is a PVMD by Theorem 2.5. Since $aD^{(S)} \cap bD^{(S)}$ has finite type, to show that $aD^{(S)} \cap bD^{(S)}$ is invertible, it suffices to show that $aD^{(S)}{}_{\mathcal{M}} \cap bD^{(S)}{}_{\mathcal{M}} = (aD^{(S)} \cap bD^{(S)})_{\mathcal{M}}$ is principal in $D^{(S)}{}_{\mathcal{M}}$ for each maximal ideal $\mathcal{M}$ of $D^{(S)}$. And to do this, it is enough to show that $D^{(S)}{}_{\mathcal{M}}$ is a GCD domain for each maximal ideal $\mathcal{M}$ of $D^{(S)}$. First, suppose that $\mathcal{M} \cap S = \varnothing$. Then $D^{(S)}{}_{\mathcal{M}} = D_S[X]_{\mathcal{M}_S}$. But D_S is a GGCD domain, and thus so is $D_S[X]$ [5, Theorem 2]. Hence $D_S[X]_{\mathcal{M}_S}$ is a GCD domain. Next, suppose that

$\mathcal{M} \cap S \neq \varnothing$. Then $\mathcal{M} = M + XD_S[X]$, where M is a maximal ideal of D. Now $D^{(S)}{}_{\mathcal{M}}$ is a localization of $D_M + X(D_M)_S[X] = (D_M)^{(S)}$. Here D_M is a GCD domain. We show that $\bar{S}$, the saturation of S in D_M, is a splitting set of D_M. Thus by the result [4, Corollary 1.5] mentioned in the Introduction, $(D_M)^{(S)}$ is a GCD domain, and hence $D^{(S)}{}_{\mathcal{M}}$ is a GCD domain. Since S is a d-splitting set, $dD_S \cap D$ is invertible for each $d \in D^*$, and thus $d(D_M)_S \cap D_M = (dD_S \cap D)_M$ is principal. Hence $x(D_M)_{\bar{S}} \cap D_M$ is principal for each $x \in (D_M)^*$, and thus $\bar{S}$ is a splitting set in D_M. □

Remark 3.4. We sketch another proof of Theorem 3.3. Note that a PVMD D is a GGCD domain if and only if $\mathrm{Pic}(D) = \mathrm{Cl}_t(D)$. Also, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(D) & \longrightarrow & \mathrm{Pic}(D^{(S)}) \\ \downarrow & & \downarrow \\ \mathrm{Cl}_t(D) & \longrightarrow & \mathrm{Cl}_t(D^{(S)}) \end{array}$$

where the vertical maps are inclusions and the horizontal maps are isomorphisms, when D is integrally closed ([23, page 113] and [16, Corollary 4.5]). Suppose that $D^{(S)}$ is a GGCD domain. Then $D^{(S)}$ is a PVMD with $\mathrm{Pic}(D^{(S)}) = \mathrm{Cl}_t(D^{(S)})$. Thus D is a PVMD with $\mathrm{Pic}(D) = \mathrm{Cl}_t(D)$. Hence D is a GGCD domain. Since S is a t-splitting set and each t-invertible t-ideal of D is invertible, S is a d-splitting set. Conversely, suppose that D is a GGCD domain and S is a d-splitting set. Then S is also a t-splitting set and D is a PVMD. So $D^{(S)}$ is a PVMD with $\mathrm{Cl}_t(D^{(S)}) = \mathrm{Cl}_t(D) = \mathrm{Pic}(D) = \mathrm{Pic}(D^{(S)})$. Hence $D^{(S)}$ is a GGCD domain.

We end this section with the d-analogs of our previous results (*i.e.*, considering $(r, X)_d$ rather than $(r, X)_t$).

Lemma 3.5. *Let D be an integral domain and S a multiplicatively closed subset of D. Then the following statements are equivalent for $d \in D^*$.*

(1) *(d, X) is principal in $D^{(S)}$.*
(2) *(d, X) is invertible in $D^{(S)}$.*
(3) *$d \in \bar{S}$, where $\bar{S}$ is the saturation of S.*

Proof.

(1) ⇒ (2) Clear.

(2) ⇒ (3) Since $(d, X)^{-1} = d^{-1}D^{(S)} \cap X^{-1}D^{(S)} = (\frac{1}{d}D \cap D_S) + XD_S[X]$, we have that (d, X) is invertible in $D^{(S)}$ if and only if $d(\frac{1}{d}D \cap D_S) = D$, if and only if $\frac{1}{d} \in D_S$, if and only if $d \in \bar{S}$.

(3) ⇒ (1) $d \in \bar{S} \Rightarrow (d, X) = dD^{(S)}$. □

Theorem 3.6. *Let D be an integral domain with quotient field K and S a multiplicatively closed subset of D. Then the following statements are equivalent.*

(1) *$D^{(S)}$ is a Prüfer (resp., Bezout) domain.*
(2) *D is a Prüfer (resp., Bezout) domain and (d, X) is invertible (resp., principal) in $D^{(S)}$ for all $d \in D^*$.*
(3) *D is a Prüfer (resp., Bezout) domain and $D_S = K$.*

Proof.

(1) ⇒ (2) Clear.

(2) ⇒ (3) Lemma 3.5.

