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Bezout rings and their subrings

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Introduction. The importance of principal ideal domains (PIDs), both in algebra itself and elsewhere in mathematics is undisputed. By contrast, Bezout rings,‡ although they represent a natural generalization of PIDs, play a much smaller role and are far less well known. It is true that many of the properties of PIDs are shared by Bezout rings, but the practical value of this observation is questioned by many on the grounds that most of the Bezout rings occuring naturally are in fact PIDs. However, there are several fairly natural methods of constructing Bezout rings from other rings, leading to wide classes of Bezout rings which are not PIDs, and it is the object of this paper to discuss some of these methods.

The best known examples of Bezout rings occurring 'in nature' are the ring of entire functions and the ring of all algebraic integers. These have been much discussed (especially the former) and we shall say no more about them.

Perhaps the most significant property of the class of Bezout rings which is not shared by the class of PIDs is that they can be defined by elementary sentences. It follows that the class of Bezout rings is closed under the formation of ultraproducts; in this way, starting from PIDs, we can obtain Bezout rings which are not PIDs or even ascending unions of PIDs. This method is outlined and illustrated in section 4.

A second construction which is rather more specific is described in section 3. To state the result we recall that an HCF-*ring* is an integral domain in which every pair of elements has a highest common factor (HCF). Further a ring R is said to be *inertly embedded* in a ring S (with the same 1) if the factorizations of an element of R are the same in S as in R. Thus, e.g. every inertly embedded subring of a Bezout ring is an HCF-ring (Theorem 3·1, Corollary 1). In the opposite direction we prove (Theorems 3·3 and 3·4):

To every HCF-ring R there corresponds a Bezout ring $\mathscr{B}(R)$ such that R is inertly embedded in $\mathscr{B}(R)$.

Moreover, $\mathscr{B}(R)$ is a PID if and only if R is a UFD. Thus a corresponding embedding theorem holds for UFDs and PIDs, giving in effect a method of constructing PIDs.

Technically it is advantageous to widen the class of HCF-rings to include all integrally closed integral domains in which any two factorizations of a given element have a common refinement; these rings are called *Schreier rings*. Their study is the subject of section 2. They show a very close similarity to UFDs, to which they reduce in the presence of the maximum condition for principal ideals. Nevertheless, they form a

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‡ See the beginning of section 1. All rings are understood to be commutative.

wider class than HCF-rings, as an example in section 2 shows. In view of this fact it is of interest to note that the Schreier property together with the Prüfer property is equivalent to the Bezout property (Theorem 2·8).

In section 1 some of the basic properties of Bezout rings are listed; these are intended for the reader's orientation and are for the most part well known, but no convenient reference seemed available.

It is a pleasure to acknowledge a stimulating conversation on the subject of this paper with Irving Kaplansky. I am also indebted to George M. Bergman for his helpful criticism of an earlier version.

1. *General properties.* Throughout, all rings are associative and commutative, with unit-element, written 1 and subrings have the same 1. By a *Bezout ring* we understand an integral domain in which the sum of two principal ideals is again principal. By induction it follows that every finitely generated ideal is principal. Moreover, the intersection of two principal ideals is also principal, so that we may speak of the highest common factor (HCF) and least common multiple (LCM) of two elements; in fact Bezout rings are a special case of HCF-rings, discussed in section 2.

An element of an integral domain is called an *atom* if it is a non-unit which cannot be written as a product of two non-units. If every element of a ring R which is not a unit or 0 can be written as a product of atoms, R is said to be *atomic*. The following result is easily verified:

PROPOSITION 1·1. *An integral domain is atomic if and only if it satisfies the maximum condition on principal ideals.*

The most important Bezout rings are of course PIDs and from Proposition 1·1 we obtain

PROPOSITION 1·2. *A Bezout ring is a principal ideal domain if and only if it is atomic.*

It is well known, and easily proved, that every Bezout ring is integrally closed (cf. e.g. (1) section 1, Ex. 20), but it need not be totally integrally closed (see section 4 below). A useful property of Bezout rings is the fact that their overrings in their quotient field are rather easily described.

Q.R Property

THEOREM 1·3. *Let R be a Bezout ring and K its quotient field. Then any ring T between R and K is again a Bezout ring, of the form $T = R_S$, where*

$$S = \{x \in R \mid x^{-1} \in T\}. \tag{1}$$

Proof. The set S defined by (1) is clearly multiplicative and contains 1, and the corresponding ring of quotients R_S is contained in T. Now let $x \in T$, say $x = ab^{-1}$ and write $aR + bR = kR$, then $a = ka_1$, $b = kb_1$ and $a_1u - b_1v = 1$ for some $u, v \in R$, hence $x = ab^{-1} = a_1b_1^{-1}$ and $b_1^{-1} = xu - v \in T$, therefore $b_1 \in S$ and $x \in R_S$. This shows that $T = R_S$; to prove that R_S is Bezout, let $x, y \in R_S$. Bringing x and y to a common denominator, we can write $x = ac^{-1}$, $y = bc^{-1}$ $(c \in S)$, then $aR + bR = kR$, say, hence $aR_S + bR_S = kR_S$ and so $xR_S + yR_S = kc^{-1}R_S$. Thus R_S is also a Bezout ring, as asserted.

Jensen (5) has shown that Prüfer rings† may be characterized as rings in which the lattice of ideals is distributive. Since every Bezout ring is a Prüfer ring, we deduce

† Recall that a Prüfer ring is an integral domain in which all finitely generated ideals are projective.

⇒ Theo: If R is a PID and Q its ~~the~~ field of quotients then every ring $S \ni R \subset S \subset Q$ is ~~again~~ a Bezout ring and is of the form R_T. where T is a multiplicative closed set with unity.

