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RESEARCH ARTICLE

A Note on Almost GCD Monoids

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Abstract

A commutative cancellative monoid H (with 0 adjoined) is called an almost GCD (AGCD) monoid if for x, y in H, there exists a natural number $n = n(x, y)$ so that x^n and y^n have an LCM, that is, $x^nH \cap y^nH$ is principal. We relate AGCD monoids to the recently introduced inside factorial monoids (there is a subset Q of H so that the submonoid F of H generated by Q and the units of H is factorial and some power of each element of H is in F). For example, we show that an inside factorial monoid H is an AGCD monoid if and only if the elements of Q are primary in H, or equivalently, H is weakly Krull, distinct elements of Q are v-coprime in H, or the radical of each element of Q is a maximal t-ideal of H. Conditions are given for an AGCD monoid to be inside factorial and the results are put in the context of integral domains.

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Throughout this note a *monoid* means a commutative cancellative semigroup with identity with zero adjoined. We will represent the semigroup operation by ordinary multiplication notation and use 1 to denote the identity of the semigroup. We will follow Halter-Koch's [8] convention of allowing a zero element 0 with the property that $0x = 0$; yet $xy = 0$ implies $x = 0$ or $y = 0$. A good representative example of a monoid is the multiplicative monoid of an integral domain. This somewhat close relationship has made it very natural to study results of a multiplicative nature known for integral domains in the monoid setting. Halter-Koch's book [8] is good source of multiplicative ideal theoretic results on monoids (and on integral domains).

Recall that an integral domain D is an *almost GCD (AGCD) domain* if for all $x, y \in D^* = D \setminus \{0\}$ there exists a natural number $n = n(x, y)$ such that $x^nD \cap y^nD$ is a principal ideal [11]. Using this definition and assuming familiarity with the notion of a monoid, we define a monoid M to be an *almost GCD (AGCD) monoid* if for all $x, y \in M^\bullet = M \setminus \{0\}$ there exists a natural number $n = n(x, y)$ and an element $z \in M$ such that $x^nM \cap y^nM = zM$. Armed with this definition we study conditions under which inside factorial monoids recently discussed in [6] are AGCD monoids. In the course of our study we show that the root closure of an inside factorial monoid is an AGCD inside factorial monoid. This, of course, indicates how intimately inside factorial monoids are related to AGCD monoids. We shall also give necessary and sufficient conditions under which an AGCD monoid is an inside factorial monoid and relate these results to known facts about integral domains that are AGCD and inside factorial as monoids. We also give an example of an inside factorial monoid/domain that was included in [11] as an example of a generalized almost factorial domain, and hence predates the concept of inside factorial monoids. But this requires an introduction to monoids and to the notion of inside factorial monoid. We have decided to write a somewhat detailed description of the notions involved. We hope that this description will serve as an introduction to the multiplicative ideal theory of monoids.

1. Basic terminology and preparatory results

As mentioned above there is a striking similarity between the study of multiplicative monoids and the study of the multiplicative aspects of an integral domain. In view of this similarity, we follow Halter-Koch in keeping the ring-theoretic notation wherever possible. Thus if H is a monoid and $a, b \in H$ we write $a \mid_H b$ to denote a divides b in H, that is, $b = ac$ for some $c \in H$. We call $a, b \in H$ *associates* if $a \mid_H b$ and $b \mid_H a$. Associates of 1 in H are called *units* and the set of units of H is denoted by $H^\times$. Now H is said to be *reduced* if $H^\times = \{1\}$. The units of H can be shown to be invertible elements of H. Thus $H^\times$ is a subgroup of H and we can consider the quotient monoid $H/H^\times$ which is obviously reduced and is denoted by H_{red}. In place of a quotient field that we have for a domain, we have a quotient groupoid of a monoid H. We shall reserve $G(H)$ for the quotient groupoid of H. For completeness note that $G(H) = \{c^{-1}h \mid h \in H \text{ and } c \in H^\bullet\}$. Just as in ring theory we can define a multiplicatively closed subset S of a monoid H and call $S^{-1}H = \{s^{-1}h \mid h \in H \text{ and } s \in S\}$. Of course to avoid trivialities we assume that $0 \notin S$. As in the case of integral domains, if $S = H \setminus P$ the complement of a prime ideal P we write H_P for H_S. A *factorial monoid* is defined in the same manner as a UFD is defined. A *monoid homomorphism* is defined in the usual manner and is assumed to preserve identities. A monoid homomorphism $\varphi\colon H \to K$ from a monoid H to another monoid K is called a *divisor homomorphism* if for $a, b \in H$, $\varphi(a) \mid_K \varphi(b)$ implies $a \mid_H b$. Note that if φ is a divisor homomorphism and if $\varphi(a)$ and $\varphi(b)$ are associates in K, then a and b are associates in H. A monoid H is called an *inside factorial monoid* if there is a factorial monoid D and a divisor homomorphism $\varphi\colon D \to H$ such that for each $h \in H$ there exists a natural number $n = n(h)$ such that $h^{n(h)} \in \varphi(D)$. An extension $K \subseteq H$ of monoids is called a *root extension* if for each $h \in H$ there exists a natural number $n = n(h)$ such that $h^{n(h)} \in K$. Thus the last part of the definition of an inside factorial monoid just says that $\varphi(D) \subseteq H$ is a root extension.

