SOME REMARKS ON DISTINGUISHED DOMAINS

D.D. ANDERSON, DONG JE KWAK, AND MUHAMMED ZAFRULLAH

ABSTRACT. Heitmann and McAdam defined an integral domain R to be *distinguished* if for $0 \neq z \in K$, the quotient field of R, $(1):(z) \not\subseteq Z(R/(1):(z^{-1}))$. They showed that a Prüfer domain and a Krull domain are distinguished. We investigate the relationship between distinguished domains and PVMD's. We show that a two-dimensional distinguished domain is a PVMD, but give an example of a three-dimensional distinguished domain that is not a PVMD. We define an integral domain R to be *super distinguished* if for $a, b \in R - \{0\}$, there exist $r \in (a):(b)$ and $s \in (b):(a)$ with $(r,s)_t = R$ and show that a super distinguished domain is distinguished and a PVMD.

Throughout R denotes an integral domain with quotient field K. For an R-module M, $Z(M)$ denotes the set of zero divisors of R with respect to M and for $x, y \in K$, $(x):(y) = \{r \in R \mid ry \in (x)\}$. Heitmann and McAdam [1] introduced the notion of a distinguished domain. We recall their definition.

Definition 1. An integral domain R (with quotient field K) is *distinguished* if for each $0 \neq z \in K$, $(1):(z) \not\subseteq Z(R/(1):(z^{-1}))$.

We first give several conditions equivalent to R being distinguished.

Proposition 2. *For an integral domain R with quotient field K, the following conditions are equivalent.*

(1) *R is distinguished.*
(2) *For $a, b \in R - \{0\}$, $(b):(a) \not\subseteq Z(R/(a):(b))$.*
(3) *For $0 \neq z \in K$, we can write $z = a/b$, $a, b \in R$, where $b \notin Z(R/(a):(b))$.*
(4) *For $0 \neq z \in K$, we can write $z = a/b$, $a, b \in R$, where $(a):(b) = (a):(b^2)$.*

Proof. (1) $\Rightarrow$ (2) This immediately follows since if $z = a/b$, $(1):(z) = (b):(a)$. (2) $\Rightarrow$ (3) Let $z = r/s$ where $r, s \in R - \{0\}$. So $(s):(r) \not\subseteq Z(R/(r):(s))$. Let $b \in (s):(r) - Z(R/(r):(s))$. So $br = as$ for some $a \in R$. Then $z = r/s = a/b$ and $(r):(s) = (1):(z^{-1}) = (a):(b)$, so $b \notin Z(R/(a):(b))$. (3) $\Rightarrow$ (4) Write $z = a/b$ where $b \notin Z(R/(a):(b))$. Suppose $r \in (a):(b^2)$, so $rb \in (a):(b)$. Since $b \notin Z(R/(a):(b))$, $r \in (a):(b)$. Since the reverse containment always holds, $(a):(b) = (a):(b^2)$. (4) $\Rightarrow$ (1) Write $z = a/b$ where $(a):(b) = (a):(b^2)$. Then $b \in (1):(z)$. Now $br \in (1):(z^{-1}) = (1):(b/a) \Rightarrow b^2 r \in (a) \Rightarrow r \in (a):(b^2) = (a):(b) = (1):(z^{-1})$. So $b \notin Z(R/(1):(z^{-1}))$. □

We next summarize and slightly extend some of the results from [1] concerning distinguished domains. Theorem 3 shows that the class of distinguished domains is quite large. Note that a Noetherian domain is distinguished if and only if it is integrally closed. The results given in Theorem 3 indicate that there might be some

1991 *Mathematics Subject Classification.* 13A15, 13F99.
Key words and phrases. distinguished domain, Prüfer v-multiplication domain.

connection between distinguished domains and PVMD's. The investigation of this connection is the main purpose of this paper.