PROPOSITION 1·4. *In any Bezout ring, the lattice of ideals is distributive.*

This is also easily proved directly: we need only show that

$$\mathfrak{a} \cap (\mathfrak{b}+\mathfrak{c}) \subseteq \mathfrak{a} \cap \mathfrak{b} + \mathfrak{a} \cap \mathfrak{c},$$

for any ideals $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$. Let $a \in \mathfrak{a} \cap (\mathfrak{b}+\mathfrak{c})$, say $a = b+c$, where $b \in \mathfrak{b}$, $c \in \mathfrak{c}$, then

$$a \in aR \cap (bR+cR) = aR \cap bR + aR \cap cR,$$

because the lattice of principal ideals in R is distributive (every lattice group is distributive.)

Prüfer.

As a consequence Jensen shows that if R is a Bezout ring, then $R_{\mathfrak{M}}$ is a valuation ring† for every maximal ideal $\mathfrak{m}$ of R. More generally, R_S is a valuation ring if the complement of S in R is a prime ideal. The precise connexion between Bezout and valuation rings is given by Jaffard

PROPOSITION 1·5 (Krull). *An integral domain is a valuation ring if and only if it is a Bezout ring and a local ring.*‡ Prüfer ring and a local ring (invertible ideals in a local ring are principal)

Proof. Clearly a valuation ring is Bezout and local. Conversely if R is a ring satisfying these conditions, let $a, b \in R$, then $aR+bR = kR$ and $a = ka_1$, $b = kb_1$, whence $a_1R+b_1R = R$. Since R is local, either a_1 or b_1 is a unit, i.e. $aR \supseteq bR$ or $aR \subseteq bR$. Thus R is a valuation ring.

2. HCF-*rings and Schreier rings.* Let R be any integral domain with quotient field K, then the principal fractional ideals of R form a group which is partially ordered by taking the 'positive cone' to consist of the integral ideals. If this group is lattice-ordered, the ring R is called an *HCF-ring*. This means in effect that every pair of elements of R has an HCF and an LCM (cf. (4)). Since we are in an ordered group, it is enough to assume that either (i) any two elements have an HCF, or (ii) any two elements have an LCM, to ensure that we have an HCF-ring. However, an interesting asymmetry becomes apparent if this equivalence between (i) and (ii) is examined in detail. Let R be any integral domain and let a, b be two elements of R which have an LCM, m say. Then they necessarily have an HCF, namely $d = ab/m$. For, $d|a$, because $a/d = am/ab = m/b \in R$ and likewise $d|b$. Now if $d'|a$, $d'|b$, then ab/d' is a common multiple of a and b and hence of m, i.e. $md'|ab$, whence $d'|d$. The converse is false, i.e. two elements a, b may well have an HCF without having an LCM, as the following example shows:§ let R be the ring of polynomials in x with integer coefficients and even coefficient of x. The elements 2 and $2x$ have HCF 1 in R, but they have no LCM, for both $4x$ and $2x^3$ are common multiples, so the only possible LCM would be a common factor of $4x$ and $2x^3$ and it is easily seen that none of these fits. These results may be summarized as

THEOREM 2·1. *Let R be an integral domain; then* LCMs *exist in R if and only if* HCFs *exist. Moreover, if a, b have an* LCM, *then they have an* HCF, *but the converse does not always hold.*

† Recall that an integral domain is a *valuation ring* if its principal ideals are totally ordered by inclusion. ⟺ Any two ideals a, b are ⊇ a ⊂ b or b ⊂ a.

‡ This term is understood in its most general sense: a *local ring* is a ring in which the non-units form an ideal.

§ This answers a question raised by Jaffard (4), p. 81.

Instead of a Prüfer ring, with zero divisors, we stick to the definition: "A ring R such that if P is a regular prime ideal then R_P (or $R_{[P]}$) is a valuation ring (for every reg. prime ...)."

Theorem An H.C.F domain D is a U.F.D iff $\forall$ non unit $x \in D^*$ x has an pi chain.

It is easily verified that a unique factorization domain (UFD) can be characterized as an atomic HCF-ring. However, to test for a UFD it is enough to check an apparently weaker condition. Let us call an element p of an integral domain R *prime* if pR is a prime ideal in R. Clearly a prime is always an atom, but the converse need not hold. In fact an atomic integral domain is a UFD precisely if every atom is prime. Our object is to generalize this property to the case where no maximum conditions are imposed. It turns out that we get a wider class than that of HCF-rings in this way. Let us define an element c of an integral domain to be *primal* if

$$c|a_1a_2 \quad \text{implies that} \quad c = c_1c_2 \quad \text{such that} \quad c_1|a_1, c_2|a_2. \tag{2}$$

Clearly an atom is primal if and only if it is prime. Now we make the

DEFINITION. A *Schreier ring* is an integrally closed integral domain in which every element is primal.

The reason for the name will become apparent soon, when it is shown that these rings may also be characterized by the fact that the Schreier refinement property holds for factorizations. Clearly a unit is always primal, as is zero, so in proving that a ring is Schreier it is enough to verify that the non-units different from zero are primal. In the factorizations which occur below, the factors will be tacitly assumed to be different from zero.

THEOREM 2·2. *A ring R is a Schreier ring if and only if it is an integrally closed integral domain such that for any two factorizations of an element $a\,(\neq 0)$ of R,*

$$a = p_1 \dots p_m = q_1 \cdots q_n \tag{3}$$

there exist elements $r_{ij}\ (i = 1, \dots, m,\ j = 1, \dots, n)$ such that

$$p_i = \prod_j r_{ij}, \quad q_j = \prod_i r_{ij}. \tag{4}$$

In other words: any two factorizations of a have a common refinement.

