FACTORIZATION OF CERTAIN SETS OF POLYNOMIALS IN AN INTEGRAL DOMAIN

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ABSTRACT. Let D be an integral domain with quotient field K and let S be a set of nonconstant polynomials of $D[X]$. We say S satisfies property (P) (resp., the extension $D[X] \subseteq K[X]$ is S-inert) if whenever $f \in S$ factors as $f = gh$ in $K[X]$ where $\deg g, \deg h \geq 1$, then $f = \alpha\beta$ where $\alpha, \beta \in D[X]$ with $\deg \alpha, \deg \beta \geq 1$ (resp., there is a $0 \neq u \in K$ with $ug, u^{-1}h \in D[X]$). Then D is integrally closed (resp., integrally closed with $\mathrm{Pic}(D) = 0$, integrally closed with t-class group $\mathrm{Cl}_t(D) = 0$, Schreier) $\Leftrightarrow$ the extension $D[X] \subseteq K[X]$ is S-inert $\Leftrightarrow$ S satisfies property (P), where S is the set of monic polynomials of $D[X]$ (resp., $S = \{f \in D[X] \mid \deg f \geq 1 \text{ and } A_f = D\}$, $S = \{f \in D[X] \mid \deg f \geq 1 \text{ and } A_f^{-1} = D\}$, S is the set of all nonconstant polynomials of $D[X]$). Here A_f is the ideal of D generated by the coefficients of f.

Throughout this note D will be an integral domain with quotient field K. It is known that D is integrally closed if and only if each irreducible monic polynomial of $D[X]$ is prime ([2, Theorem 3.2], [7, Theorem]). This is easily seem to be equivalent to the condition that each monic polynomial of $D[X]$ that has a nontrivial factorization in $K[X]$ has a nontrivial factorization in $D[X]$. The purpose of this paper is to prove similar results for other sets of nonconstant polynomials. For example, we show that D is integrally closed with $\mathrm{Pic}(D) = 0$ if and only if each nonconstant irreducible polynomial of $D[X]$ with unit content is prime if and only if each nonconstant polynomial of $D[X]$ with unit content that factors nontrivially in $K[X]$ also factors nontrivially in $D[X]$. To state our results we need the following definition.

Definition 1. Let D be an integral domain with quotient field K and let S be a set of nonconstant polynomials of $D[X]$. Then the set S satisfies *property (P)* (resp., the extension $D[X] \subseteq K[X]$ is *S-inert*) if for each $f \in S$, whenever $f = gh$ in $K[X]$ where $\deg g, \deg h \geq 1$, then $f = \alpha\beta$ where $\alpha, \beta \in D[X]$ with $\deg \alpha, \deg \beta \geq 1$ (resp., there exists $0 \neq u \in K$ with $ug, u^{-1}h \in D[X]$).

For $f \in K[X]$, the content A_f of f is the (fractional) ideal of D generated by the coefficients of f. Recall that for a nonzero fractional ideal I of D, $I^{-1} = [D : I] = \{x \in K \mid xI \subseteq D\}$ and $I_v = (I^{-1})^{-1}$. Also, recall that if D is integrally closed, then $(A_{fg})_v = (A_f A_g)_v$ for all nonzero $f, g \in K[X]$ [6, Proposition 34.8]. For a survey of results concerning the contents of polynomials see [1]. We next show that under certain conditions the set S satisfies property (P) if and only if each irreducible element of S is prime.

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Proposition 2. *Let D be an integral domain with quotient field K and let $S \subseteq D[X]$ be a set of nonconstant polynomials where each $f \in S$ satisfies $A_f^{-1} = D$ (or equivalently, $(A_f)_v = D$). Then S satisfies property (P) if and only if each irreducible element of S is prime.*

Proof. ($\Rightarrow$) Suppose that S satisfies property (P). Let $f \in S$ be irreducible in $D[X]$. Then property (P) gives that f is irreducible (and hence prime) in $K[X]$. We can then quote [9, Theorem A], but we prefer to sketch the proof. Since $A_f^{-1} = D$, it is easily checked that $fK[X] \cap D[X] = fD[X]$ [9, Lemma 1]. Since $fK[X]$ is a prime ideal, so is $fD[X]$. ($\Leftarrow$) Assume that each irreducible element of S is prime. Suppose that some $f \in S$ has a factorization $f = gh$ in $K[X]$ where $\deg g, \deg h \geq 1$, but no such factorization exists in $D[X]$. Since $A_f^{-1} = D$, f has no nonunit constant factor, so f is irreducible in $D[X]$. Thus f is prime in $D[X]$ and hence prime and irreducible in $K[X]$, a contradiction. □

The next theorem gives a number of conditions equivalent to D being integrally closed.