Theorem 3. (1) *A distinguished domain is integrally closed.*

(2) *A Krull domain is distinguished. More generally, if R is a Krull domain with quotient field K and L is a field extension of K, then the integral closure of R in L is distinguished.*

(3) *A Prüfer domain is distinguished. In fact, a distinguished domain R is a Prüfer domain if and only if for each maximal ideal M of R, the prime ideals of R_M form a chain.*

(4) *If R is a distinguished domain, so is $R[X]$.*

(5) *If R is a distinguished domain and S is a multiplicatively closed subset of R, then R_S is distinguished.*

(6) *Let R be a domain and Q a prime ideal of R such that for each ideal I of R, $I \subseteq Q$ or $Q \subseteq I$. Then R is distinguished if and only if R/Q and R_Q are distinguished. In particular, if (T, Q) is a quasilocal domain and D is an integral domain contained in T/Q, then $R = \{t \in T \mid t + Q \in D\}$ is a distinguished domain if and only if T and D are distinguished and D has quotient field T/Q.*

Proof. (1) [1, Proposition 1.8]. (2) The first statement is [1, Proposition 1.1] while the second statement in the case where L/K is algebraic is [1, Proposition 1.7]. However, it is easily seen that we do not need to assume that L/K is algebraic. (3) [1, Theorem 1.2]. (4) [1, Theorem 3.2]. (5) The case where $S = R - P$, P a prime ideal of R, is remarked on the top of page 182 of [1]. The proof of the general case is identical. (6) The first statement is [1, Theorem 1.3]. To prove the second statement, it suffices by [1, Corollary 1.4] to show that if D does not have quotient field T/Q, then R is not distinguished. Suppose that D has quotient field $k \subsetneq T/Q$. Choose $t \in T$ with $\bar{t} = t + Q \notin k$. So $t \notin Q$ and hence t is a unit. So $t, t^{-1} \in T - R$. We show $(1) : (t) \subseteq Z(R/(1) : (t^{-1}))$. Let $r \in (1) : (t)$, so $rt \in R$. Then $\bar{r}\bar{t} \in D \subseteq k \Rightarrow \bar{r} = \bar{0}$, i.e., $r \in Q$. But then $rt^{-1} \in Q \subseteq R$, so $r \in (1) : (t^{-1})$. Hence $r \in Z(R/(1) : (t^{-1}))$. □

Let R be an integral domain and let $\text{Assp}(R) = \{P \in \text{Spec}(R) \mid P$ is minimal over some $(a) : (b)$, $a, b \in R - \{0\}\}$, the set of *associated primes* of R. Note that $\text{Assp}(R_S) = \{Q_S \mid Q \in \text{Assp}(R),\ Q \cap S = \emptyset\}$ for each multiplicatively closed subset S of R. Recall [4] that R is a *P-domain* if R_P is a valuation domain for each $P \in \text{Assp}(R)$. The *t-operation* on an integral domain R is given by $I \to I_t = \bigcup\{(a_1, \dots, a_n)_v \mid a_1, \dots, a_n \in I - \{0\}\}$ where as usual, $I_v = (I^{-1})^{-1}$ for I a nonzero (fractional) ideal of R. An ideal I is a *t-ideal* if $I = I_t$. If $P \in \text{Assp}(R)$, then P is a prime t-ideal. Recall that R is a *Prüfer v-multiplication domain* (*PVMD*) if R_M is a valuation domain for each maximal t-ideal M of R, or equivalently, if $(II^{-1})_t = R$ for each nonzero finitely generated ideal I of R, i.e., I is t-invertible. Hence a PVMD is a P-domain, but not conversely [4]. Note that R is a P-domain if and only if R_M is a P-domain for each maximal ideal M of R and that if R is a P-domain, so is R_S for each multiplicatively closed subset S of R. Recall that an integral domain R is *essential* if $R = \bigcap R_{P_\alpha}$ where each R_{P_α} is a valuation domain. An integral domain R is a P-domain if and only if each localization of R is essential. We next define U-primes and V-primes which were studied in [1].

Definition 4. Let $0 \neq P$ be a prime ideal of the integral domain R. Then P is a *U-prime* if $R_P = \bigcap\{R_Q \mid Q \in \mathrm{Spec}(R) \text{ with } Q \subsetneq P\}$ and P is a *V-prime* if there exists a prime ideal Q directly below P such that for $0 \neq \omega \in R_Q$, ω or $\omega^{-1} \in R_P$. And R is a *UV-domain* if each nonzero prime ideal of R is either a U-prime or a V-prime.