By an ideal of a monoid H we mean a non-empty subset A of H such that for all $h \in H$ and for all $a \in A$ we have $ha \in A$. Thus for $k \in H$,

$kH = \{kh \mid h \in H\}$ is the principal ideal generated by k. The ideal generated by a non-empty set X can be regarded as $\langle X \rangle = \bigcup_{x \in X} xH$ and the ideal generated by the empty set ϕ is $\{0\}$. As in the integral domain case, a proper ideal P of H is a *prime ideal* if for $x, y \in H$, $xy \in P$ implies $x \in P$ or $y \in P$. What may be noted with interest is the fact that if A and B are two ideals of a monoid H, then $AB = \{ab \mid a \in A, b \in B\}$ is an ideal of H. This indicates that ideal-theoretic results proved for monoids may not be of use in ring theory. A good example of this discrepancy has been discussed in [12, page 1901]. The reader may also want to look up [1] and [5] where the authors study integral domains in which the product of two ideals is equal to their monoid product. This in our opinion makes the study of monoids to get a grip on multiplicative ideal theory in integral domains a somewhat risky endeavor. To make up for this deficiency the introduction of ideal systems becomes a must.

As in the case of integral domains we can also define various ideal systems on a monoid H. This fact has been amply demonstrated in [8]. For the sake of completeness we include a definition here. For the properties of these ideal systems the reader may consult [8, Chapter 2]. An ideal system r on a monoid H is a map on $P(H)$, the power set of H, defined by $X \longmapsto X_r$ such that for all $X, Y \in P(H)$ and $c \in H$ the following conditions hold:

(1) $X \cup \{0\} \subseteq X_r$,

(2) $X \subseteq Y_r$ implies $X_r \subseteq Y_r$,

(3) $cH \subseteq \{c\}_r$, and

(4) $(cX)_r = cX_r$.

An ideal I is called an r-*ideal* if $I = I_r$ and is r-*finitely generated* if $I = J_r$ for a finitely generated ideal J of H. From (1) it follows that for every r-system we have $H_r = H$ and from (1) we have that every principal ideal is an r-ideal. If A is an r-ideal and X is any subset of H then the set $(A : X) = \{x \in H \mid xX \subseteq A\}$ is an r-ideal and $(A : X) = (A : X_r)$. An ideal system r on H is said to be *finitary* if for each $X \in P(H)$, $X_r = \bigcup F_r$ where F ranges over the finite subsets of X.

The simplest ideal system is the d-system given by $X \longmapsto X_d = \langle X \rangle$. One of the ideal systems of interest to us is the ideal system induced by a set of overmonoids $\{H_\alpha\}$ of H such that $H = \bigcap_\alpha H_\alpha$, defined by $X \mapsto X_r = \bigcap \langle XH_\alpha \rangle$ where by $\langle XH_\alpha \rangle$ we mean the submonoid of H_α generated by X. For ideal systems induced by a defining system of overmonoids the reader may consult [8, Chapter 21]. Here by an overmonoid H_α we mean a submonoid of $G(H)$ that contains H as a submonoid. (As in the case of rings a submonoid of a monoid K must contain the identity of the monoid K.) Two other ideal systems of interest to us are the v-system and the t-system. The v-system is defined by $X_v = \bigcap cH$ where $c \in G(H)$ such that $X \subseteq cH$. The t-system is defined by $X_t = \bigcup \{Y_v \mid Y$ ranges over finite subsets of $X\}$. For the v-system of ideals the reader may consult [8, Chapter 11] and for the t-system the reader may want to note that the t-system is a finitary system. Let us recall that if r is an ideal system on H, a *maximal r-ideal*, if it exists, is an r-ideal $P \neq H$ such that there is no r-ideal J with $P \subsetneq J \subsetneq H$ [8, page 57]. A maximal r-ideal, if it exists, is a prime ideal and if r is finitary, then each proper r-ideal A of H is contained in a maximal r-ideal of H [8, Theorem 6.3]. The set of maximal r-ideals of H will be denoted by $r-\max(H)$. Next if $\{P_\alpha\}_{\alpha\in I}$ is a family of prime r-ideals of H then $H = \bigcap_{\alpha\in I} H_{P_\alpha}$ if and only if for each pair $a,b \in H$, either $a \mid b$ or $(aH{:}b)$ is contained in at least one P_α [8, page 234]. Now if $(aH{:}b) \neq H$ then being a t-ideal $(aH:b)$ must be contained in at least one maximal t-ideal of H and this leads to $H = \bigcap_{P\in r-\max(H)} H_P$. A monoid H is said to be *weakly Krull* if $H = \bigcap H_P$ where the intersection ranges over the minimal prime ideals of H and is *locally finite*, that is, each nonzero element of H is a unit in almost all H_P. As in the integral domain case, ideal systems can and do work for fractional ideals. In the case of a monoid H a *fractional ideal* F is a subset of $G(H)$ such that, for every $a \in H$, $aF \subseteq F$ and, for some $a \in H^\bullet$, aF is an ideal of H. As in the case of fractional ideals of integral domains, we can define for each nonzero fractional ideal A of a monoid H, the inverse $A^{-1} = \{x \in G(H) \mid xA \subseteq H\}$ which is again a fractional ideal. It can be easily checked that for $x,y \in H^\bullet$ we have $\{x,y\}^{-1} = (xH \cup yH)^{-1} = x^{-1}H \cap y^{-1}H = x^{-1}y^{-1}(xH \cap yH)$, that $\{x,y\}_v = (\{x,y\}^{-1})^{-1}$ and that $(\{x,y\}_v)^{-1} = \{x,y\}^{-1}$.