(3) ⇒ (1) The Prüfer domain case is from [1, Corollary 4.15]; while the Bezout domain case is from [1, Corollary 4.13]. □

Note that $D^{(S)}$ is a Dedekind domain if and only if $D(=K)$ is a field, and in this case, $D^{(S)} = K[X]$ is a PID.

4. *t*-SPLITTING D

Suppose that D is an integral domain and S is a t-splitting set in D . Let $\mathcal{T} = \{A_1 \cdots A_n \mid \text{each } A_i = d_i D_S \cap D$ for some $d_i \in D^*\}$ and $D_{\mathcal{T}} = \{x \in K | xA \subseteq D \text{ for some } A \in \mathcal{T}\}$. Note that $D = D_S \cap D_{\mathcal{T}}$. As the containment $\subseteq$ is clear, let $x \in D_S \cap D_{\mathcal{T}}$. Then there exists $s \in S$ with $sx \in D$ and ideals $A_1, \ldots, A_n$ with each $A_i = d_i D_S \cap D$ such that $xA_1 \cdots A_n \subseteq D$. Now each $(A_i, s)_t = D$, and hence $(A_1 \cdots A_n, s)_t = D$. Thus $xD = x(A_1 \cdots A_n, s)_t = (xA_1 \cdots A_n, sx)_t \subseteq D$, and hence $x \in D$. We next give a complement to Lemma 2.1.

Lemma 4.1. *Let D be an integral domain, S a t-splitting set of D, and $d \in D^*$. Suppose that $(d) = (AB)_t$, where A and B are integral ideals with $A_t \cap sD = sA_t$ for all $s \in S$ and $B_t \cap S \neq \varnothing$. Then*

$$B_t = d(dD_S \cap D)^{-1} = dD_{\mathcal{T}} \cap D.$$

Moreover,

$$(d) = (dD_S \cap D) \cap (dD_{\mathcal{T}} \cap D) = \big((dD_S \cap D)(dD_{\mathcal{T}} \cap D)\big)_t.$$

Proof. By Lemma 2.1, $A_t = dD_S \cap D$ and $B_t = dA^{-1}$. So just take $A = dD_S \cap D$. Then $A^{-1}A \subseteq D$; so $A^{-1} \subseteq D_{\mathcal{T}}$, and hence $dA^{-1} \subseteq dD_{\mathcal{T}}$. Also, $dA^{-1} \subseteq D$, so $B_t = dA^{-1} \subseteq dD_{\mathcal{T}} \cap D$. Now $(d) = d(D_S \cap D_{\mathcal{T}}) = dD_S \cap dD_{\mathcal{T}} = (dD_S \cap D) \cap (dD_{\mathcal{T}} \cap D) \supseteq (dD_S \cap D)(dD_{\mathcal{T}} \cap D) \supseteq dAA^{-1}$. Thus $(d) \supseteq \big((dD_S \cap D)(dD_{\mathcal{T}} \cap D)\big)_t \supseteq (dAA^{-1})_t = (d)$. Hence $(d) = \big((dD_S \cap D)(dD_{\mathcal{T}} \cap D)\big)_t$. Then $(dD_{\mathcal{T}} \cap D)_t = dA^{-1} = B_t$, and hence $B_t = dD_{\mathcal{T}} \cap D$. □

Let D be an integral domain and S a multiplicatively closed subset of D . Recall that a prime ideal Q of D with $Q \cap S \neq \varnothing$ is said to *intersect S in detail* if $P \cap S \neq \varnothing$ for each nonzero prime ideal $P \subseteq Q$.

Lemma 4.2. *Let D be an integral domain and let S be a t-splitting set of D. Let Q be a prime t-ideal of D with $Q \cap S \neq \varnothing$. Then Q intersects S in detail.*

Proof. Let $0 \neq P \subseteq Q$ be a prime ideal of D. Let $0 \neq x \in P$. Then we can shrink P to a prime ideal minimal over (x) which is necessarily a t-ideal. Thus we can assume that P is a t-ideal. Assume $P \cap S = \varnothing$. Let $0 \neq x \in P$, so $(x) = (AB)_t$, where $B_t \cap S \neq \varnothing$ and $(A, s)_t = D$ for each $s \in S$. Now $A_t B_t \subseteq (AB)_t = (x) \subseteq P$ and $B_t \not\subseteq P$ since $B_t \cap S \neq \varnothing$. Thus $A_t \subseteq P$. Suppose $s \in Q \cap S$. Then $D = (A, s)_t \subseteq Q$, a contradiction. □

Let D be an integral domain and S a t-splitting set of D. Let $F = \{P \in t\text{-Max}(D) \mid P \cap S = \varnothing\}$ and $G = \{P \in t\text{-Max}(D) \mid P \cap S \neq \varnothing\} = \{P \in t\text{-Max}(D) \mid P \text{ intersects } S \text{ in detail}\}$. Note that t-Max(D) is the disjoint union of F and G and that for $P \in F$ and $Q \in G$, $P \cap Q$ contains no nonzero prime ideal since Q intersects S in detail. However, such a splitting of t-Max(D) does not force S to be a t-splitting set, even when $D_{\mathcal{T}}$ is a quotient ring of D. For let E be the ring of entire functions and let S be the multiplicatively closed set generated by the principal primes of E. Then since E is Bezout, $F = \{P \in t\text{-Max}(D) \mid P \cap S = \varnothing\} = \{P \in \text{Max}(D) \mid \text{ht}\, P > 1\}$ and $G = \{P \in t\text{-Max}(D) \mid P \cap S \neq \varnothing\} = \{P \in \text{Max}(D) \mid \text{ht}\, P = 1\}$. Clearly $\text{Max}(D) = t\text{-Max}(D)$ is the disjoint union of F and G, and for $P \in F$ and $Q \in G$, $P \cap Q$ contains no nonzero prime ideal. But S is not a t-splitting set by Theorem 2.5 since $E^{(S)}$ is not a PVMD [4, Example 2.6]. Also, note that $E_S\big(\bigcap_{Q\in G} E_Q\big) \neq K$, the quotient field of E, since $\bigcap_{Q\in G} E_Q = E$ (see Theorem 4.3(3)).