Proof. Suppose first that the refinement property holds in R, and let $c|a_1a_2$, say

$$a_1a_2 = bc,$$

then by hypothesis there exist $d_{ij}\ (i,j = 1, 2)$ such that

$$a_1 = d_{11}d_{12}, \quad a_2 = d_{21}d_{22}, \quad b = d_{11}d_{21}, \quad c = d_{12}d_{22}.$$

Now the conclusion of (2) follows if we put $c_i = d_{i2}$. Hence c is primal, and this shows R to be a Schreier ring. Conversely, assume R to be a Schreier ring and let

$$a_1a_2 = b_1b_2,$$

then $b_1|a_1a_2$ and hence there is a factorization

$$b_1 = c_{11}c_{21} \quad \text{such that} \quad c_{i1}|a_i \quad (i = 1, 2). \tag{5}$$

Write $a_i = c_{i1}c_{i2}$, then $\qquad b_1b_2 = a_1a_2 = c_{11}c_{12}c_{21}c_{22}.$

If we cancel b_1 (using (5)), we find that $b_2 = c_{12}c_{22}$ and this proves the refinement property when $m = n = 2$. To prove it generally we use double induction, on m and n.

Let
$$a = p_1 \dots p_m = q_1 \dots q_n.$$
By what has been proved, there exist c_{ij} $(i, j = 1, 2)$ such that
$$p_1 = c_{11}c_{12}, \quad q_1 = c_{11}c_{21}, \quad p_2 \dots p_m = c_{21}c_{22}, \quad q_2 \dots q_n = c_{12}c_{22}.$$
By induction there exist u_i, v_i $(i = 2, \dots, m)$ such that
$$p_i = u_i v_i, \quad c_{21} = \Pi u_i, \quad c_{22} = \Pi v_i,$$
and there exist x_j, y_j $(j = 2, \dots, n)$ such that
$$q_j = x_j y_j, \quad c_{12} = \Pi x_j, \quad c_{22} = \Pi y_j.$$
Hence
$$c_{22} = v_2 \dots v_m = y_2 \dots y_n.$$
Again by induction we find r_{ij} $(i = 2, \dots, m, j = 2, \dots, n)$ such that
$$\prod_i r_{ij} = y_j, \quad \prod_j r_{ij} = v_i.$$
Moreover, it follows that
$$p_1 = c_{11} \prod_j x_j, \quad p_i = u_i \prod_j r_{ij} \quad (i = 2, \dots, m),$$
$$q_1 = c_{11} \prod_i u_i, \quad q_j = x_j \prod_i r_{ij} \quad (j = 2, \dots, n).$$
If we set $r_{11} = c_{11}$, $r_{1j} = x_j$, $r_{i1} = u_i$ we obtain (4) and so the refinement property holds generally. This completes the proof.

It is clear that the Schreier rings form a local class, i.e. a ring R is Schreier whenever it can be expressed as the union of a directed family of Schreier rings. More generally, any direct limit of Schreier rings is a Schreier ring. Further, any UFD is a Schreier ring; more precisely, we have G.U.F.D

THEOREM 2·3. *A ring R is a unique factorization domain if and only if it is atomic and a Schreier ring.* : generalized Krull and Schreier

Proof. That a UFD satisfies these conditions is clear. Conversely, when R satisfies the conditions, every non-unit different from 0 is a product of atoms and if we are given two complete factorizations of a, as in (3) say, where the p_i and q_j are now atoms, then the matrix (r_{ij}) such that (4) holds has just one non-unit in each row and each column. By permuting $q_1, \dots, q_n$ we can permute the columns so as to have non-units down the main diagonal. This shows the matrix (r_{ij}) to be square and p_i to be associated to q_i. Hence R is a UFD.

The sufficiency could also be proved by observing that in a Schreier ring every atom is prime and using the fact proved later (Lemma 2·5) that any product of primal elements is again primal.

The condition for R to be a Schreier ring, when expressed in terms of the ordered group of fractional principal ideals of R, becomes just the Riesz interpolation property:

$R_{m,n}$. *Given $a_1, \dots, a_m, b_1, \dots, b_n \in R$ such that $a_i | b_j$, there exists $c \in R$ such that $a_i | c | b_j$.*

Thus in contrast to an HCF-ring, where we can find a *highest common factor* of a finite set of elements, in a Schreier ring we can find a *higher common factor* than any

if a, b are co-prime in a Schreier ring Then $aR \cap bR = ~~abR~~$.
supposing on the contrary
here $abR \in aR \cap bR$.
let $x \in aR \cap bR$, Then $x = x_1 a = x_2 b$.
Now $b | x_1 a$ ∴ by definition $b = b_1 b_2$
where $b_1 | x_1$, $b_2 | a$ but as $(b, a) = 1$
∴ $b | x_1$ ∴ $x \in abR$.

given finite set of common factors. As in the case of HCF-rings it is enough to assume $R_{m,n}$ for $m = n = 2$. For this fact and other equivalent forms of $R_{m,n}$ we refer to Fuchs (3). In particular, every lattice-ordered group has the Riesz property, a fact which for rings is expressed by