Let H be a monoid. Call $x,y \in H$, *v-coprime* if $\{x,y\}_v = H$. This means that $\{x,y\}^{-1} = H$, that is, $x^{-1}y^{-1}(xH \cap yH) = H$ which happens if and only if $xH \cap yH = xyH$. It may be noted that as in the case of integral domains if x and y are v-coprime in a monoid H and if $x \mid_H yz$ then $x \mid_H z$ ($x \mid_H yz \Rightarrow xH = \{x,yz\}_v = \{x,xz,yz\}_v = \{x,\{xz,yz\}_v\}_v = \{x,z\}_v \Rightarrow x \mid_H z$). It is easily checked (see [11, Lemma 1.1] for the integral domain case) that for x,y in a monoid H, $\{x,y\}_v = H$ if and only if $\{x^r,y^s\}_v = H$ for natural numbers r,s. Call a monoid homomorphism $\varphi{:}\, H \to D$ a *pseudo-injection* if for $a,b \in H$, $\varphi(a) = \varphi(b)$ implies that a and b are associates in H. Both divisor homomorphisms and injective homomorphisms are examples of pseudo-injections. Note that if φ is pseudo-injective, then $\varphi_{\text{red}}{:}\, H_{\text{red}} \to D_{\text{red}}$ is injective. If $\varphi{:}\, H \to K$ is a pseudo-injection of monoids, call a monoid K *2-t-linked over H with respect to φ* or *simply 2-t-linked over H* if for $x,y \in H$, $\{xH \cup yH\}^{-1} = H$ implies that $\{\varphi(x)K \cup \varphi(y)K\}^{-1} = K$, that is, $xH \cap yH = xyH$ implies that $\varphi(x)K \cap \varphi(y)K = \varphi(xy)K$. Now if φ is a pseudo-injection, then $\varphi(H)$ is 2-t-linked over H. For if x and y are v-coprime in H consider $\varphi(x)\varphi(H) \cap \varphi(y)\varphi(H)$. Then for $u \in \varphi(x)\varphi(H) \cap \varphi(y)\varphi(H)$ we have $u = \varphi(x)\varphi(a) = \varphi(y)\varphi(b)$ for some $a,b \in H$. But then $\varphi(xa) = \varphi(yb)$ which implies that xa and yb are associates. This leads to $xa = \epsilon yb \in xH \cap yH = xyH$, which forces $a = yr$ and $b = xs$. But then $\varphi(a) = \varphi(yr) = \varphi(y)\varphi(r)$ and $\varphi(b) = \varphi(xs) = \varphi(x)\varphi(s)$. So $u = \varphi(x)\varphi(a) = \varphi(x)\varphi(y)\varphi(r) = \varphi(y)\varphi(b) = \varphi(x)\varphi(y)\varphi(s)$. Thus $u \in \varphi(x)\varphi(H) \cap \varphi(y)\varphi(H)$ implies that $u \in \varphi(xy)\varphi(H)$. The concepts of v-coprimality and 2-t-linkedness are translations from ring theoretic multiplicative ideal theory. The concept of v-coprimality was used in [11] and has often been of use in studying various concepts in ring theory and 2-t-linkedness is a translation of what is known as R_2-stableness. The reader may consult Uda [10] for a better appreciation of the concept.

Recall now the definition of AGCD monoids as given in the introduction above. Using the notion of the v-system we can restate the definition as follows. A monoid M is an *almost GCD (AGCD) monoid* if for all $x, y \in M^\bullet = M \setminus \{0\}$ there exists a natural number $n = n(x, y)$ and an element $z \in M$ such that $\{x^n, y^n\}_v = zM$. Note that M is an AGCD monoid if and only if M_{red} is an AGCD monoid. When using the v-system for two distinct monoids we will use the same symbol v yet indicate the monoid in which this operation is taking place. For instance to indicate that $\{x, y\}_v$ is taking place in M we shall use $(xM \cup yM)_v$ in place of $\{x, y\}_v$. The following proposition, which is an interesting result in its own right, will be of help in proving that many inside factorial monoids are in fact AGCD monoids.

Proposition 1.1. *Let D be an AGCD monoid. Suppose $\varphi\colon D \to H$ is a pseudo-injection from D to a monoid H. If H is 2-t-linked over D with respect to φ and if $\varphi(D) \subseteq H$ is a root extension, then H is an AGCD monoid.*