Theorem 4.3. *Let D be an integral domain with quotient field K and S a t-splitting set of D. Then*

(1) $D_S = \bigcap_{P\in F} D_P$*, where* $F = \{P \in t\text{-Max}(D) \mid P \cap S = \varnothing\}$*,*

(2) $D_{\mathcal{T}} = \bigcap_{P\in G} D_P$*, where* $G = \{P \in t\text{- Max}(D) \mid P$ *intersects* S *in detail*$\}$*, and*

(3) $D_S D_{\mathcal{T}} = K$*.*

Proof.

(1) Now $P \cap S = \varnothing$ gives $D_S \subseteq D_P$, and hence $D_S \subseteq \bigcap_{P\in F} D_P$. Let $x \in \bigcap_{P\in F} D_P$. Then $(D : x) \not\subseteq P$ for each $P \in F$. Suppose $(D : x) \cap S = \varnothing$. Then there is a prime t-ideal Q with $(D : x) \subseteq Q$ and $Q \cap S = \varnothing$. Enlarge Q to a maximal t-ideal P. Since $P \supseteq (D : x)$, $P \in G$, and thus $P \cap S \neq \varnothing$. Hence $Q \cap S \neq \varnothing$ by Lemma 4.2, a contradiction. Thus $(D : x) \cap S \neq \varnothing$, and hence $x \in D_S$.

(2) Let $x \in D_{\mathcal{T}}$; so $xA_1 \cdots A_n \subseteq D$, where each $A_i = d_i D_S \cap D$. Let $P \in G$. We show that $(D : x) \not\subseteq P$, and hence $x \in D_P$. Assume $(D : x) \subseteq P$ for some $P \in G$. Now $A_1 \cdots A_n \subseteq P$ implies some $A_i \subseteq P$. Let $s \in P \cap S$. Then $D = (A_i, s)_t \subseteq P$, a contradiction. Conversely, let $\frac{a}{b} \in \bigcap_{P\in G} D_P$. Then $\frac{a}{b}(bD_S \cap D) \subseteq \bigcap_{P\in G} D_P$ and $\frac{a}{b}(bD_S \cap D) = aD_S \cap \frac{a}{b} D \subseteq D_S$. Thus $\frac{a}{b}(bD_S \cap D) \subseteq D_S \cap \big(\bigcap_{P\in G} D_P\big) = \bigcap_{P\in F} D_P \cap \bigcap_{P\in G} D_P = D$. Hence $\frac{a}{b} \in D_{\mathcal{T}}$.

(3) It suffices to show that $dD_S D_{\mathcal{T}} = D_S D_{\mathcal{T}}$ for each $d \in D^*$. Write $(d) = (AB)_t$, where $(A, s)_t = D$ for each $s \in S$ and $B \cap S \neq \varnothing$. Note that for each $Q \in G$, $A \not\subseteq Q$, and hence $AD_Q = D_Q$. Now $dD_S D_{\mathcal{T}} = D_S\big(\bigcap_{Q\in G} dD_Q\big) = D_S\big(\bigcap_{Q\in G} (AB)_t D_Q\big) \supseteq D_S\big(\bigcap_{Q\in G} ABD_Q\big) = D_S\big(\bigcap_{Q\in G} BD_Q\big) \supseteq D_S\big(B\big(\bigcap_{Q\in G} D_Q\big)\big) = BD_S D_{\mathcal{T}} = D_S D_{\mathcal{T}} \supseteq dD_S D_{\mathcal{T}}$, and hence we have equality. □