So P is a U-prime of $R \Leftrightarrow P_P$ is a U-prime of R_P. Recall [3, Exercise 20, page 42] that for a set $\{Q_\alpha\}$ of primes of R, $R = \bigcap R_{Q_\alpha} \Leftrightarrow$ for each $(a):(b) \neq R$, $(a):(b) \subseteq Q_\alpha$ for some α. Hence P_P is a U-prime of $R_P \Leftrightarrow P_P \notin \mathrm{Assp}(R_P)$. So P is a U-prime of $R \Leftrightarrow P_P$ is a U-prime of $R_P \Leftrightarrow P_P \notin \mathrm{Assp}(R_P) \Leftrightarrow P \notin \mathrm{Assp}(R)$. Thus R is a UV-domain $\Leftrightarrow$ each $P \in \mathrm{Assp}(R)$ is a V-prime. In [1, Remark (c), page 186] it was shown that a distinguished domain is a UV-domain, but not conversely [1, Example, page 191].

Thus a height-one prime ideal P of R is a V-prime $\Leftrightarrow R_P$ is a valuation domain. Note that P is a V-prime of $R \Leftrightarrow P_P$ is a V-prime of R_P. And thus R is a UV-domain $\Leftrightarrow R$ is locally a UV-domain. If P is a V-prime of R and Q is as in Definition 4, then the ideals of R_P are comparable to Q_P and hence Q is the unique prime ideal directly below P. (We can assume that R is quasilocal with maximal ideal P. Suppose I is an ideal with $I \nsubseteq Q$, so let $i \in I - Q$. For $q \in Q$, $q/i \in R_Q$. Now $i/q \notin R \Rightarrow q/i \in R \Rightarrow q \in iR \subseteq I$.) Also, R_P/Q_P is a rank-one valuation domain. Recall that a prime ideal P of an integral domain R is *essential* if R_P is a valuation domain.

Proposition 5. *Let P be an essential prime ideal of an integral domain R. Then the following are equivalent:*

(1) *P is a V-prime;*
(2) *there is a prime ideal Q directly below P;*
(3) *P is a minimal over a principal ideal;*
(4) *$P \in \mathrm{Assp}(R)$, and;*
(5) *P is not a U-prime.*

Proof. (1) $\Rightarrow$ (2) Clear. (2) $\Rightarrow$ (3) Let $c \in P - Q$. Since R_P is a valuation domain, P_P is minimal over cR_P. Hence P is minimal over cR. (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are always true. (5) $\Rightarrow$ (1) R_P is a valuation domain and P_P is not a U-prime of R_P. Hence P_P is not the union of the primes properly below P_P and hence there is a prime Q directly below P_P. It is now immediate that P_P is a V-prime. □

Proposition 6. *A P-domain (and hence a PVMD) R is a UV-domain. For a P-domain R, a nonzero prime ideal P of R is a V-prime if and only if $P \in \mathrm{Assp}(R)$.*

Proof. Let R be a P-domain. If $P \in \mathrm{Assp}(R)$, then P is essential and hence is a V-prime by Proposition 5. So R is a UV-domain.

Suppose that R is a P-domain. We have already shown that if $P \in \mathrm{Assp}(R)$, then P is a V-prime. Conversely, suppose that P is a V-ideal. Let Q be the unique prime ideal directly below P. Now $R_P \subsetneq R_Q = \bigcap\{R_N \mid N \subsetneq P \text{ is prime}\} = \bigcap\{R_N \mid N \subsetneq P_P \text{ is prime}\}$. So for some $(a/1):(b/1)$ in R_P, $(a/1):(b/1) \nsubseteq N$ for any prime $N \subseteq Q_P$. So P_P is minimal over $(a/1):(a/1)$. Hence P is minimal over $(a):(b)$. So $P \in \mathrm{Assp}(R)$. □

We next give a result related to Theorem 3(3). This result and Proposition 6 are then used to give a number of conditions equivalent to a distinguished domain being a PVMD.

Theorem 7. *An integral domain R is a Prüfer domain if and only if (1) R is distinguished and (2) for $P, Q \in \operatorname{Assp}(R)$, either $P \subseteq Q$, $Q \subseteq P$, or $P + Q = R$.*

Proof. ($\Rightarrow$) Clear. ($\Leftarrow$) It suffices to show that $(a):(b)+(b):(a) = R$ for $a, b \in R-\{0\}$. Suppose $(a):(b) + (b):(a) \subseteq M$ for some maximal ideal M. Then we can shrink M down to prime ideals P and Q with P minimal over $(a):(b)$ and Q minimal over $(b):(a)$. Thus $P, Q \in \operatorname{Assp}(R)$. Now $P + Q \neq R$, so say $P \subseteq Q$. Now Q minimal over $(b):(a)$ gives $Q \subseteq Z(R/(b):(a))$. So $(a):(b) \subseteq P \subseteq Q \subseteq Z(R/(b):(a))$, a contradiction. Hence $(a):(b) + (b):(a) = R$. □

Of course, in Theorem 7 "$P, Q \in \operatorname{Assp}(R)$" could be replaced by "$P$ and Q are V-primes" or "P and Q are prime t-ideals".