Proof. Let $h, k \in H^\bullet$, then $h^r, k^s \in \varphi(D)$ for some natural numbers r, s. Let $m = \operatorname{lcm}(r, s)$, then $h^m, k^m \in \varphi(D)$. Then there exist $a, b \in D$ such that $h^m = \varphi(a)$ and $k^m = \varphi(b)$. Since D is an AGCD monoid there is a natural number n such that $\{a^n D \cup b^n D\}_v = dD$ for some $d \in D$. But then, $a^n = \alpha d$ and $b^n = \beta d$ where α and β are v-coprime in D. But then by hypothesis $\varphi(\alpha)$ and $\varphi(\beta)$ are v-coprime in H. That is, $(\varphi(\alpha)H \cup \varphi(\beta)H)_v = H$. Multiplying both sides by $\varphi(d)$ and using the properties of the v-system we get $(\varphi(\alpha)\varphi(d)H \cup \varphi(\beta)\varphi(d)H)_v = \varphi(d)H$ or $(\varphi(\alpha d)H \cup \varphi(\beta d)H)_v = \varphi(d)H$. Substituting for αd and βd we get $(\varphi(a^n)H \cup \varphi(b^n)H)_v = \varphi(d)H$. Using the fact that φ is a homomorphism we get $((\varphi(a))^n H \cup (\varphi(b))^n H)_v = \varphi(d)H$. Then substituting for $\varphi(a)$ and $\varphi(b)$ we have $(h^{mn}H \cup k^{mn}H)_v = \varphi(d)H$. Thus for each pair $h, k \in H^\bullet$ we have a natural number $q = mn$ such that $(h^q H \cup k^q H)_v$ is principal. ∎

Corollary 1.2. *Let D and H be monoids with divisor homomorphism $\varphi\colon D \to H$ and suppose that D is an AGCD monoid. If H is 2-t-linked over D with respect to φ and if $\varphi(D) \subseteq H$ is a root extension, then H is an AGCD monoid.*

Corollary 1.3. *Let D and H be monoids with divisor homomorphism $\varphi\colon D \to H$ and suppose that D is a factorial monoid. If H is 2-t-linked over D with respect to φ and if $\varphi(D) \subseteq H$ is a root extension, then H is an AGCD monoid.*

2. Inside factorial monoids that are weakly Krull

The aim of this section is to prepare the reader to see, among other things, the fact that to each inside factorial monoid there is associated an inside factorial weakly Krull monoid that happens also to be an AGCD monoid.

We refer the reader to [6] for a detailed study of inside factorial monoids. For our purposes we shall present, in the following, the notions and results that are useful in proving the intended results as mentioned above.

Definition 2.1. [6, Definition 2] Let H be a monoid. A subset $Q \subseteq H$ is called a *Cale basis for* H, if the submonoid $F = H^\times[Q]$ generated by Q and $H^\times$ has the following two properties.

(1) If $\varepsilon \prod_{q \in Q} q^{v(q)} = \varepsilon' \prod_{q\in Q} q^{v'(q)}$ where $\varepsilon, \varepsilon' \in H^\times$, $v(q), v'(q)$ are nonnegative integers and $v(q) = v'(q) = 0$ for almost all $q \in Q$, then $v(q) = v'(q)$ for all $q \in Q$.

(2) For every $x \in H$, there exists some natural number $n = n(x)$ such that $x^n \in F$, i.e., $F \subseteq H$ is a root extension.

Proposition 2.2. [6, Proposition 4] *A monoid H is inside factorial if and only if it possesses a Cale basis. More precisely if Q is a Cale basis for H, then $F = H^\times[Q]$ is a factorial monoid and the injection $F \hookrightarrow H$ is a divisor homomorphism.*

Here are some properties of the elements of Q that will be useful.

Notes 2.3.

Note 2.3.1. As noted in [6] F is a factorial monoid and Q is a complete system of mutually nonassociated primes of F. (However, note that these "primes" may not even be irreducible in H.)

Note 2.3.2. Given any $x \in H$, either no power of q divides any power of x or $x \in \sqrt{qH}$ (but not both). (Reason: For some natural number n, $x^n \in F$ and if q is not present in the prime factorization of x^n for minimal n then no integral power of x has q as a factor. However, if q is in the factorization of some $x^n \in F$ then $x \in \sqrt{qH}$.) Here, for any ideal A of H, $\sqrt{A}$ represents the radical ideal of A defined the same way as in the case of rings.

Note 2.3.3. For $q \in Q$, $q \in \sqrt{xH}$ if, and only if, $x^k = \varepsilon q^s$ for some natural numbers k and s and for some $\varepsilon \in H^\times$ [6, Lemma 2(e)].

Note 2.3.4. If $q \in Q$ and $q \mid xy$, then there exists a natural number n such that $q \mid x^n$ or $q \mid y^n$ [6, Lemma 2(f)].

Note 2.3.5. By Note 2.3.4, for each $q \in Q$, $\sqrt{qH}$ is a prime ideal. An easy argument shows that $\sqrt{qH}$ is a minimal prime ideal of H and hence a prime t-ideal [8]. In fact, the proof of [6, Theorem 2] shows that $\{\sqrt{qH} \mid q \in Q\}$ is the set of height-one prime ideals of H and every nonzero prime ideal of H contains a height-one prime ideal.

According to [6, Example 6.2] members of Q may not be primary in H. However if we assume that the members of Q are all primary in H then we can draw some interesting conclusions.

