t-LOCAL DOMAINS AND VALUATION DOMAINS

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Abstract. In a valuation domain (V, M) every nonzero finitely generated ideal J is principal and so, in particular, $J = J^t$, hence the maximal ideal M is a t-ideal. Therefore, the t-local domains (i.e., the local domains, with maximal ideal being a t-ideal) are "cousins" of valuation domains, but, as we will see in detail, not so close. Indeed, for instance, a localization of a t-local domain is not necessarily t-local, but of course a localization of a valuation domain is a valuation domain.

So it is natural to ask under what conditions is a t-local domain a valuation domain? The main purpose of the present paper is to address this question, surveying in part previous work by various authors containing useful properties for applying them to our goal.

Dedicated to David F. Anderson

1. Introduction

We begin by reviewing the notion of a t-local domain.

Let D be an integral domain with quotient field K, let $\boldsymbol{F}(D)$ be the set of non-zero fractional ideals of D, and let $\boldsymbol{f}(D)$ be the set of all nonzero finitely generated D-submodules of K (obviously, $\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D)$). For $E \in \boldsymbol{F}(D)$, let $E^{-1} := \{x \in K \mid xE \subseteq D\}$. The functions on $\boldsymbol{F}(D)$ defined by $E \mapsto E^v := (E^{-1})^{-1}$ and $E \mapsto E^t := \bigcup\{F^v \mid 0 \neq F \text{ is a finitely generated subideal of } E\}$, called respectively the *v-operation* and the *t-operation* on the integral domain D, come under the umbrella of star operations (briefly recalled in Section 2), discussed in Sections 32 and 34 of [18], where the reader can find proofs of the basic statements made here about the v-, t- and, more generally, the star operations.

Recall that a nonzero fractional ideal E of D is a *v-ideal*, or a *divisorial ideal*, (resp., a *t-ideal*) if $E = E^v$ (resp., $E = E^t$) and a *v-ideal* (resp., a *t-ideal*) *of finite type* if $E = E^v = F^v$ (resp., $E = E^t = F^t$) for some finitely generated $F \in \boldsymbol{f}(D)$ and, obviously, $F \subseteq E$. Next, the t-operation is a star operation of finite type on the integral domain D, in the sense that $E \in \boldsymbol{F}(D)$ is a t-ideal if and only if for each finitely generated nonzero subideal F of E we have $F^v = F^t \subseteq E$ and it is easy to see that if F is principal $F^v = F = F^t$.

An integral ideal of D maximal with respect to being an integral t-ideal is called a *maximal t-ideal* of D and it is always a prime ideal. We denote by $\mathrm{Max}^t(D)$ the

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set of all the maximal t-ideals of D. This set is non empty, since every t-ideal is contained in a maximal t-ideal, thanks to the definition of the t-operation and to Zorn's Lemma. An integral domain is called a *t-local domain* if it is local and its maximal ideal is a t-ideal.

The purpose of this article is to survey the notion indicating what t-local domains are, where they may or may not be found and what their uses are.

The first example of a t-local domain that comes to mind is a valuation domain, i.e., a local domain (V, M) in which every nonzero finitely generated ideal is principal. In this case, we can say that for each $F \in \boldsymbol{f}(V)$ with $F \subseteq M$ we have $F = (a) \in M$ and so $F^t = (a)^t = (a) \subseteq M$. But, of course, t-local domains are much more general than that. We can, for example, show that if P is a height one prime ideal of an integral domain D, then D_P is a t-local domain. We can show, as well will in more generality, that if $M = pD$ is a prime ideal generated by a prime element of a domain D then M is a maximal t-ideal and D_M is a t-local domain. However, we cannot just take a prime t-ideal P of D and claim that D_P is a t-local domain, as there are examples of some domains D with prime t-ideals P such that D_P is not a t-local domain. In Section 2, we discuss cases of prime t-ideals P with D_P a t-local domain and cases of domains that have prime t-ideals P with D_P non t-local, indicating also that if D is t-local then, for some multiplicative set S of D, D_S the ring of fractions may not be a t-local domain.

Now localization may not always produce t-local domains, but there are elements of a special kind whose presence in a domain D ensures that D is a t-local domain. In Section 3, we record the results related to the fact that the presence of a nonzero nonunit comparable element (definition recalled later) in an integral domain D makes D into a t-local domain. The related results include for instance (1) the effects the presence of a nonzero nonunit comparable element on different kinds of domains, (2) the presence of a nonzero comparable element in some domains would make them into valuation domains, if D is Noetherian then the presence of a nonzero nonunit comparable element in D makes D a DVR (= discrete valuation ring), (3) a t-local domain may not have a comparable element, and so on, the list continues.

Citing Krull, P.M. Cohn [10] showed that D is a valuation domain if and only if D is a Bézout domain and a local domain. (In fact, in this result "Bézout"can be replaced by "Prüfer"; here D is Bézout –respectively, Prüfer– if every nonzero finitely generated ideal of D is principal –respectively, invertible–.) In Section 4, we show that D is a valuation domain if and only if D is a GCD domain and a t-local domain, and point out that if, in the above statement, we replace "GCD domain" by "PvMD" the result would still be a characterization of a valuation domain (here, D is a PvMD, if for each pair $0 \neq a, b \in D$ we have $((a,b)\frac{(a)\cap(b)}{ab})^t = D$). But of course we do not stop here, we point to situations where recognizing the fact that the domain in question is a t-local domain makes proving that it is a valuation domain easier.