Theorem 3. *Let D be an integral domain with quotient field K and let S be the set of nonconstant monic polynomials of $D[X]$. Then the following conditions are equivalent.*

(1) *D is integrally closed.*
(2) *The extension $D[X] \subseteq K[X]$ is S-inert.*
(3) *The set S satisfies property (P).*
(4) *Each irreducible element of S is prime.*
(5) *Each element of S is a product of prime monic polynomials.*
(6) *The elements of S have unique factorization into monic irreducible polynomials.*

Proof. (1) $\Rightarrow$ (2) Let $f \in D[X]$ be monic. Suppose that $f = gh$ in $D[X]$ where $\deg g, \deg h \geq 1$. Choose $0 \neq u \in K$ so that $ug, u^{-1}h$ are monic in $K[X]$. Now $A_f = D$, so $D = (A_f)_v = (A_{ugu^{-1}h})_v = (A_{ug}A_{u^{-1}h})_v \supseteq A_{ug}A_{u^{-1}h} \supseteq A_{ugu^{-1}h} = A_f = D$. Since $A_{ug}, A_{u^{-1}h} \supseteq D$, $A_{ug} = A_{u^{-1}h} = D$. Hence $ug, u^{-1}h \in D[X]$. (2) $\Rightarrow$ (3) Clear. (3) $\Rightarrow$ (4) Proposition 2. (4) $\Rightarrow$ (1) Let $\alpha \in K$ be integral over D. Let f be the monic polynomial in $D[X]$ of least degree with $f(\alpha) = 0$. Certainly f is irreducible. Thus f is prime in $D[X]$ and hence prime and irreducible in $K[X]$. But since $X - \alpha$ is a factor of f in $K[X]$, we must have $f = X - \alpha$. Thus $\alpha \in D$. (4) $\Rightarrow$ (5) This follows since each monic polynomial is a product of irreducible monic polynomials. (5) $\Rightarrow$ (6) This follows from the well known fact that factorization into primes is unique. (6) $\Rightarrow$ (1) [7, Theorem]. □

Concerning Theorem 3, we remark that the implication (1) $\Leftrightarrow$ (5) is given in [2, Theorem 3.2] and the implication (1) $\Leftrightarrow$ (6) is given in [7, Theorem].

For the next theorem, our main result, we need some remarks on star operations. For an introduction to star operations, see [6]. Let $F(D)$ be the set of nonzero fractional ideals of D. A closure operation $*$ on $F(D)$ is called a *star operation* if $D^* = D$ and $(aA)^* = aA^*$ for each $0 \neq a \in K$ and $A \in F(D)$. Further, $*$ has finite character if $A^* = \bigcup\{B^* \mid 0 \neq B \subseteq A \text{ is finitely generated}\}$. The function $A \to A_v$ is a star operation. Two examples of finite character star operations are the d-operation $A \to A_d = A$ and the t-operation $A \to A_t = \bigcup\{B_v \mid 0 \neq B \subseteq A \text{ is finitely generated}\}$. Observe that if A is finitely generated, then $A_t = A_v$; in particular for a polynomial $f \in K[X]$, $(A_f)_t = (A_f)_v$. An ideal $A \in F(D)$ is $*$-*invertible* if $(AB)^* = D$ for some $B \in F(D)$; we can then take $B = A^{-1}$. If $*$ has finite character and A is $*$-invertible, then A has finite type, that is, $A^* = (c_1, \ldots, c_n)^*$ for some $c_1, \ldots, c_n \in K$. The ideal A is a $*$-ideal if $A = A^*$. The set of $*$-invertible $*$-ideals forms a group under the product $A * B = (AB)^*$ and this group modulo its subgroup of principal fractional ideals is called the $*$-class group of D and is denoted by $\mathrm{Cl}_*(D)$. For the case $* = d$, we get the usual Picard group $\mathrm{Pic}(D)$ and for the case $* = t$, we get the t-class group $\mathrm{Cl}_t(D)$.