Proposition 2.4. *Let H be an inside factorial monoid with Cale basis Q. Then the following conditions are equivalent.*

(1) *For each $q \in Q$, $\sqrt{qH}$ is a maximal t-ideal, i.e., $\{\sqrt{qH} \mid q \in Q\} = r - \max(H)$.*

(2) *H is an AGCD monoid.*

(3) *Each $q \in Q$ is a primary element.*

(4) *H is a weakly Krull monoid.*

(5) *Distinct elements of Q are v-coprime.*

Proof. (2.4)$\Rightarrow$(2.4) Let $x, y \in H^\bullet$. Then there is an $n \geq 1$ with $x^n, y^n \in F = H^\times[Q]$. Let $d = \gcd_F(x^n, y^n)$; we claim that $\{x^n, y^n\}_v = dH$. Indeed, otherwise there would be some $q \in Q$ such that $d^{-1}x^n, d^{-1}y^n \in \sqrt{qH}$ and hence $q \mid d^{-k}x^{nk}$, $q \mid d^{-k}y^{nk}$ for some $k \geq 1$ in H and thus in F, since $F \hookrightarrow H$ is a divisor homomorphism. But this is a contradiction.

(2.4)$\Rightarrow$(2.4) Suppose H is an AGCD monoid. For $q \in Q$, suppose $q \mid xy$ in H; then $q \mid x^n$ or $q \mid y^n$ (Note 2.3.4). Suppose that $q \nmid x^n$ for any n. Since H is an AGCD monoid, there exists an m such that $\{q^m, x^m\}_v = dH$ for some $d \in H$. But then $d \mid q^m$ and if d is not a unit then $q \in \sqrt{dH}$. But then $d^s = \varepsilon q^t$ as given by Note 2.3.3, and this forces $q \mid x^{ms}$ a contradiction. So d is a unit and hence $\{q^m, x^m\}_v = H$. Thus $\{q, x\}_v = H$ and so $q \mid xy$ and $q \nmid x$ gives $q \mid y$.

(2.4)$\Rightarrow$(2.4) Let P be a prime t-ideal of H. Then $\sqrt{qH} \subseteq P$ for some $q \in Q$. Hence the primary t-invertible t-ideal qH lies in P. By [8, Theorem 22.7g] H is weakly Krull.

(2.4)$\Rightarrow$(2.4) By [8, Theorem 22.5ii] every nonzero prime t-ideal of H is a maximal t-ideal.

(2.4)$\Rightarrow$(2.4) For distinct $q, q' \in Q$, $\sqrt{qH}$ and $\sqrt{q'H}$ are distinct maximal t-ideals. Hence $\{q, q'\}_v = H$.

(2.4)$\Rightarrow$(2.4) Now $\sqrt{qH}$ is a proper t-ideal, so it suffices to show that $x \notin \sqrt{qH}$ implies that $\{x, q\}_v = H$. Now $x \notin \sqrt{qH}$ gives that $x^n = \varepsilon q_1 \cdots q_s$ for some $n \geq 1$, $q_i \in Q$ with each $q_i \neq q$ and unit ε. Then $\{q_i, q\}_v = H$ gives $\{q_1 \cdots q_s, q\}_v = H$ and hence $\{x^n, q\}_v = H$. Thus $\{x, q\}_v = H$. ■

Recall that a monoid H is *root closed* if for every $x \in G(H)$ and for any natural number n, $x^n \in H$ implies that $x \in H$. The above proposition ensures that if H is an inside factorial weakly Krull monoid then H is indeed an AGCD monoid. Having established that, we note that according to [6, Theorem 3] every root closed inside factorial monoid H is an AGCD monoid. Indeed, the root closure of an inside factorial monoid is an inside factorial weakly Krull monoid [6, Theorem 4]. This leads us to the following statement.

Corollary 2.5. *The root closure of an inside factorial monoid is an AGCD weakly Krull monoid.*

3. AGCD inside factorial monoids

Having seen that AGCD monoids are so intimately related to inside factorial monoids one may want to know the conditions under which an AGCD monoid is inside factorial. Here is how we proceed.

Definition 3.1. Call a monoid H an *almost valuation monoid* if for all $x, y \in H$ there exists a natural number $n = n(x, y)$ such that $x^n \mid y^n$ or $y^n \mid x^n$.

The reader may want to compare this definition with that of almost valuation domains of [11].

Proposition 3.2. *Let H be an AGCD monoid. Then the following are equivalent.*

(1) *No two nonzero nonunits of H are v-coprime.*

(2) *For every pair of elements x, y of H there exists a natural n such that $x^n \mid y^n$ or $y^n \mid x^n$.*

Proof. (3.2)$\Rightarrow$(3.2). If either of x, y is 0 or a unit, then $x^n \mid y^n$ or $y^n \mid x^n$ holds trivially for $n = 1$. So, it is enough to concentrate on nonzero nonunit x, y. Since H is an AGCD domain, there exists $n = n(x, y)$ such that $\{x^n, y^n\}_v = dH$. But then $\{x^n/d, y^n/d\}_v = H$ forcing at least one of $x^n/d, y^n/d$ to be a unit.

(3.2)$\Rightarrow$(3.2) Obvious. ∎

Indeed every valuation monoid, as defined in [8] is an almost valuation monoid. Before deciding which almost valuation monoids are inside factorial it seems pertinent to digress a little and get some terminology at hand.