Theorem 8. *Let R be a distinguished domain. Then the following conditions are equivalent.*

(1) *R is a PVMD.*
(2) *R is a P-domain.*
(3) *For each V-prime P of R, R_P is a valuation domain.*
(4) *For each V-prime P of R, $(x):(y) + (y):(x) \nsubseteq P$ for $x, y \in R - \{0\}$.*
(5) *For prime t-ideals P and Q of R, either $P \subseteq Q$, $Q \subseteq P$, or $(P + Q)_t = R$.*
(6) *For $P, Q \in \operatorname{Assp}(R)$, either $P \subseteq Q$, $Q \subseteq P$, or $(P + Q)_t = R$.*

Proof. (1) $\Rightarrow$ (2) This always holds. (2) $\Rightarrow$ (3) Proposition 6. (3) $\Rightarrow$ (4) Clear. (4) $\Rightarrow$ (1) For $(x):(y)$ (resp., $(y):(x)$) choose $b_1, b_2 \in (x):(y)$ (resp., $c_1, c_2 \in (y):(x)$) such that if P is a V-prime ideal with $(x):(y) \nsubseteq P$ (resp., $(y):(x) \nsubseteq P$), then $(b_1, b_2) \nsubseteq P$ (resp., $(c_1, c_2) \nsubseteq P$) [1, Proposition 2.3]. Put $A = (b_1, b_2) + (c_1, c_2)$. Let P be a V-prime. Now $(x):(y) + (y):(x) \nsubseteq P$. Hence $(x):(y) \nsubseteq P$ or $(y):(x) \nsubseteq P$. Thus $A \nsubseteq P$. Now $I \to I^* = \bigcap\{IR_P \mid P \text{ is a } V\text{-prime of } R\}$ is a star-operation on R since $R = \bigcap\{R_P \mid P \text{ is a } V\text{-prime of } R\}$. So $A^* = R$. Hence $A_t = A_v = R$. So $((x):(y)+(y):(x))_t = R$. Hence R is a PVMD. (1) $\Rightarrow$ (5) This holds for any PVMD. (5) $\Rightarrow$ (6) This holds for any integral domain since an associated prime is a prime t-ideal. (6) $\Rightarrow$ (1) Let M be a maximal t-ideal of R. Then R_M is distinguished and the set $\operatorname{Assp}(R_M)$ is totally ordered. By Theorem 7, R_M is a valuation domain. So R is a PVMD. □

Corollary 9. *Let R be a two-dimensional distinguished domain. Then R is a PVMD.*

Proof. Let P be a V-prime of R. If $\operatorname{ht} P = 2$, then there is a unique prime ideal directly below P; so the prime ideals of R_P form a chain. Hence R_P is a valuation domain by Theorem 3(3). If $\operatorname{ht} P = 1$, then R_P is a valuation domain by the definition of a V-prime. By Theorem 8, R is a PVMD. □

Theorems 7 and 8 may be generalized as follows. We leave the proof to the reader. Let R be an integral domain with the property that every associated prime is a V-prime. Then R is a Prüfer domain (resp., PVMD) if and only if for $P, Q \in \operatorname{Assp}(R)$, either $P \subseteq Q$, $Q \subseteq P$, or $P + Q = R$ (resp., $(P + Q)_t = R$).

Of course, for a one-dimensional domain the notions of Prüfer domain, PVMD, P-domain, distinguished domain, and UV-domain all coincide. And by Corollary 9, a two-dimensional distinguished domain is a PVMD.

We next give an example of a three-dimensional distinguished domain that is not a PVMD.

Example 10. For $n \geq 3$, an n-dimensional quasilocal distinguished domain that is not essential and hence not a P-domain nor a PVMD.