Suppose that D is an integral domain and S is a splitting set of D with m-complement T. Let $T(D)$ be the monoid of fractional t-ideals of D with the t-product $A * B = (AB)_t$ and ordered by reverse inclusion. So $T_t(D) = \{A \in T(D) \mid 0 \leq A\} = \{A \in T(D) \mid A \subseteq D\}$. In [2, Theorem 3.7], we showed that the map $\theta : T(D) \to T(D_S) \times_c T(D_T)$ (cardinal product, *i.e.*, the direct product with the order $(A, B) \leq (C, D) \Leftrightarrow A \leq C$ and $B \leq D$), given by $\theta(A) = (AD_S, AD_T)$, is a monoid order-isomorphism. Moreover, for $A \in T(D)$, A is integral (resp., principal, of finite type, t-invertible) if and only if both AD_S and AD_T are integral (resp., principal, of finite type, t-invertible). Thus by [2, Corollary 3.8], the map θ induces the group isomorphism $\bar{\theta} : \text{Cl}_t(D) \to \text{Cl}_t(D_S) \times \text{Cl}_t(D_T)$, given by $\bar{\theta}([A]) = ([AD_S], [AD_T])$. The proof of [2, Theorem 3.7] is based on the following decomposition of a nonzero integral ideal A of D. Let $A = (\{a_\alpha\})$, where each $a_\alpha \neq 0$. Write $a_\alpha = s_\alpha t_\alpha$, where $s_\alpha \in S$ and $t_\alpha \in T$. Then $(AD_S)_t \cap D = (\{t_\alpha\})_t$, $(AD_T)_t \cap D = (\{s_\alpha\})_t$, and $A_t = \big((\{s_\alpha\})(\{t_\alpha\})\big)_t = (\{s_\alpha\})_t \cap (\{t_\alpha\})_t = (AD_S)_t \cap (AD_T)_t = \big((AD_S)_t \cap D\big) \cap \big((AD_T)_t \cap D\big) = \big(((AD_S)_t \cap D)((AD_T)_t \cap D)\big)_t$. Moreover, $A_t D_S = (AD_S)_t$ (resp., $A_t D_T = (AD_T)_t$) is a t-ideal of D_S (resp., D_T); so the localization of a t-ideal is again a t-ideal. Our next goal is to partially extend these results to t-splitting sets. A key tool is the next theorem which is of independent interest.

Theorem 4.4. *Let $\star$ be a star-operation on an integral domain D and let A be a nonzero fractional ideal of D. Then the following statements are equivalent.*

(1) *A is $\star$-invertible.*

(2) $\big(A(\bigcap_\alpha B_\alpha)\big)^\star = \big(\bigcap_\alpha AB_\alpha\big)^\star$ *for each nonempty collection* $\{B_\alpha\}$ *of fractional (or just integral) ideals of* D *with* $\bigcap_\alpha B_\alpha \neq 0$.

(3) $\big(A(\bigcap_\alpha B_\alpha^\star)\big)^\star = \bigcap_\alpha (AB_\alpha)^\star$ *for each nonempty collection* $\{B_\alpha\}$ *of fractional (or just integral) ideals of* D *with* $\bigcap_\alpha B_\alpha \neq 0$.

Proof. The proof is almost identical to the proof of [24, Theorem 1]. □

Lemma 4.5. *Let D be an integral domain, S a t-splitting set of D, and $d_1, \dots, d_n \in D^*$. Write each* $(d_i) = (A_iB_i)_t$, *where* $A_i = d_iD_S \cap D$ *and* $B_i = d_iA_i^{-1}$. *Then* $(A_1B_1, \dots, A_nB_n)_v = \big((A_1 + \cdots + A_n)(B_1 + \cdots + B_n)\big)_v$.

Proof. Put $A = A_1 \cdots A_n$, $\tilde{A}_i = A_1 \cdots \hat{A}_i \cdots A_n$, $B = B_1 \cdots B_n$, and $\tilde{B}_i = B_1 \cdots \hat{B}_i \cdots B_n$. Now since each $B_j \cap S \neq \varnothing$, $(A_i, B_j)_t = D$ for each $1 \leq i, j \leq n$. Hence $(\tilde{A}_i, \tilde{B}_j)_t = D$, and so $(\tilde{A}_i\tilde{B}_j)_t = \tilde{A}_{it} \cap \tilde{B}_{jt}$. Then using Theorem 4.4, $(A_1B_1 + \cdots + A_nB_n)^{-1} = (A_1B_1)^{-1} \cap \cdots \cap (A_nB_n)^{-1} = (A_1^{-1}B_1^{-1})_t \cap \cdots \cap (A_n^{-1}B_n^{-1})_t = \big((A^{-1}B^{-1})(\tilde{A}_1\tilde{B}_1)\big)_t \cap \cdots \cap \big((A^{-1}B^{-1})(\tilde{A}_n\tilde{B}_n)\big)_t = \big((A^{-1}B^{-1})((\tilde{A}_1\tilde{B}_1)_t \cap \cdots \cap (\tilde{A}_n\tilde{B}_n)_t)\big)_t = \big((A^{-1}B^{-1})(\tilde{A}_{1t} \cap \tilde{B}_{1t} \cap \cdots \cap \tilde{A}_{nt} \cap \tilde{B}_{nt})\big)_t$. Likewise, $\big((A_1 + \cdots + A_n)(B_1 + \cdots + B_n)\big)^{-1} = \bigcap_{1 \leq i,j \leq n} (A_iB_j)^{-1} = \bigcap_{1 \leq i,j \leq n} (A_i^{-1}B_j^{-1})_t = \big((A^{-1}B^{-1})\ (\bigcap_{1 \leq i,j \leq n} (\tilde{A}_i\tilde{B}_j)_t)\big)_t = \big((A^{-1}B^{-1})(\bigcap_{1 \leq i,j \leq n} (\tilde{A}_{it} \cap \tilde{B}_{jt}))\big)_t = \big((A^{-1}B^{-1})(\tilde{A}_{1t} \cap \tilde{B}_{1t} \cap \cdots \cap \tilde{A}_{nt} \cap \tilde{B}_{nt})\big)_t$. Hence $(A_1B_1 + \cdots + A_nB_n)^{-1} = \big((A_1 + \cdots + A_n)(B_1 + \cdots + B_n)\big)^{-1}$, and thus $(A_1B_1, \dots, A_nB_n)_v = \big((A_1 + \cdots + A_n)(B_1 + \cdots + B_n)\big)_v$. □

Lemma 4.6. *Let D be an integral domain, S a t-splitting set of D, and let $A = (\{a_\alpha\})$ be an integral ideal of D, where each $a_\alpha \neq 0$. For each α, let* $(a_\alpha) = (A_\alpha B_\alpha)_t$, *where* $A_\alpha = a_\alpha D_S \cap D$ *and* $B_\alpha = a_\alpha A_\alpha^{-1}$. *Then* $A_t = \big((\sum A_\alpha)(\sum B_\alpha)\big)_t$.