THEOREM 2·4. *Every* HCF-*ring is a Schreier ring.*

The converse is false, as the following example shows, of a Schreier ring which is a union of HCF-rings but not itself an HCF-ring. Let G be the semigroup of all pairs of non-negative rationals and $F[G]$ the semigroup algebra over a field F. Then $F[G]$ is an HCF-ring (and hence a Schreier ring); for we can write G as a union of semigroups $N \times N$ (where N is the additive semigroup of non-negative integers) and hence $F[G]$ is a union of polynomial rings $F[x, y]$ over F. For any positive integer n, let Γ_n be the subsemigroup of G consisting of all α, β satisfying $\beta/n \leqslant \alpha \leqslant \beta n$. Each Γ_n is isomorphic to G, under the mapping defined by

$$\begin{pmatrix} 1 & n \\ n & 1 \end{pmatrix},$$

hence $F[\Gamma_n]$ is again a Schreier ring; writing $\Gamma = \bigcup \Gamma_n$ we see that $F[\Gamma]$ is also a Schreier ring (as union of Schreier rings). However, it is not an HCF-ring; for Γ consists of all pairs of *positive* rationals together with $(0, 0)$ and so, e.g. $(1, 2)$ and $(2, 1)$ have no HCF in $F[\Gamma]$.

We now consider localization for Schreier rings. An element of an integral domain R is said to be *completely primal* if all its factors are primal. A subset S of a ring R is said to be *multiplicative* if it is a subsemigroup of the multiplicative semigroup of R, i.e. if S is closed under multiplication and contains the unit-element; if moreover, any factor of an element of S again lies in S, then S is said to be *saturated.*

LEMMA 2·5. *In an integral domain, any product of* (*completely*) *primal elements is* (*completely*) *primal. Moreover, the set of all completely primal elements is saturated.*

Proof. Let p, q be primal and assume that $pq|a_1a_2$, then $p = p_1p_2$ and $p_i|a_i$. Writing $a_i = p_ir_i$, we have

$$a_1a_2 = p_1r_1p_2r_2 = pqs, \text{ say},$$

hence $r_1r_2 = qs$, i.e. $q|r_1r_2$, whence $q = q_1q_2$ and $q_i|r_i$. Thus $pq = p_1q_1p_2q_2$ and

$$p_iq_i|p_ir_i = a_i,$$

which shows pq to be primal. By induction it follows that any product of primal elements is again primal. In particular, the product of any two completely primal elements is primal and it only remains to consider factors of such products. Thus let p, q be completely primal and

$$pq = ab,$$

Then $p = p'p''$, $p'|a$, $p''|b$, say

$$a = p'q', \quad b = p''q'';$$

on multiplying out we find that $q'q'' = q$. Since p, q are completely primal, it follows that p', q' are primal and by the first part, so is a. This shows pq to be completely primal, and again the result extends to any number of factors by induction. The last assertion is clear from the definition.

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The behaviour of Schreier rings under localization is described by the next result, which is analogous to Nagata's theorem for UFDs (cf. (6)):

THEOREM 2·6. *Let R be an integrally closed integral domain and S a multiplicative subset of R. Then*

(i) *if R is a Schreier ring, so is R_S,*

(ii) *if R_S is a Schreier ring and S is generated by completely primal elements of R, then R is a Schreier ring.*

Of course in (ii) we could simply have assumed that S is a multiplicative set consisting of completely primal elements, by Lemma 2·5, but the hypothesis actually stated may be easier to verify in practice.

Proof. (i) Let $c|a_1a_2$ in R_S, say $a_1a_2 = bc$. Then there exist $s_1, s_2, t_1, t_2 \in S$ such that $s_1a_1, s_2a_2, t_1b, t_2c \in R$ and

$$(t_1s_1a_1)(t_2s_2a_2) = s_1s_2t_1bt_2c.$$

Now $s_1s_2t_1t_2c \in R$, hence by the Schreier property for R,

$$s_1s_2t_1t_2c = c_1c_2, \quad \text{where} \quad c_i|s_it_ia_i \quad (i = 1, 2).$$

Write $c_i' = c_i/s_it_i$, then $c_i' \in R_S$, $c_1'c_2' = c$ and $c_i'|a_i$ (in R_S). Thus every element of R_S is primal.

(ii) By Lemma 2·5, every element of S is completely primal; we have to show that every element of R is primal. Given

$$c|a_1a_2 \quad \text{in } R,$$

by hypothesis we can write $c = c_1c_2 (c_i \in R_S)$ such that $a_i/c_i \in R_S$. Hence

$$c_i = c_i'/s_i\,(c_i' \in R, s_i \in S), \quad s_ic_i = c_i'$$

and so

$$c_1'c_2' = s_1s_2c_1c_2 = s_1s_2c.$$

Write $s = s_1s_2$, then s is primal, hence $s = t_1t_2$ and $t_i|c_i'$ in R. Thus if we put $c_i'' = c_i'/t_i$, then $c_i'' \in R$ and

$$c_1''c_2'' = c. \tag{6}$$

Moreover, since $a_i/c_i \in R_S$ and $t_ic_i'' = c_i' = s_ic_i$, it follows that $a_i/c_i'' \in R_S$, i.e.

$$u_ia_i = c_i''b_i \quad (u_i \in S, b_i \in R). \tag{7}$$

Omitting the primes from c_i'' we have thus obtained a factorization

$$c = c_1c_2, \quad u_ia_i = c_ib_i \quad (b_i \in R, u_i \in S, i = 1, 2). \tag{8}$$

Further, $a_1a_2 = ck$, for some $k \in R$, therefore $u_1u_2a_1a_2 = u_1u_2ck = c_1c_2b_1b_2$ and so

$$u_1u_2k = b_1b_2.$$

Now u_1 is primal and $u_1|b_1b_2$, hence $u_1 = u_{11}u_{12}$, $u_{1i}|b_i$, say $b_1 = u_{11}b_1'$. Inserting this value in (8), we obtain

$$u_{11}u_{12}a_1 = u_{11}b_1'c_1,$$

hence $u_{12}|b_1'c_1$. Here u_{12} is again primal, so $u_{12} = vw$, $v|b_1'$, $w|c_1$. Thus $u_1 = u_{11}vw$, $u_{11}v|b_1$, $w|c_1$ and $w|b_2$ (because $w|u_{12}|b_2$). We have now the following factorizations in R:

$$a_1 = (c_1/w)(b_1/u_{11}v), \quad c = (c_1/w)(c_2w), \quad u_2a_2 = (c_2w)(b_2/w).$$

(i/δ) Theorem 2·6 can be restated as:
If R_S is an H.C.F ring so is R where S is mo-
generated by prime quanta. No there is a snag
here
(NO.)