It was shown in [11] that if D is an AGCD domain with quotient field K then an element $x \in K$ is integral over D if and only if $x^n \in D$ for some natural n [11, Theorem 3.1]. Luckily, in monoids root closure is the only substitute for ordinary integral closure of integral domains and some authors keep the domain terminology in monoids. So, if H is a monoid and $x \in G(H)$ we shall say that x is *integral over* H if $x^n \in H$ for some natural number n. Similarly, an overmonoid K of H is *integral* over H if for each $x \in K$ there is a natural number n such that $x^n \in H$, i.e., $H \subseteq K$ is a root extension. The *integral closure* or the *root closure* of a monoid H is denoted by $\tilde{H}$ and defined as $\tilde{H} = \{x \in G(H) \mid x^n \in H$ for some natural $n\}$. Obviously the integral closure is integrally closed, i.e., for each $x \in G(H)$, $x^m \in \tilde{H}$, for some natural number

m implies that $x \in \tilde{H}$. Indeed, another way of characterizing an integrally closed monoid is to say that K is integrally closed if and only if for all $x, y \in K$, $x^n \mid_K y^n$ for some natural number n implies that $x \mid_K y$.

Proposition 3.3. *Let H be a monoid. Then $\tilde{H}$ is 2-t-linked over H with respect to inclusion. Consequently, if H is an AGCD monoid, then so is $\tilde{H}$.*

Proof. Suppose that there exist $x, y \in H$ such that $xH \cap yH = xyH$ but that $x\tilde{H} \cap y\tilde{H} \neq xy\tilde{H}$. Let $u \in x\tilde{H} \cap y\tilde{H} \setminus xy\tilde{H}$. We can write $u = xr = ys$ where $r, s \in \tilde{H}$. But then there exists a natural number k such that r^k, $s^k \in H$ and hence $u^k = (xr)^k = (ys)^k \in H$. This forces $x^k \mid_H y^k s^k$ and since x and y are v-coprime in H we have $x^k \mid_H s^k$ and hence $x^k \mid_{\tilde{H}} s^k$. But then $x \mid_{\tilde{H}} s$, so that $u \in xy\tilde{H}$, a contradiction. The second statement of the proposition now follows from Proposition 1.1. The fact that H AGCD implies $\tilde{H}$ AGCD has a simple alternative argument. Let $x, y \in \tilde{H}^\bullet$; so $x^n, y^n \in H$ for some $n \geq 1$ and $x^{nm}H \cap y^{nm}H = dH$ for some $m \geq 1$ and $d \in H$. Certainly $d\tilde{H} \subseteq x^{nm}\tilde{H} \cap y^{nm}\tilde{H}$. Suppose $z \in x^{nm}\tilde{H} \cap y^{nm}\tilde{H}$; so $x^{nm}, y^{nm} \mid_{\tilde{H}} z$. Then for some $l \geq 1$, $z^l \in H$ and $x^{nml}, y^{nml} \mid_H z^l$. Thus $d^l \mid_H z^l$. So $d^l \mid_{\tilde{H}} z^l$ and hence $d \mid_{\tilde{H}} z$. So $x^{nm}\tilde{H} \cap y^{nm}\tilde{H} \subseteq d\tilde{H}$. ■

Observe that the converse of the second part of Proposition 3.3 does not hold: $\tilde{H}$ an AGCD monoid does not force H to be an AGCD monoid. The monoid $H = \{\pm 2^m 3^n \mid m + n \geq 1\} \cup \{0, 1\}$ mentioned in Note 2.3.5 is not an AGCD monoid ($\{2^n, 3^n\}_v = H \setminus \{1\}$ is never principal), but its root closure $\tilde{H} = H \cup \{-1\} = \{\pm 2^m 3^n \mid m + n \geq 0\} \cup \{0\}$ is, in fact, it is even factorial. For an integral domain example, see [2, Example 3.1].

Coming back to almost valuation monoids we note that if H is an almost valuation monoid then $\tilde{H}$ is a valuation monoid. Now to see the conditions under which an AGCD monoid H is an inside factorial almost valuation monoid let us introduce a notion that will facilitate further work.

Definition 3.4. Call two nonzero nonunits x, y of a monoid H, *power associates* of each other if there exist natural numbers m, n and a unit ε of H such that $x^m = \varepsilon y^n$.

If H is an almost valuation inside factorial monoid, then any two distinct $q, q' \in Q$ are v-coprime since H is an AGCD monoid (Proposition 2.4). But since H is an almost valuation monoid, no two elements are v-coprime, hence $|Q| = 1$. On the other hand, let H be a monoid with the property that there exists at least one nonzero nonunit q such that for each nonzero nonunit $h \in H$ we have $h^m = \varepsilon q^n$ for some natural numbers m, n and for some unit ε of H. Then as each nonzero nonunit of H is a power associate of q we conclude that H is an almost valuation domain and that $\{q\}$ is a Cale basis for H. Note that the representation $h^m = \varepsilon q^n$ induces an isomorphism $[h] \to n/m$ from H_{red} to an additive subgroup of $\mathbb{Q}_{\geq 0}$. Conversely, if we have such an isomorphism, then for any fixed nonunit $q \in H$ and $h \in H$ we have $h^m = \varepsilon q^n$ for some $m > 0$ and $n \geq 0$.

The above discussion gives rise to the following observation.

Proposition 3.5. *A monoid H is an inside factorial almost valuation monoid if and only if there exists a nonzero nonunit q in H such that every nonzero nonunit of H is a power associate of q, or equivalently, H_{red} is isomorphic to an additive subgroup of $\mathbb{Q}_{\geq 0}$.*

Next we note that if H is an AGCD inside factorial monoid then every primary element of H is a power associate of some member of Q. Indeed, if h is a primary element of H, then by the inside factorial property $h^m = \varepsilon q_1^{n_1} \cdots q_r^{n_r}$. But h being primary requires exactly one n_i to be positive.