Let $(V, (p))$ be a rank $n-1$ discrete valuation ring where $n \geq 3$. Let $Q = \bigcap_{n=1}^{\infty} p^n V$, $R_0 = V + XV_Q[X]$, $Q_0 = Q + XV_Q[X]$, and $P_0 = pR_0 = pV + XV_Q[X]$. Now $R_0/P_0 = V/(p)$, so P_0 is a maximal ideal of R_0. Put $R = R_{0\,P_0}$, so R is a quasilocal domain. Now $\bigcap_{n=1}^{\infty} p^n R_0 = Q + XV_Q[X] = Q_0$, so there are no prime ideals properly between pR_0 and Q_0. Hence there are no prime ideals properly between $P_{0\,P_0}$ and $Q_{0\,P_0}$. So $Q_{0\,P_0} = \bigcap_{n=1}^{\infty} p^n R$. If I is an ideal of R, then either $I \subseteq \bigcap_{n=1}^{\infty} p^n R = Q_{0\,P_0}$ or $Q_{0\,P_0} \subset I = p^n R$ for some $n \geq 0$. So $Q_{0\,P_0}$ is comparable to each ideal of R. Now $R_{Q_{0\,P_0}} = (V + XV_Q[X])_{Q+XV_Q[X]}$ is a localization of the GCD domain $V_Q[X]$ and hence is distinguished (see Theorem 15 or use the fact that the valuation domain V_Q is distinguished and hence so is the localization of a polynomial ring over it (Theorem 3)). Also, $R/Q_{0\,P_0} = R_{0\,P_0}/Q_{0\,P_0}$ is a localization of $R_0/Q_0 = V/Q$, a valuation domain. So $R/Q_{0\,P_0}$ is a valuation domain and hence is distinguished. By Theorem 3(6), R is distinguished.

We next show that $\dim R = n$. Now V_Q is an $(n-2)$-dimensional valuation domain, so $\dim V_Q[X] = n-1$. Hence $R_{Q_{0\,P_0}}$ being a localization of $V_Q[X]$ gives $\dim R_{Q_{0\,P_0}} \leq n-1$ and so $\dim R \leq n$. But if $0 \subsetneq Q_{n-1} \subsetneq \cdots \subsetneq Q \subsetneq (p)$ are the prime ideals of V, then $0 \subsetneq XV_Q[X]_{P_0} \subsetneq (Q_{n-1} + XV_Q[X])_{P_0} \subsetneq \cdots \subsetneq (Q + XV_Q[X])_{P_0} = Q_{0\,P_0} \subsetneq P_{0\,P_0}$ is a chain of prime ideals in R. Hence $\dim R = n$.

Suppose that R is essential. Since the maximal ideal of R is principal, R must actually be a valuation domain. Then $R_{(Q_{n-1}+XV_Q[X])_{P_0}} = R_{0\,Q_{n-1}+XV_Q[X]}$ is a two-dimensional valuation domain. But $R_{0\,Q_{n-1}+XV_Q[X]}$ is a localization of $V_{Q_{n-1}}[X]$ which is a UFD since $V_{Q_{n-1}}$ is a rank-one DVR. So $R_{0\,Q_{n-1}+XV_Q[X]}$ is a UFD, a contradiction.

We next show that a two-dimensional domain is a UV-domain if and only if it is a P-domain.

Theorem 11. *For a two-dimensional integral domain R, the following conditions are equivalent.*

1. R is a UV-domain.
2. For each maximal ideal M of R, either R_M is a valuation domain or $R_M = \bigcap\{R_P \mid P \subsetneq M\}$ and each R_P is a valuation domain.
3. R is a P-domain.

Proof. (1) $\Rightarrow$ (2) Let R be a two-dimensional UV-domain. Let P be a prime ideal of R with $\operatorname{ht} P = 1$. Then P must be a V-prime and hence R_P is a valuation domain. Let M be a maximal ideal of R. If $\operatorname{ht} M = 1$, R_M is a valuation domain. So assume $\operatorname{ht} M = 2$. Suppose that M is a U-prime. Then the second case of (2) occurs. So suppose that M is a V-prime. Then there is a height-one prime $P \subsetneq M$ so that for $0 \neq \omega \in R_P$, ω or $\omega^{-1} \in R_M$. Let $0 \neq x \in K$, the quotient field of R. Since R_P is a valuation domain, x or $x^{-1} \in R_P$. Hence x or $x^{-1} \in R_M$. So R_M is a valuation domain. (2) $\Rightarrow$ (3) Let $P \in \operatorname{Assp}(R)$. If $\operatorname{ht} P = 1$, R_P is a valuation domain. If $\operatorname{ht} P = 2$, then the second case of (2) cannot occur. So again R_P is a valuation domain. (3) $\Rightarrow$ (1) Proposition 6. □

Recall that an integral domain is a *Mori domain* if it satisfies ACC on divisorial ideals. According to Theorem 3(2), a Krull domain is a distinguished Mori domain. We next prove the converse.