Therefore if in (8) we replace b_1, b_2, c_1, c_2 by $b_1/u_{11}v$, b_2/w, c_1/w, c_2w respectively, we obtain the same set of equations with u_1 replaced by 1 and u_2 left unchanged. Repeating the procedure we can reduce u_2 also to 1, and then we obtain the system

$$c = c_1 c_2, \quad a_i = c_i b_i. \tag{9}$$

This shows c to be primal in R. Since c was any non-zero element of R, this proves R to be a Schreier ring.

By imposing the maximum condition on principal ideals we obtain again Nagata's theorem, in the following slightly more general form (cf. (6), p. 31):

COROLLARY. *Let R be an atomic integral domain and S a multiplicative subset. If R_S is a* UFD *and S is generated by primes, then R is itself a* UFD.

Of course this result may also be proved directly without any difficulty, if we use the remarks following Theorem 2·1 above. We need only verify that an atom of R either divides an element of S (and hence is prime because S is generated by primes) or it stays an atom in R_S (and hence is prime because R_S is a UFD).

As an application of Theorem 2·6 we show that the Schreier property is preserved by polynomial extension; this result will be needed later.

THEOREM 2·7. *Let R be a Schreier ring and x an indeterminate, then $R[x]$ is again a Schreier ring.*

Proof. It is well known (cf. (4), p. 99) that $R[x]$ is again an integrally closed integral domain. Let K be the quotient field of R and S the multiplicative subset consisting of all non-zero elements of R. Then $K = R_S$ and

$$R[x]_S = R_S[x] = K[x]$$

is a *UFD*. So the result will follow by Theorem 2·6 if we can show that S is generated by completely primal elements. Let $c \in S$ and $c | fg$, where $f, g \in R[x]$, say $f = \Sigma a_i x^i$, $g = \Sigma b_j x^j$. Since R is integrally closed, we can apply Kronecker's lemma ((4), p. 99) and conclude that $c | a_i b_j$ for all i, j, i.e.

$$1, c/a_i | c, b_j.$$

By the Riesz interpolation property there exists $d \in K$ such that

$$1, c/a_i | d | c, b_j, \quad \text{i.e.} \quad d, c/d \in R \quad \text{and} \quad c/d | a_i, \quad d | b_j.$$

Thus $c/d | f$, $d | g$ and this shows c to be primal in $R[x]$. Since every factor of c is again in S, c is completely primal and the result follows.

To explore the relation between Schreier rings and Bezout rings further, let us recall that PIDs may be characterized as UFDs which are also Dedekind rings. Analogously, Bezout rings may be characterized as HCF-rings which are also Prüfer rings† (cf.(1), section 2, Ex. 17). This follows also from the next result, which is slightly more general:

THEOREM 2·8. *A ring is a Bezout ring if and only if it is a Schreier ring and a Prüfer ring.*

† Recall that a Prüfer ring is an integral domain in which all finitely generated ideals are invertible.

Proof. Clearly any Bezout ring is both Schreier and Prüfer. Conversely, let R be Schreier and Prüfer; we have to show that any ideal generated by two elements is principal. Since R is a Prüfer ring, such an ideal is projective as R-module, so we need only show that any 2-generator projective R-module is free. Such a module is determined by its projection from R^2, i.e. by an idempotent 2×2 matrix, and it suffices to show that every such matrix which is different from O and I is similar to

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let

$$E = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

be an idempotent matrix different from O and I. Since E is certainly similar to E_1 over the quotient field of R, E and E_1 have the same trace and the same determinant, i.e.

$$a + d = 1, \quad ad - bc = 0.$$

Since R is a Schreier ring, we have $a = a_1 a_2$, $d = d_1 d_2$, $b = a_1 d_2$, $c = a_2 d_1$, i.e.

$$E = \begin{pmatrix} a_1 a_2 & a_1 d_2 \\ a_2 d_1 & d_1 d_2 \end{pmatrix} \quad \text{where} \quad a_1 a_2 + d_1 d_2 = 1,$$

$$= \begin{pmatrix} a_1 & -d_2 \\ d_1 & a_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_2 & d_2 \\ -d_1 & a_1 \end{pmatrix},$$

thus E is similar to E_1 as asserted.

3. *Bezout rings from Schreier rings.* Our objective is to find a fairly wide class of rings which can be embedded in Bezout rings. Of course some limitation on the embedding is necessary, for every integral domain can be embedded in a field and this is certainly a Bezout ring. We shall therefore restrict ourselves to embeddings which do not change the possible factorizations of existing elements. To be precise, let us make the following

Definition. Given a ring R, a subring P is said to be *inertly embedded* in R and R/P is an *inert extension*† if any factorization in R of an element of P also lies in P, i.e. given $a \in P$, if $a = a_1 a_2$, where $a_i \in R$, then $a_i \in P$.