Proof. Now $A_t = (\{a_\alpha\})_t = (\{(A_\alpha B_\alpha)_t\})_t = (\sum A_\alpha B_\alpha)_t \subseteq \big((\sum A_\alpha)(\sum B_\alpha)\big)_t$. Let $0 \neq x \in \big((\sum A_\alpha)(\sum B_\alpha)\big)_t$; so there exist $\alpha_1, \dots, \alpha_n$ with $x \in \big((\sum_{i=1}^n A_{\alpha_i})(\sum_{i=1}^n B_{\alpha_i})\big)_v$. Then by Lemma 4.5, $x \in (\sum_{i=1}^n A_{\alpha_i}B_{\alpha_i})_v \subseteq \big(\sum_{i=1}^n (A_{\alpha_i}B_{\alpha_i})_t\big)_v = \big(\sum_{i=1}^n (a_{\alpha_i})\big)_v \subseteq A_t$. □

Lemma 4.7. *Let D be an integral domain, S a t-splitting set of D, and let $A = (\{a_\alpha\})$ be an integral ideal of D, where for each α,* $0 \neq (a_\alpha) = (A_\alpha B_\alpha)_t$, *with* $A_\alpha = a_\alpha D_S \cap D$ *and* $B_\alpha = a_\alpha A_\alpha^{-1}$. *Then* $(AD_S)_t \cap D = (\sum A_\alpha)_t$. *Hence* $\big((\sum A_\alpha)D_S\big)_t \cap D = (\sum A_\alpha)_t$ *and* $(\sum A_\alpha)_t D_S = \big((\sum A_\alpha)D_S\big)_t$.

Proof. Now $(\sum A_\alpha)_t \subseteq (\sum A_\alpha)_t D_S \cap D \subseteq \big((\sum A_\alpha)D_S\big)_t \cap D = \big((\sum A_\alpha B_\alpha)D_S\big)_t \cap D = \big((\sum A_\alpha B_\alpha)_t D_S\big)_t \cap D = (A_tD_S)_t \cap D = (AD_S)_t \cap D$. Let $0 \neq x \in (AD_S)_t \cap D$; so $x \in \big((\sum A_\alpha)D_S\big)_t$, and hence $x \in \big((A_{\alpha_1} + \cdots + A_{\alpha_n})D_S\big)_v \cap D$ for some finite subset $\{\alpha_1, \dots, \alpha_n\}$. Let $(x) = (A_0B_0)_t$ be the "canonical t-splitting"; so $A_0 \subseteq \big((A_{\alpha_1} + \cdots + A_{\alpha_n})D_S\big)_v$. Thus $A_0\big((A_{\alpha_1} + \cdots + A_{\alpha_n})^{-1}D_S\big) = A_0\big((A_{\alpha_1} + \cdots + A_{\alpha_n})D_S\big)^{-1} \subseteq D_S$, and hence $A_0(A_{\alpha_1}^{-1}D_S \cap \cdots \cap A_{\alpha_n}^{-1}D_S) \subseteq D_S$. So $A_0A_{\alpha_1} \cdots A_{\alpha_n}(A_{\alpha_1}^{-1}D_S \cap \cdots \cap A_{\alpha_n}^{-1}D_S) \subseteq A_{\alpha_1} \cdots A_{\alpha_n}D_S$, and hence $\big(A_{\alpha_1} \cdots A_{\alpha_n}A_0(A_{\alpha_1}^{-1}D_S \cap \cdots \cap A_{\alpha_n}^{-1}D_S)\big)_t \subseteq (A_{\alpha_1} \cdots A_{\alpha_n}D_S)_t = (A_{\alpha_1} \cdots A_{\alpha_n})_tD_S$. Thus by Theorem 4.4 (since $A_0A_{\alpha_1} \cdots A_{\alpha_n}$ is t-invertible), $(A_0A_{\alpha_2} \cdots A_{\alpha_n})_tD_S \cap \cdots \cap (A_0A_{\alpha_1} \cdots A_{\alpha_{n-1}})_tD_S \subseteq (A_{\alpha_1} \cdots A_{\alpha_n})_tD_S$. So $\big((A_0A_{\alpha_2} \cdots A_{\alpha_n})_tD_S \cap D\big) \cap \cdots \cap \big((A_0A_{\alpha_1} \cdots A_{\alpha_{n-1}})_tD_S \cap D\big) \subseteq (A_{\alpha_1} \cdots A_{\alpha_n})_tD_S \cap D$. But since $(xa_{\alpha_2} \cdots a_{\alpha_n}) = \big((A_0A_{\alpha_2} \cdots A_{\alpha_n})_t(B_0B_{\alpha_2} \cdots B_{\alpha_n})_t\big)_t$ is the "canonical t-splitting", we get $(A_0A_{\alpha_2} \cdots A_{\alpha_n})_tD_S \cap D = (A_0A_{\alpha_2} \cdots A_{\alpha_n}D_S)_t \cap D = (A_0A_{\alpha_2} \cdots A_{\alpha_n}B_0B_{\alpha_2} \cdots B_{\alpha_n}D_S)_t \cap D = (A_0A_{\alpha_2} \cdots A_{\alpha_n}B_0B_{\alpha_2} \cdots B_{\alpha_n})_tD_S \cap D = xa_{\alpha_2} \cdots a_{\alpha_n}D_S \cap D = (A_0A_{\alpha_2} \cdots A_{\alpha_n})_t$. So $(A_0A_{\alpha_2} \cdots A_{\alpha_n})_t \cap \cdots \cap (A_0A_{\alpha_1} \cdots A_{\alpha_{n-1}})_t \subseteq (A_{\alpha_1} \cdots A_{\alpha_n})_t$. Multiplying both sides by $(A_{\alpha_1} \cdots A_{\alpha_n})^{-1}$ and using Theorem 4.4 gives $D \supseteq A_0(A_{\alpha_1}^{-1} \cap \cdots \cap A_{\alpha_n}^{-1}) = A_0(A_{\alpha_1} + \cdots + A_{\alpha_n})^{-1}$. Thus $A_0 \subseteq (A_{\alpha_1} + \cdots + A_{\alpha_n})_v = (A_{\alpha_1} + \cdots + A_{\alpha_n})_t$, and hence $x \in (A_0B_0)_t \subseteq (A_{\alpha_1} + \cdots + A_{\alpha_n})_t$. □