Theorem 12. *A Mori UV-domain R (e.g., a distinguished Mori domain) is a Krull domain.*

Proof. Let P be a maximal t-ideal of R, so $P = (a) : (b)$. Hence P is a V-prime. Suppose that $\operatorname{ht} P > 1$. Let Q be the unique prime ideal directly below P. Choose $c \in PR_P - QR_P$. Then $c^n R_P \nsubseteq QR_P$, so $QR_P \subsetneq c^n R_P$. Thus $\bigcap_{n=1}^{\infty} c^n R_P \supseteq QR_P \neq 0$. So R_P does not satisfy ACCP, a contradiction. Hence $\operatorname{ht} P = 1$. So R_P is a valuation domain, necessarily a DVR. Hence R is completely integrally closed. Thus R is a Krull domain. □

To get more examples of distinguished domains, we introduce the concept of a super distinguished domain.

Definition 13. An integral domain R is *super distinguished* if for $x, y \in R - \{0\}$, there exist $r \in (x) : (y)$ and $s \in (y) : (x)$ such that $(r, s)_t = D$.

We first note that a super distinguished domain is distinguished and is a PVMD. Since Example 10 gives an example of a distinguished domain that is not a PVMD, a distinguished domain need not be super distinguished.

Theorem 14. *A super distinguished domain R is a distinguished PVMD.*

Proof. Suppose that R is super distinguished. Let $a, b \in R - \{0\}$. By hypothesis, there exist $r \in (a) : (b)$ and $s \in (b) : (a)$ such that $(r, s)_t = R$. Hence $R \subseteq (r, s)_t \subseteq ((a) : (b) + (b) : (a))_t \subseteq R$, so $((a) : (b) + (b) : (a))_t = R$. Thus R is a PVMD. To show that R is distinguished, we need that $(b) : (a) \nsubseteq Z(R/(a) : (b))$. It suffices to show that $s \notin Z(R/(a) : (b))$. Suppose that $sd \in (a) : (b)$. Also, $rd \in (a) : (b)$, so $d \in dR = d(r, s)_t = (dr, ds)_t \subseteq ((a) : (b))_t = (a) : (b)$. □

We next give three important classes of PVMD's that are super distinguished and hence distinguished. Let R be a PVMD. The *t-class group* $\mathrm{Cl}_t(R)$ of R is the group of t-invertible t-ideals under the t-product modulo its subgroup of principal ideals. Hence R is a GCD domain $\Leftrightarrow \mathrm{Cl}_t(R) = \{0\}$. Now $R = \bigcap_{M \in t-\max(R)} R_M$ where t-$\max(R)$ is the set of maximal t-ideals of R. If the intersection has finite character, R is called a *ring of Krull type.* Certainly a Krull domain is a ring of Krull type.

Theorem 15. *Let R be an integral domain. If R is a Prüfer domain, a ring of Krull type, or a PVMD with $\mathrm{Cl}_t(R)$ torsion (e.g., a GCD domain), then R is super distinguished and hence distinguished.*

Proof. First suppose that R is a Prüfer domain. For $x, y \in R - \{0\}$, $(x) : (y) + (y) : (x) = R$. So there exist $r \in (x) : (y)$ and $s \in (y) : (x)$ with $r + s = 1$. Hence $R = (r, s) = (r, s)_t$. So R is super distinguished. Next suppose that R is a ring of Krull type. Let $x, y \in R - \{0\}$. Let $P_1, \dots, P_n$ be the maximal t-ideals containing $(x) : (y)$ and let $Q_1, \dots, Q_m$ be the maximal t-ideals containing $(y) : (x)$. Since R is a PVMD, $(x) : (y) + (y) : (x)$ is not contained in any prime t-ideal. Choose $r \in (x) : (y) - Q_1 \cup \cdots \cup Q_m$. Let $P_1, \dots, P_n, P_1', \dots, P_k'$ be the maximal t-ideals containing r. Now $(y) : (x) \nsubseteq P_1 \cup \cdots \cup P_n \cup P_1' \cup \cdots \cup P_k'$, so there exists $s \in (y) : (x) - P_1 \cup \cdots \cup P_n \cup P_1' \cup \cdots \cup P_k'$. Then $(r, s)_t = R$. Thus R is super distinguished. Finally, suppose that R is a PVMD with $\mathrm{Cl}_t(R)$ torsion. Let $x, y \in R - \{0\}$. Choose $n \geq 1$ so that $(((x) : (y))^n)_t = (r)$ and $(((y) : (x))^n)_t = (s)$. Then $r \in (x) : (y)$, $s \in (y) : (x)$, and $(r, s)_t = R$. Hence R is super distinguished. □