It follows from this definition that R and P have the same units. Our first aim is to prove that the HCF-property is preserved by inert embeddings.

Theorem 3·1. *Let R be an* HCF-*ring, then any inertly embedded subring P is again an* HCF-*ring, and the* HCF *of any two elements of P is the same in P as in R.*

Proof. Suppose that P is inertly embedded in R, and let $a, b, \in P$ have an HCF d in R:

$$a = da_0, \quad b = db_0 \quad \text{and} \quad d'|a, \quad d'|b \quad \text{in} \quad R \text{ implies} \quad d'|d.$$

By inertia the factors d, a_0, b_0 all lie in P. Now let d' be a common factor of a, b in P, then since d is the HCF in R, we have $d = d'u$ in R and hence in P. Thus $d'|d$ in P and this shows d to be the HCF in P also.

† Strictly speaking this should be called a *strongly inert extension* in contrast to an extension in which $a = a_1 a_2$ $(a_i \in R)$ implies $a_1 u$, $u^{-1} a_2 \in P$ for some unit u in P. This reduces to the definition given in the text if we add the assumption that every unit of R lies in P.

Since a Bezout ring is HCF and an HCF-ring is Schreier, we have

COROLLARY 1. *Any inertly embedded subring of a Bezout ring is an* HCF-*ring and hence a Schreier ring.*

COROLLARY 2. *Any inertly embedded subring of a* PID *is a* UFD.

For by Corollary 1, an inertly embedded subring P of a PID R is an HCF-ring. Further, any non-unit $\neq 0$ of P is a product of atoms in R and hence in P. Therefore P is atomic and so is a UFD.

Our aim will be to prove that conversely, any UFD can be embedded in a PID and any HCF-ring can be embedded in a Bezout ring. It is convenient to prove a slightly more general result, starting from Schreier rings. A pair of elements a, b of a ring R is said to be *coprime*, if a, b have no common factor in R (apart from units); if moreover there exist $c, d \in R$ such that $ad - bc = 1$, then a, b are said to be *comaximal*. Clearly any comaximal pair is coprime; the converse is not true in general, though it does hold in Bezout rings, as is easily seen. We now define a *pre-Bezout ring* to be an integral domain in which every coprime pair is comaximal. These rings are related to Bezout rings by the following proposition, whose proof may be left to the reader:

PROPOSITION 3·2. *A ring is a Bezout ring if and only if it is a pre-Bezout ring and an* HCF-*ring.*

This result does not hold with 'HCF-ring' replaced by 'Schreier ring', as we shall see later (Theorem 3·4, Corollary 2).

We can now state our main result:

THEOREM 3·3. *Given any Schreier ring R, there exists a pre-Bezout ring $\mathscr{B}(R)$ such that $\mathscr{B}(R)/R$ is an inert extension; moreover, $\mathscr{B}(R)$ is again a Schreier ring.*

Proof. Let a, b be a coprime pair in R and consider the polynomial ring $S = R[x, y]$; by Theorem 2·7, this is again a Schreier ring. Write

$$z = ax - by \tag{10}$$

and consider the ring of quotients with respect to the multiplicative set generated by z, say $S_{(z)}$. This is again a Schreier ring by Theorem 2·6; it is obtained by adjoining x, y, z, z^{-1} to R, subject to the relation (10). Let T be the subring of $S_{(z)}$ consisting of elements of degree zero in x, y, z. We assert that (i) T is Schreier and (ii) T/R is an inert extension. Thus assume that

$$cd = e_1 e_2 \quad \text{in } T,$$

then c, d, e_1, e_2 are all homogeneous of degree 0. Since $S_{(z)}$ is Schreier, $c = c_1 c_2$ and $c_i | e_i$ in $S_{(z)}$. Clearly c_1, c_2 are homogeneous and their degrees add up to 0, so on replacing c_1, c_2 by $c_1 z^n$, $c_2 z^{-n}$ for suitable n, we can ensure that c_1, c_2 are both of degree 0. Let $e_i = c_i d_i$, then d_i has degree 0 and hence lies in T, therefore $c_i | e_i$ in T; this shows T to be a Schreier ring.

We have
$$R \subseteq R[x, y] = S \subseteq S_{(z)} \quad (R \subseteq T).$$

Thus R is embedded in T. Every element of T has the form

$$f z^{-n}, \tag{11}$$

Proof positive that my children and I used to read together ... and occasionally they would help.

where f is a form of degree n in x and y. Now assume that an element c of R can be factorized in T:

$$c = c_1 c_2, \quad c_i = f_i z^{-n_i} \quad (c \in R), \tag{12}$$

where f_i is a form of degree n_i in x and y which is not divisible by $z = ax - by$. Multiplying up, we obtain the equation

$$cz^{n_1+n_2} = f_1 f_2$$

in S and since a and b are coprime in S, $z = ax - by$ is an atom in S; hence z is prime (because S is Schreier) and so, if $n_1 + n_2 > 0$, then $z|f_1$ or $z|f_2$. This contradicts the choice of the f_i, hence $n_1 = n_2 = 0$ and (12) is a factorization in R. This shows T/R to be an inert extension.

By transfinite induction (or by carrying out all the necessary adjunctions simultaneously) we reach a Schreier ring R_1 such that R_1/R is an inert extension and every coprime pair in R is comaximal in R_1. Repeating the process, we obtain a chain

$$R = R_0 \subseteq R_1 \subseteq R_2 \subseteq \dots \tag{13}$$

whose union U is again a Schreier ring (because being Schreier is a local condition). Secondly, U/R is an inert extension, for if $c \in R$ has a factorization $c = c_1 c_2$ in U, choose the least n such that $c_i' \in R_n$ $(i = 1, 2)$; if $n > 0$, then since R_n/R_{n-1} is inert, $c_i \in R_{n-1}$. This shows that in fact $n = 0$, i.e. $c_i \in R$. Thirdly, U is pre-Bezout, for if a, b are coprime in U, then $a, b \in R_n$ say, hence a, b are comaximal in R_{n+1} and so also in U. This completes the proof.