Lemma 4.8. *Let D be an integral domain, S a t-splitting set of D, and let A be a nonzero ideal of D. Suppose $A = (\{a_\alpha\})$, where for each α, $0 \neq (a_\alpha) = (A_\alpha B_\alpha)_t$ with $A_\alpha = a_\alpha D_S \cap D$ and $B_\alpha = a_\alpha A_\alpha^{-1}$. Then $A_t = \big((\sum A_\alpha)(\sum B_\alpha)\big)_t = (\sum A_\alpha)_t \cap (\sum B_\alpha)_t$.*

Proof. By Lemma 4.6, $A_t = \big((\sum A_\alpha)(\sum B_\alpha)\big)_t$. Since $(\sum A_\alpha, \sum B_\alpha)_t = D$, $(\sum A_\alpha)_t \cap (\sum B_\alpha)_t = \big((\sum A_\alpha)(\sum B_\alpha)\big)_t$. The result follows. □

Our next theorem generalizes [2, Corollary 3.5] to t-splitting sets.

Theorem 4.9. *Let D be an integral domain and S a t-splitting set of D. If B is an (integral) t-ideal of D, then BD_S is an (integral) t-ideal of D_S. In fact, for a nonzero ideal A of D, $A_t D_S = (AD_S)_t$. If E is a t-ideal of D_S, then $E \cap D$ is a t-ideal of D.*

Proof. As in Lemma 4.8, for an integral ideal A of D, $A_t = \big((\sum A_\alpha)(\sum B_\alpha)\big)_t = (\sum A_\alpha)_t \cap (\sum B_\alpha)_t$. Hence $A_t D_S = (\sum A_\alpha)_t D_S \cap (\sum B_\alpha)_t D_S = (\sum A_\alpha)_t D_S = \big((AD_S)_t \cap D\big) D_S = (AD_S)_t$, where the third equality follows from Lemma 4.7. Dividing through by an appropriate element shows the equality also holds for fractional ideals as well. It is well known that if E is a t-ideal of D_S for any multiplicatively closed set S, then $E \cap D$ is a t-ideal of D. □

We next show that the t-operation on D is induced by the t-operations on D_S and $D_{\mathcal{T}}$.

Theorem 4.10. *Let D be an integral domain and S a t-splitting set of D. Then $A_t = (AD_S)_t \cap (AD_{\mathcal{T}})_t$ for each nonzero fractional ideal A of D. Hence, if A is a nonzero integral ideal of D, then $A_t = \big((AD_S)_t \cap D\big) \cap \big((AD_{\mathcal{T}})_t \cap D\big) = \big(((AD_S)_t \cap D)((AD_{\mathcal{T}})_t \cap D)\big)_t$.*

Proof. Since $D = D_S \cap D_{\mathcal{T}}$, the function $A \to A^\star = (AD_S)_t \cap (AD_{\mathcal{T}})_t$, A a nonzero fractional ideal of D, is a finite character star-operation on D [25, Theorem 2]. Hence $A^\star \subseteq A_t$ for each nonzero fractional ideal A of D. To show that $\star = t$, it is enough to show that $A_t \subseteq A^\star$ for each nonzero integral ideal A of D. But $A^\star = (AD_S)_t \cap (AD_{\mathcal{T}})_t = \big((AD_S)_t \cap D\big) \cap \big((AD_{\mathcal{T}})_t \cap D\big)$ is a t-ideal because $(AD_S)_t \cap D$ and $(AD_{\mathcal{T}})_t \cap D$ are both t-ideals, being the contractions of t-ideals of D_S and $D_{\mathcal{T}}$, respectively. (One reference for the fact that E a t-ideal of $D_{\mathcal{T}}$ implies that $E \cap D$ is a t-ideal of D is [26, Propositions 1.5 and 1.8].) Hence $A_t \subseteq (A^\star)_t = A^\star$. The equality $\big((AD_S)_t \cap D\big) \cap \big((AD_{\mathcal{T}})_t \cap D\big) = \big(((AD_S)_t \cap D)((AD_{\mathcal{T}})_t \cap D)\big)_t$ follows from the fact that $\big((AD_S)_t \cap D, (AD_{\mathcal{T}})_t \cap D\big)_t = D$. Indeed, if $0 \neq a \in A$, then by Lemma 4.1, $(aD_S \cap D, aD_{\mathcal{T}} \cap D)_t = D$ and hence $\big((AD_S)_t \cap D, (AD_{\mathcal{T}})_t \cap D\big)_t = D$. □