So, to find sufficient conditions for an AGCD monoid to be inside factorial we can start with an AGCD monoid that is weakly Krull. We need a quick definition and lemma. Call a nonzero element p of a prime ideal P a *base for* P if $P = \sqrt{pH}$. Thus if pH is P-primary, p is a base for P.

Lemma 3.6. *Let H be an AGCD weakly Krull monoid. Let $P, P_1, \ldots, P_r$ be height-one primes of H with $P \subseteq P_1 \cup \ldots \cup P_r$. Then $P = P_i$ for some i.*

Proof. For the case of integral domains, this follows from the well known result that a prime ideal contained in a finite union of prime ideals is contained in one of them. However, this need not hold for monoids. In our case we argue as follows. Suppose that $P \neq P_i$ for each i. Now $\{P, P_1 \cdots P_r\}_t = H$, so there exist $y_1, \ldots, y_s \in P$ and $z_1, \ldots, z_l \in P_1 \cdots P_r$ with $\{y_1, \ldots, y_s, z_1, \ldots, z_l\}_v = H$. Hence for $t \geq 1$, $H = \{y_1^t, \ldots, y_s^t, z_1^t, \ldots, z_l^t\}_v = \{\{y_1^t, \ldots, y_s^t\}_v, \{z_1^t, \ldots, z_l^t\}_v\}_v$. Since H is an AGCD monoid, we can choose t so that $\{y_1^t, \ldots, y_s^t\}_v = yH$ and $\{z_1^t, \ldots, z_l^t\}_v = zH$ are principal. Then $y \in P$, $z \in P_1 \cap \cdots \cap P_r$ and $\{y, z\}_v = H$. Hence $y \notin P_1 \cup \cdots \cup P_r$. ∎

Proposition 3.7. *Let H be an AGCD weakly Krull monoid. Then for every height-one prime ideal P of H and for every nonzero $x \in P$ we have $x^r = x_1 b$ where b is a base for P, r a natural number, and $\{x_1, b\}_v = H$.*

Proof. Let P be a height-one prime ideal of H and let $x \in P$. If x belongs to no other height-one prime, then x is a primary element and hence a base for P and we have nothing to prove. So let us assume that x belongs to at least one more height-one prime. Then, as H is weakly Krull, the set of all height-one primes containing x is finite, say $\{P, P_2, \ldots, P_r\}$. Now by Lemma 3.6 we can find $y \in P \setminus (P_2 \cup \cdots \cup P_r)$. Since H is AGCD and P is a t-ideal, for some natural number n, $(x^n, y^n)_v = dH \subseteq P$. But then $d \in P$ but d is in no other height-one prime. Using the fact that P is a height-one prime we can see that d is a base for P. Indeed, it is easy to see that nonunit divisors of all powers of d are again in P. We can write for some natural number n, $x^n = cd$. Now c is a nonunit by our assumption that x is in more than one height-one prime. Note that we cannot stop d, at this stage, from being non v-coprime with c. To be on the safe side let m be a natural number for which $x^n \mid d^m$ properly in H_P (The existence of such an m is assured by the fact that P is a minimal prime.) Now since H is an AGCD monoid there is a natural number k such that $\{x^{nk}, d^{mk}\}_v = bH$. Let $x^{nk} = x_1 b$ and $d^{mk} = y_1 b$ where $\{x_1, y_1\}_v = H$. We know that x_1 is a nonunit by our choice of the number of primes containing x and y_1 is a nonunit by our choice of m. Since $y_1 \mid d^{mk}$ we conclude that $x_1 \notin P$ and so x_1 is v-coprime with every base for P. ∎

Using the proof of Proposition 3.7 we can show that for each nonzero nonunit x in a weakly Krull AGCD monoid H we can express some power of x, uniquely, as a product of bases for height-one primes that contain x. Now if each height-one prime P of H has a fixed base b so that every other base of P is a power associate of b, then we can conclude that H has a Cale basis. Thus we have the following theorem.

Theorem 3.8. *For a weakly Krull monoid H, the following statements are equivalent.*

(1) *H is inside factorial.*

(2) *H is an AGCD monoid such that every height-one prime P of H has a base b such that every other base for P is a power associate of b.*

Proof. (3.8)⇒(3.8). By Proposition 2.4 H is an AGCD monoid. The facts that each height-one prime P has a base and that any other base for P is a power associate follow from [6, Theorem 2] and [6, Corollary 2], respectively.