We shall call a ring $\mathscr{B}(R)$ satisfying the conditions of Theorem 3·3 a *pre-Bezout hull* of R. In general there will be more than one pre-Bezout hull. Thus suppose for a moment that a, b are already comaximal in R, say

$$ad - bc = 1. \tag{14}$$

If x and y are again indeterminates adjoined to R, then the transformation

$$x' = ax - by,$$

$$y' = cx - dy,$$

is invertible, by (14). Thus $R[x, y] = R[x', y']$ and writing again $z = ax - by$ we have $R[x, y, z^{-1}] = R[x', y', x'^{-1}]$. Thus the ring T constructed in the proof consists of all elements $f(x,' y')/x'^n$, where f is a form of degree n in x' and y'. In other words, T has the form $R[t]$, where $t = y'/x'$ is an indeterminate.

This observation suggests that different pre-Bezout hulls are obtained from a fixed 'minimal' one by adjoining indeterminates and again forming a pre-Bezout hull, but we have not been able to prove this supposition. However, this is immaterial in what follows. To apply our theorem we look at pre-Bezout hulls in the case of HCF-rings and UFDs. The situation here is described in the next theorem.

THEOREM 3·4. *Let R be a Schreier ring and $\mathscr{B}(R)$ a pre-Bezout hull; then*
(i) *$\mathscr{B}(R)$ is a Bezout ring if and only if R is an* HCF*-ring,*
(ii) *$\mathscr{B}(R)$ is a* PID *if and only if R is a* UFD.

Proof. If $\mathscr{B}(R)$ is Bezout, then R is an HCF-ring by Theorem 3·1, Corollary 1. Conversely, if R is an HCF-ring, then so is the polynomial ring $R[x]$ (cf. (1) section 1, Ex. 23 or (4), p. 100). Now consider the ring T constructed in the proof of Theorem 3·3. To show that this is an HCF-ring, let us take any two elements of T; they have the form fz^{-m}, gz^{-n} and in fact we may take $m = n$, without loss of generality. Now

$$f, g \in S = R[x, y],$$

they have an HCF in S, say d, and if this is of degree k then $dz^{-k} \in T$ and this is easily verified to be the HCF of fz^{-n} and gz^{-n} in T. Thus T is an HCF-ring, and by transfinite induction, so is $U = \mathscr{B}(R)$. Together with the pre-Bezout property this shows $\mathscr{B}(R)$ to be a Bezout ring, by Proposition 3·2.

To prove (ii) assume that $\mathscr{B}(R)$ is a PID, then any non-unit $\neq 0$ in R is still a non-unit in $\mathscr{B}(R)$ and so may be written as a product of atoms. These factors must lie in R and are again atoms in R. Hence R is atomic and being a Schreier ring it is a UFD (by Theorem 2·3). Conversely, if R is a UFD, it is atomic and we need only check that this property is preserved in the construction of Theorem 3·3. Clearly the ring T constructed there is atomic and U is the union of an ascending well-ordered system of atomic rings. Let $a \in U$ and take an inert atomic subring of U containing a. Then a can be written as a product of atoms in this subring and this atomic factorization still holds good in U. Hence $U = \mathscr{B}(R)$ is atomic and Bezout, i.e. it is a PID, as asserted.

This theorem gives us our construction of Bezout rings; in detail we have

COROLLARY 1. *Every* HCF-*ring can be inertly embedded in a Bezout ring and every* UFD *can be inertly embedded in a* PID.

We can now also substantiate the remark made after Proposition 3·2:

COROLLARY 2. *There exists a pre-Bezout ring which is a Schreier ring but not a Bezout ring.*

For let R be a Schreier ring which is not an HCF-ring and form its pre-Bezout hull $\mathscr{B}(R)$. By Theorem 3·3 this is pre-Bezout and Schreier; if it were a Bezout ring, then by Theorem 3·3 (i), R would be an HCF-ring, which is not the case.

Thus we have the following relation between Schreier rings and their pre-Bezout hulls, in increasing order of generality:

Ring:	UFD	HCF-ring	Schreier ring
pre-Bezout hull:	PID	Bezout ring	pre-Bezout ring

An inert subring of a pre-Bezout ring need not be Schreier. This is shown by the following example (due to G. M. Bergman) of a pre-Bezout ring, which is not Schreier. Let G be the additive semigroup of all rationals $\geqslant 0$ and reals $\geqslant 1$, form the semigroup algebra $F[G]$ and let $F(G)$ be the ring obtained by adjoining inverses of all elements with non-zero constant term. Then no two non-units of $F(G)$ are coprime: any two have a common factor of the form (α) for sufficiently small α. But the Schreier property fails: consider the equation $(1)+(2) = (\sqrt{2})+(3-\sqrt{2})$ in G. If (1) were primal, say $(1) = (a)+(1-a)$, $(2) = (\sqrt{2}-a)+(2-\sqrt{2}+a)$, then a must be rational, hence

$$\sqrt{2}-a > 1, \quad 2-\sqrt{2}+a > 1, \quad \text{i.e.} \quad \sqrt{2}-a < 1,$$

a contradiction. Thus G does not have the Schreier property, hence neither does $F(G)$.