Corollary 4.11. *Let D be an integral domain and S a t-splitting set of D. Let $A = (\{a_\alpha\})$ be an integral ideal of D with each $a_\alpha \neq 0$. For each α, let $A_\alpha = a_\alpha D_S \cap D$ and $B_\alpha = a_\alpha A_\alpha^{-1} = a_\alpha D_{\mathcal{T}} \cap D$. Then $(\sum B_\alpha)_t = (AD_{\mathcal{T}})_t \cap D = \big((\sum B_\alpha) D_{\mathcal{T}}\big)_t \cap D = (\sum B_\alpha)_t D_{\mathcal{T}} \cap D$.*

Proof. By Theorem 4.10, $(\sum B_\alpha)_t = \big(((\sum B_\alpha)D_S)_t \cap D\big) \cap \big(((\sum B_\alpha)D_{\mathcal{T}})_t \cap D\big)$. But $(\sum B_\alpha) \cap S \neq \varnothing$; so $\big((\sum B_\alpha)D_S\big)_t \cap D = D$, and hence $(\sum B_\alpha)_t = \big((\sum B_\alpha)D_{\mathcal{T}}\big)_t \cap D$. But $A \subseteq \sum B_\alpha = \sum (a_\alpha D_{\mathcal{T}} \cap D) \subseteq (\sum a_\alpha D_{\mathcal{T}}) \cap D = AD_{\mathcal{T}} \cap D$; so $AD_{\mathcal{T}} = (\sum B_\alpha) D_{\mathcal{T}}$. Hence $\big((\sum B_\alpha)D_{\mathcal{T}}\big)_t \cap D = (AD_{\mathcal{T}})_t \cap D$. It remains to show that $(\sum B_\alpha)_t D_{\mathcal{T}} \cap D = (\sum B_\alpha)_t$. The containment $\supseteq$ is clear. Let $d \in (\sum B_\alpha)_t D_{\mathcal{T}} \cap D$. So $d = b_1 x_1 + \cdots + b_n x_n$, where each $b_i \in (\sum B_\alpha)_t$ and each $x_i \in D_{\mathcal{T}}$. Choose $C \in \mathcal{T}$ with $Cx_i \subseteq D$ for each i. Then $dC \subseteq (\sum B_\alpha)_t$; so $d(C, \sum B_\alpha)_t \subseteq (\sum B_\alpha)_t$. But $(\sum B_\alpha) \cap S \neq \varnothing$; so $(C, \sum B_\alpha)_t = D$. Hence $d \in d(C, \sum B_\alpha)_t \subseteq (\sum B_\alpha)_t$. □

Let S be a t-splitting set for the integral domain D. Let us call S a *complemented t-splitting set* if $D_{\mathcal{T}} = D_T$ for some multiplicatively closed set T. We then call $\bar{T}$, the saturation of T, the *t-complement of S*. In this case,

$\bar{T} = \{x \in D^* \mid x^{-1} \in D_{\mathcal{T}}\} = \{x \in D \mid x \text{ is a unit in each } D_Q, Q \in G\} = D - \bigcup_{Q\in G} Q$. Note that by Lemma 4.1, for each $d \in D^*$, $dD_T \cap D = dD_{\mathcal{T}} \cap D = B_t$, where $(d) = (AB)_t$ with $A_t = dD_S \cap D$ and $B_t = dA^{-1}$. Thus for each $d \in D^*$, $dD_T \cap D$ is t-invertible. Hence by Corollary 2.3, T is also a t-splitting set. Moreover, the decomposition of t-Max(D) given by S shows that $\bar{S}$ is the t-complement for T.

However, a t-splitting set need not be t-complemented. For example, let D be a Dedekind domain with a prime ideal P such that no power of P is principal. Then $S = D - P$ is a t-splitting set (as every multiplicatively closed subset of a Dedekind domain is a t-splitting set), but S is not t-complemented. For $D_{\mathcal{T}} = \bigcap\{D_Q \mid Q \in \text{Max}(D) - \{P\}\} = \bigcup_{n\geq 0} P^{-n}$ is not a quotient ring of D. We do have the following result, the proof of which is left to the reader. Let D be a Krull domain. Then every multiplicative set of D is a t-complemented t-splitting set if and only if the divisor class group of D is torsion.

Our next theorem is the promised partial extension of [2, Theorem 3.7] .