(3.8)⇒(3.8). For each height-one prime P_α, let b_α be a base for P_α so that every other base for P_α is a power associate of P_α. We claim that $\{b_\alpha\}$ is a Cale basis for H which gives that H is inside factorial. Suppose that $\varepsilon b_{\alpha_1}^{n_1} \cdots b_{\alpha_s}^{n_s} = \varepsilon' b_{\alpha_1}^{n'_1} \cdots b_{\alpha_s}^{n'_s}$ where $\varepsilon, \varepsilon'$ are units and each $n_i, n'_i \geq 0$. Then $b_{\alpha_i}^{n_i} H_{P_{\alpha_i}} = \varepsilon b_{\alpha_1}^{n_1} \cdots b_{\alpha_s}^{n_s} H_{P_{\alpha_i}} = \varepsilon' b_{\alpha_1}^{n'_1} \cdots b_{\alpha_s}^{n'_s} H_{P_{\alpha_i}} = b_{\alpha_i}^{n'_i} H_{P_{\alpha_i}}$ and so $n_i = n'_i$. Next let x be a nonzero nonunit of H. Suppose that $P_1, \ldots, P_n$ are the height-one primes containing x. By Proposition 3.7 we have $x^r = x_1 b$ where b is a base for P_1, $r \geq 1$ and $\{x_1, b\}_v = H$. By hypothesis, b and b_1 are power associates, so $b^{s_1} = \varepsilon b_1^{t_1}$ for some unit ε and $s_1, t_1 \geq 1$. So $x^{rs_1} = \varepsilon x_1^{s_1} b_1^{t_1}$ where $x_1^{s_1}$ is contained in the height-one primes $P_2, \ldots, P_n$. By induction on n, $(x_1^s)^l = \varepsilon' b_2^{t_2} \cdots b_n^{t_n}$ where $l \geq 1$, ε' is a unit and each $t_i \geq 1$. Then $x^{rs_1 l} = \varepsilon^l (x_1^s)^l b_1^{lt_1} = \varepsilon^l \varepsilon' b_1^{lt_1} b_2^{t_2} \cdots b_n^{t_n}$ ∎

4. Integral domains with a Cale basis

In this short section we switch our focus from monoids to integral domains in order to relate some of the results of the previous sections to earlier work on integral domains. We begin by recalling the following result.

Theorem 4.1. [3, Theorem 3.4] *For an integral domain D the following are equivalent.*

(1) *D is almost weakly factorial, i.e., for each nonzero nonunit $x \in D$, there exists a natural number $n(x)$ such that $x^{n(x)}$ is a product of primary elements.*

(2) *For P a prime ideal minimal over a proper principal ideal (x), there exists a natural number $n = n(x,P)$ so that $x^n D_P \cap D$ is principal.*

(3) *$D = \bigcap_{P \in X^{(1)}(D)} D_P$ has finite character (i.e., D is weakly Krull) and $Cl_t(D)$ is torsion.*

Here $X^{(1)}(D)$ is the set of height-one prime ideals of D and $Cl_t(D)$ is the t-class group of D, that is, the group of t-invertible t-ideals (under t-multiplication) modulo its subgroup of principal fractional ideals. The term "$Cl_t(D)$ is torsion" means that for every t-invertible t-ideal I there is a natural number n such that $(I^n)_t$ is principal. Now [4, Theorem 3.4] an AGCD domain has torsion t-class group and the same proof also works for monoids. Thus by Theorem 4.1 a weakly Krull AGCD domain is almost weakly factorial. Proposition 4.2 below gives a condition under which an almost weakly factorial domain is an AGCD domain.

The integral domains characterized by Theorem 4.1 are called *almost weakly factorial domains*. Keeping this terminology for monoids, we see that if D is an almost weakly factorial monoid, then every height-one prime P of D contains a base b. Now, requiring that every other base for P is a power associate of a fixed base b we get our Cale basis.

Proposition 4.2. *Let D be an almost weakly factorial domain and suppose that every prime ideal P of D contains a base $b(P)$ such that every other base for P is a power associate of $b(P)$. Then D is an AGCD inside factorial domain.*

Proof. Let $Q = \{b = b(P) \mid P \in X^{(1)}(D)\}$. Let $F = \{\varepsilon b_1^{n_1} \cdots b_s^{n_s} \mid \varepsilon$ a unit, $b_i \in Q$, $n_i \geq 0\}$. As in the proof of Theorem 3.8, F is a factorial monoid. By Theorem 4.1 for every nonzero nonunit x there is a natural number n so that x^n is a product of mutually v-coprime base elements of primes. Say, $x^n = \beta_1 \cdots \beta_r$. Now by the hypotheses for each β_i there are m_i, n_i such that $(\beta_i)^{m_i} = \varepsilon_i (b_i)^{n_i}$ where $b_i = b(P_i)$ and ε_i is some unit. Taking $\operatorname{lcm}(\{m_i\}) = k$ we have $x^{nk} = \prod((\beta_i)^{m_i})^{k_i}$ where $k_i = k/m_i$. So, $x^{nk} = \varepsilon \prod (b_i)^{n_i k_i}$, a unique product of powers of elements of Q. So D is inside factorial. By Theorem 3.8, D is also an AGCD domain. ■

It is easy to see that Theorem 4.1 and Proposition 4.2 can be stated and proved for monoids. Leaving this to an interested reader let us close this note with an example of an AGCD inside factorial monoid.

Example 4.3. [11, Example 2.13] Let K be a field with characteristic $p \neq 0$ and let L be a purely inseparable extension of K such that $L^p \subseteq K$. Then $R = K + XL[X]$ is an inside factorial domain that is AGCD and obviously R is not integrally closed. For the AGCD part the reader may consult [11] and note that in this case $Q = \{f \mid f$ is a single representative of a nonzero prime in $K[X]\}$. This makes $F = K[X]$ and $H = R$ in the notation of Definition 2.1. Now note that for each $h \in H$, we have $h^p \in F$ and that F is a factorial monoid.

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