4. *Ultraproducts.* The construction of Bezout rings from HCF-rings in the last section was a particularly economical one. We now come to a second way of forming Bezout rings, which by comparison is particularly prodigal, namely as ultraproducts. The formation of ultraproducts preserves elementary properties (cf. e.g. (2), V. 5) and since Bezout rings can clearly be defined by elementary sentences, every ultraproduct of Bezout rings is again a Bezout ring. The interest of this method resides in the fact that starting with a family of PIDs we obtain Bezout rings from them by forming ultraproducts, and these are not in general PIDs. Examples of this phenomenon confirm the rather plausible surmise that the class of PIDs can*not* be defined by elementary sentences.

We briefly recall the definition of ultraproducts, in a form adapted to the present context. Let R_i $(i \in I)$ be any family of rings and let $P = \Pi R_i$ be their direct product, with projections $\epsilon_i : P \to R_i$. With each element $x \in P$ we associate its *zero-set*

$$N(x) = \{i \in I \mid x\epsilon_i = 0\}.$$

Let $\mathscr{D}$ be a filter on the index-set I and define a subset $\mathfrak{a}(\mathscr{D})$ of P by the equation

$$\mathfrak{a}(\mathscr{D}) = \{x \in P \mid N(x) \in \mathscr{D}\}.$$

It is easily seen that $\mathfrak{a} = \mathfrak{a}(\mathscr{D})$ is an ideal in P and the quotient $P/\mathfrak{a}$ is called the *reduced product* associated with the filter $\mathscr{D}$, and is also written $P/\mathscr{D}$. We shall only be concerned with the case where $\mathscr{D}$ is an ultrafilter on I, in which case $P/\mathscr{D}$ is called an *ultraproduct* of the R_i, or in case all the R_i are equal to R, an *ultrapower* of R. Now the basic ultraproduct theorem (cf. e.g. (2), Theorem V. 5·1) asserts that an elementary property holds in the ultraproduct $P/\mathscr{D}$ precisely if it holds in the factors R_i for i running over some set of $\mathscr{D}$. In particular, any elementary property holding in all the factors also holds in the ultraproduct. If $\mathscr{D}$ is a principal ultrafilter, $P/\mathscr{D}$ is isomorphic to one of the factors R_i, so this case gives nothing of interest and may be excluded.

Let us consider the special case of an ultrapower, $R^I/\mathscr{D}$ more closely. We remark that the diagonal mapping $R \to R^I$ combined with the canonical homomorphism $R^I \to R^I/\mathscr{D}$ gives a canonical embedding

$$R \to R^I/\mathscr{D}, \tag{15}$$

which is actually an elementary embedding ((2), VI. 3), as is easily verified. As a consequence we have

PROPOSITION 4·1. *Let R be a ring in which any element has only a finite number of factors. Then the embedding* (15) *of R in any ultrapower is inert.*

Of course this result may also be verified directly, a task which is left to the reader.

To obtain a concrete example, let R be the ultrapower of the ring Z of integers over a countable index-set (with respect to a non-principal ultrafilter) and write

$$x = (p_1, p_1^2 p_2^2, p_1^3 p_2^3 p_3^3, \ldots),$$

where $p_1, p_2, \ldots$ are the rational primes in ascending order. Then x (more precisely, its residue class in $Z^I/\mathscr{D}$) is divisible by any non-zero element of Z. Thus we obtain a subring T of R by taking all polynomials in x with rational coefficients, except for the constant term which is in Z. We observe that T is a Bezout ring. For if $f, g \in T$, we

may assume that f, g have no common factor of positive degree in x. Hence there exist $u, v, \in T$ such that
$$fu - gv = \gamma,$$
where γ is a non-zero integer. Now if the constant terms of f, g are α, β respectively, say
$$f = \alpha + f_1, \quad g = \beta + g_1,$$
where f_1, g_1 have zero constant term, then
$$\alpha = f - (f_1/\gamma)\gamma \in fT + gT,$$
and similarly $\beta \in fT + gT$, hence $\delta = (\alpha, \beta)$ divides γ. It follows that δ divides f, g and so $\delta T = fT + gT$.

The ring T is an example of a Bezout ring which cannot be expressed as an ascending union of PIDs. For if it could be so expressed, then any finite subset of T would be contained in a PID; choose a prime p and assume that there is a subring S of T which contains x and p and is a PID. Let $xS + pS = dS$, then d must be 1 or p, since these are the only factors of p, even in T. If $d = 1$, then $xu + pv = 1$ for some $u, v \in S$ and this leads to a contradiction, by equating the constant terms; hence $d = p$, i.e. $x/p \in S$. The same argument shows that if $x/p^n \in S$, then $x/p^{n+1} \in S$. Thus S contains $x, x/p, x/p^2, \ldots$, and by hypothesis the ideal generated by these elements in S is principal. Let f be a generator, then
$$xS + (x/p)S + \ldots = fS,$$
hence $xT + (x/p)T + \ldots = fT$, which is a contradiction.

We also note that T itself cannot be expressed as an ultraproduct in a non-trivial way, because it is countable, whereas any non-trivial ultraproduct is uncountable, by the results of (2), VI. 6. However, T may be expressed in terms of Steinitz numbers ('supernatural' numbers) by interpreting x as Πp_i^∞, taken over all primes. This interpretation also suggests other Bezout rings, which can be formed in this way, and it may be instructive to compare these with the inert subrings of the ultrapower $Z^I/\mathscr{D}$.

Finally, we note that the ring T constructed here is an example of a Bezout ring which is not totally integrally closed, because $xp^{-n} \in T$ for all n, yet $p^{-1} \notin T$.

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