Theorem 4.12. *Let D be an integral domain and S a complemented t-splitting set with t-complement T. Then the map $\theta : T(D) \to T(D_S) \times_c T(D_T)$ given by $\theta(A) = (AD_S, AD_T)$ is a monoid order-isomorphism. Moreover, for $A \in T(D)$, A is integral (resp., of finite type, t-invertible) if and only if both AD_S and AD_T are integral (resp., of finite type, t-invertible).*

Proof. Let $A \in T(D)$. Since S and T are both t-splitting sets, $AD_S \in T(D_S)$ and $AD_T \in T(D_T)$ by Theorem 4.9. Clearly θ is an order-preserving monoid homomorphism. If $\theta(A) \leq \theta(B)$, then $A = AD_S \cap AD_T \supseteq BD_S \cap BD_T = B$. Hence $A \leq B$. So θ is an order-monomorphism. It remains to show that θ is surjective. It suffices to show that $\theta|_{T_+(D)} : T_+(D) \to T_+(D_S) \times_c T_+(D_T)$ is onto. Let $(E, F) \in T_+(D_S) \times_c T_+(D_T)$. Then $E \cap D, F \cap D \in T_+(D)$ by Theorem 4.9. Also, $(E \cap D) \cap T \neq \varnothing$ and $(F \cap D) \cap S \neq \varnothing$ by Corollary 4.11. Hence $\theta\big((E \cap D) \cap (F \cap D)\big) = (E, F)$. The "moreover" statements are easy consequences of the fact that θ is an isomorphism. □

Remark 4.13. In the setup of Theorem 4.12, if A is a nonzero principal fractional ideal of D, then AD_S (resp., AD_T) is a nonzero principal fractional ideal of D_S (resp., D_T). Thus θ induces a surjective group homomorphism $\bar{\theta} : \text{Cl}_t(D) \to \text{Cl}_t(D_S) \times \text{Cl}_t(D_T)$, given by $\bar{\theta}([A]) = ([AD_S], [AD_T])$. However, unlike the splitting set case, $\bar{\theta}$ need not be a monomorphism, *i.e.*, AD_S and AD_T principal need not imply that A is principal. For example, let $D = \mathbb{Z}[\sqrt{-5}]_N$, where N is the multiplicative set generated by the principal primes of $\mathbb{Z}[\sqrt{-5}]$. So $\text{Cl}(D) = \text{Cl}(\mathbb{Z}[\sqrt{-5}]) = \mathbb{Z}/2\mathbb{Z}$, but each proper overring of D is a localization of D and is a PID. Let M be a nonprincipal maximal ideal of D and let $S = D - M$. Let T be the t-complement of S. Then MD_S and MD_T are both principal, but M is not principal. In fact, here D_S and D_T are both PID's, so $\text{Cl}(D_S) \times \text{Cl}(D_T) = 0$, while $\text{Cl}(D) = \mathbb{Z}/2\mathbb{Z}$.

We end with the following Nagata-type theorem (*cf.* [2, Theorem 4.4]). One may add many more properties (for example: being root closed, integrally closed, completely integrally closed, weakly Krull, or satisfying ACCP) to those listed in Theorem 4.14; we leave the precise formulation to the interested reader.

Theorem 4.14. *Let D be an integral domain and S a t-splitting set of D. Further, suppose that for each nonunit $s \in S$, sD is a t-product of height-one prime ideals. If D_S is a Mori domain (resp., Krull domain, PVMD), then D is a Mori domain (resp., Krull domain, PVMD).*

Proof. As before, let $F = \{P \in t\text{-Max}(D) | P \cap S = \varnothing\}$ and $G = \{P \in t\text{-Max}(D) | P \cap S \neq \varnothing\}$. Let $Q \in G$; so $sD \subseteq Q$ for some $s \in S$. Since sD is a t-product of height-one prime ideals, Q contains a height-one t-invertible prime t-ideal Q_0. Being t-invertible, Q_0 is a maximal t-ideal and hence $Q = Q_0$. So each $Q \in G$ is a height-one t-invertible maximal t-ideal and D_Q is a DVR. We next show that $D_{\mathcal{T}} = \bigcap_{Q\in G} D_Q$ is a Krull domain.

It suffices to show that each nonunit $x \in D^*$ is contained in only finitely many members of G. Suppose that $x \in Q \in G$. Write $xD = (AB)_t$, where $(A, s)_t = D$ for all $s \in S$ and $B_t \cap S \neq \varnothing$. Choose $s_0 \in B_t \cap S$. Write $s_0 D = (Q_1 \cdots Q_n)_t$, where each Q_i is a height-one prime ideal. Now $A_t B_t \subseteq xD \subseteq Q$ and $A_t \not\subseteq Q$, so $B_t \subseteq Q$. Hence $Q_1 \cdots Q_n \subseteq s_0 D \subseteq B_t \subseteq Q$; and so $Q = Q_i$ for some i. Thus $D_{\mathcal{T}}$ is a Krull domain.

Suppose that D_S is a Mori domain (resp., Krull domain). It is immediately apparent that $D = D_S \cap D_{\mathcal{T}}$ is a Mori domain (resp., Krull domain) since the intersection of two Mori domains (resp., Krull domains) is a Mori domain (resp., Krull domain).

Next suppose that D_S is a PVMD. Let P be a maximal t-ideal of D. We need to show that D_P is a valuation domain. We have already shown that if $P \in G$, then D_P is a DVR. So suppose $P \in F$. By Theorem 4.9, PD_S is a maximal t-ideal of D_S. Hence $D_P = (D_S)_{PD_S}$ is a valuation domain since D_S is a PVMD. □

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