We next show that the intersection of two (super) distinguished domains need not be distinguished.

Example 16. Let R be a one-dimensional integrally closed domain that is not a Prüfer domain. Then $R[X]$ is not distinguished. (Let M be a maximal ideal of R for which R_M is not a valuation domain. If $R[X]$ were distinguished, then $R_M(X) = R[X]_{M[X]}$ would also be distinguished (Theorem 3). Now $\dim R_M(X) = 2$, so $R_M(X)$ is a PVMD by Corollary 9. Hence R_M is a PVMD and thus a valuation domain, a contradiction.) But since R is integrally closed, $R[X] = R^b \cap K[X]$ where R^b is the Kronecker function ring for R and K is the quotient field of R. Now R^b is a Bezout domain and $K[X]$ a PID, so $R[X]$ is the intersection of two super distinguished domains but $R[X]$ is not distinguished.

We have been unable to determine if R a PVMD implies R is distinguished. However, we next show that if R is a PVMD, then $R[X]$ is distinguished (and, of course, is a PVMD). Heitmann and McAdam (Theorem 3(4)) showed that if R is distinguished, then so is $R[X]$, but they were unable to determine if $R[X]$ distinguished forces R to be distinguished. Note that either R a PVMD implies R is distinguished or there is a PVMD S such that S is not distinguished. Then $S[X]$ is distinguished while S is not distinguished.

Theorem 17. *Let R be a PVMD. Then $R[X]$ is distinguished.*

Proof. Let R be a PVMD with quotient field K. Then by Proposition 6, R is a UV-domain and each V-prime of R is an associated prime and hence a t-ideal. Let $0 \neq z \in K$. Now $(1):(z) = (b_1, \ldots, b_n)_t$ for some $b_1, \ldots, b_n \in R - \{0\}$. Let P be a V-prime with $(1):(z) \nsubseteq P$. Then $(b_1, \ldots, b_n) \nsubseteq P$. By the comments after Question 2 [1, page 188], $R[X]$ is distinguished. □

We end with the following diagram of implications:

$$\begin{array}{ccccc} & \nearrow & \text{distinguished} & \searrow & \\ \text{super distinguished} & & & & UV\text{-domain} \\ & \searrow & \text{PVMD} \longrightarrow P\text{-domain} & \nearrow & \end{array}$$

Heinzer and Ohm [2] gave an example of a two-dimensional essential domain R that is not a PVMD. Mott and the third author [4] showed that R is a P-domain. At each localization at a maximal ideal M, R_M is a regular local ring. Thus R is locally distinguished and a UV-domain. However, R is not distinguished since by Corollary 9 a two-dimensional distinguished domain must be a PVMD. Example 10 gives an integral domain R that is distinguished (and hence a UV-domain) but not a P-domain and hence not super distinguished. Thus there is no connection between R being distinguished and R being a P-domain.

References

[1] R.C. Heitmann and S. McAdam, Distinguished domains, *Canad. J. Math.* **34** (1982), 181–193.
[2] W. Heinzer and J. Ohm, An essential ring which is not a v-multiplication ring, *Canad. J. Math.* **25** (1973), 856–861.
[3] I. Kaplansky, *Commutative Rings,* revised edition, Polygonal Publishing House, Washington, New Jersey, 1994.
[4] J.L. Mott and M. Zafrullah, On Prüfer v-multiplication domains, *Manuscripta Math.* **35** (1981), 1–26.

Department of Mathematics, The University of Iowa, Iowa City, IA 52242
E-mail address: dan-anderson@uiowa.edu

Department of Mathematics, College of Natural Science, Kyungpook National University, Tae Gu, Korea

Department of Mathematics, The University of Arkansas, Fayetteville AR 72701
E-mail address: mzafrullah@usa.net