Theorem 4. *Let D be an integral domain with quotient field K and let $*$ be a finite character star operation on D. Let $S_* = \{f \in D[X] \mid (A_f)^* = D$ and $\deg f \geq 1\}$. Then the following conditions are equivalent.*

1. *D is integrally closed and $\mathrm{Cl}_*(D) = 0$.*
2. *The extension $D[X] \subseteq K[X]$ is S_*-inert.*
3. *The set S_* satisfies property (P).*
4. *Each irreducible element of S_* is prime.*
5. *Each element of S_* is a product of primes from S_*.*

Proof. (1) ⇒ (2) Write $f = gh$ in $K[X]$ where $f \in S_*$ and $\deg g, \deg h \geq 1$. Note that $(A_f)^* = D$ gives $(A_f)_v = (A_f)_t = D$. Now D integrally closed gives $A_g A_h \subseteq (A_g A_h)_v = (A_{gh})_v = (A_f)_v = D$. Thus $D \supseteq (A_g A_h)^* \supseteq (A_{gh})^* = (A_f)^* = D$ and so $(A_g A_h)^* = D$. Hence A_g and A_h are $*$-invertible and thus principal; say $(A_g)^* = \alpha D$ and so $(A_h)^* = \alpha^{-1} D$, $0 \neq \alpha \in K$. Then $\alpha^{-1} g, \alpha h \in D[X]$. (2) ⇒ (3) Clear. (3) ⇒ (4) Proposition 2. (4) ⇒ (5) Clear. (5) ⇒ (1) Note that each monic polynomial of $D[X]$ is a product of prime monic polynomials. By Theorem 3 D is integrally closed. Let $I \subseteq D$ be $*$-invertible. Say $I^* = (a_0, \ldots, a_n)^*$ and $I^{-1} = (b_0, \ldots, b_m)^*$ where $a_0, \ldots, a_n, b_0, \ldots, b_m \in K$. Put $f = a_0 + a_1 X + \cdots + a_n X^n$ and $g = b_0 + b_1 X + \cdots + b_m X^m$. Now since A_f is $*$-invertible $(A_{fg})^* = (A_f A_g)^* = D$. (Indeed, by the Dedekind–Mertens Theorem [6, Theorem 28.1] $A_f^n A_{fg} = A_f^n A_f A_g$ for some n. Then $(A_f^n)^{-1} A_f^n A_{fg} = (A_f^n)^{-1} A_f^n A_f A_g$ and hence $(A_{fg})^* = (A_f A_g)^*$ since A_f^n is $*$-invertible.) Now in $D[X]$ we can factor $fg = f_1 \cdots f_s$ where each f_i is prime and has $(A_{f_i})^* = D$. Hence f_i is prime in $K[X]$. By unique factorization in $K[X]$ (and re-ordering if necessary), $g = \lambda f_1 \cdots f_l$, say, where $1 \leq l < s$ and $0 \neq \lambda \in K$. Then $f_1 \ldots f_s = fg = f(\lambda f_1 \cdots f_l)$ so $f_{l+1} \cdots f_s = \lambda f$. Hence $\lambda (A_f)^* = (A_{\lambda f})^* = (A_{f_{l+1} \cdots f_s})^* = (A_{f_{l+1}} \cdots A_{f_s})^* = D$. Thus $I^* = (A_f)^* = \lambda^{-1} D$ is principal. □

The implication (1) ⇔ (5) of Theorem 4 for a special class of finite-character star operations is given in [3, Theorem 4].

The last theorem considers the case where S is the set of all nonconstant polynomials of $D[X]$. We recall the notion of a Schreier domain which was introduced by P.M. Cohn [5]. An element c of a domain D is *primal* if $c \mid a_1 a_2$ implies that $c = c_1 c_2$ such that $c_1 \mid a_1$ and $c_2 \mid a_2$. A *Schreier domain* is an integrally closed domain in which every element is primal. For a survey of Schreier domains, see [1].

Theorem 5. *Let D be an integral domain with quotient field K and let S be the set of nonconstant polynomials of $D[X]$. Then the following conditions are equivalent.*

1. *D is a Schreier domain.*
2. *The extension $D[X] \subseteq K[X]$ is an S-inert extension.*
3. *The set S satisfies property (P).*

Proof. (1) $\Leftrightarrow$ (2) [4, page 562]. (1) $\Leftrightarrow$ (3) [8, Theorem 3]. □

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