Section 5 has to do with "applications" which are essentially more efficient proofs of known results. We follow the study of the ring called Shannon's quadratic extension in [27] and point out that it is indeed a t-local domain, thus providing a shorter, more efficient proof of Theorem 6.2 of [27]. We also point to examples of maximal t-ideals Q in a particular domain D such that D_Q is not t-local.

2. Background results and t-local domains

We start with proving some important preliminary results. But, for that, we need to recall the formal definition of star operation. A *star operation* on D is a map $\ast : \boldsymbol{F}(D) \to \boldsymbol{F}(D)$, $E \mapsto E^\ast$, such that, for all $x \in K$, $x \neq 0$, and for all $E, F \in \boldsymbol{F}(D)$, the following properties hold:

($\ast_1$) $(xD)^\ast = xD$;
($\ast_2$) $E \subseteq F$ implies $E^\ast \subseteq F^\ast$;
($\ast_3$) $E \subseteq E^\ast$ and $E^{\ast\ast} := (E^\ast)^\ast = E^\ast$;

[18, Section 32]).

If $\ast$ is a star operation on D, then we can consider a map $\ast_{_f} : \boldsymbol{F}(D) \to \boldsymbol{F}(D)$ defined, for each $E \in \boldsymbol{F}(D)$, as follows:

$$E^{\ast_f} := \bigcup\{F^\ast \mid \ F \in \boldsymbol{f}(D) \text{ and } F \subseteq E\}.$$

It is easy to see that $\ast_{_f}$ is a star operation on D, called *the finite type star operation associated to* $\ast$ (or *the star operation of finite type associated to* $\ast$). A star operation $\ast$ is called a *finite type star operation* (or, *star operation of finite type*) if $\ast = \ast_{_f}$. It is easy to see that $(\ast_{_f})_{_f} = \ast_{_f}$ (that is, $\ast_{_f}$ is of finite type).

If $\ast_1$ and $\ast_2$ are two star operations on D, we say that $\ast_1 \leq \ast_2$ if $E^{\ast_1} \subseteq E^{\ast_2}$, for each $E \in \boldsymbol{F}(D)$, equivalently, if $(E^{\ast_1})^{\ast_2} = E^{\ast_2} = (E^{\ast_2})^{\ast_1}$, for each $E \in \boldsymbol{F}(D)$. Obviously, for each star operation $\ast$, we have $\ast_{_f} \leq \ast$. Clearly, $v_{_f} = t$. Let d_D (or, simply, d) be the *identity star operation on* D. Clearly, $d \leq \ast$ and, moreover, $\ast \leq v$, for all star operations $\ast$ on D [18, Theorem 34.1(4)].

Recall that an integral domain D is called a *Prüfer v-multiplication domain*, (for short, *PvMD*), if every nonzero finitely generated $F \in \boldsymbol{f}(D)$ is t-invertible, i.e., $(FF^{-1})^t = D$. Obviously, every Prüfer domain is a PvMD. It is well known (see, Griffin [22, Theorem 5]) that D is a PvMD if and only if D_Q is a valuation domain, for each maximal (or, equivalently, prime) t-ideal Q of D.

Any unexplained terminology is straightforward, well accepted, and usually comes from [33] or [18].

Lemma 2.1. (Hedstrom-Houston [25, Proposition 1.1]) *Let $\ast$ be a star operation on an integral domain D and let $\ast_{_f}$ be the finite type star operation on D canonically associated with $\ast$. If P is a minimal prime ideal over a $\ast_{_f}$-ideal of D, then P is a $\ast_{_f}$-ideal.*

Proof. Let J be a finitely generated (integral) ideal contained in P, the conclusion will follow if we show that $J^\ast \subseteq P$. Since P is minimal over some (integral) ideal I, with $I = I^{\ast_f}$, then $\mathrm{rad}(ID_P) = PD_P$ and, since J is finitely generated, there exists an integer $m \geq 1$ such that $J^m D_P \subseteq ID_P$. Therefore, for some $s \in D \setminus P$, $sJ^m \subseteq I$. Thus, $s(J^\ast)^m \subseteq s(J^m)^\ast = s(J^m)^{\ast_f} \subseteq I^{\ast_f} = I \subseteq P$, and so $J^\ast \subseteq P$, since $s \notin P$. $\square$

The next step is to apply this lemma for obtaining some sufficient conditions for a local domain to be a t-local domain (recall that an integral domain is a t-local domain if it is local and its maximal ideal is a t-ideal).

Remark 2.2. (1) Note that if D is an integral domain such that $\mathrm{Max}^t(D)$ contains only one element, then D is necessarily a t-local domain (and conversely). If not, let M be the unique t-maximal ideal of D and N be a maximal ideal of D with $N \neq M$. Let $x \in N \setminus M$, clearly, the t-ideal xD must be contained in some t-maximal ideal. In the present situation xD should be contained in M and this is a contradiction.

(2) Note that if D is a local domain with divisorial maximal ideal, then clearly D is t-local. The converse is not true: take, for instance, a valuation domain with nonprincipal maximal ideal (e.g., a 1-dimensional non-discrete valuation domain).

(3) In an integral domain D, the set of maximal divisorial ideals, $\mathrm{Max}^v(D)$, might be empty (e.g., take a 1-dimensional valuation domain with nonprincipal maximal ideal). However, if $\mathrm{Max}^v(D) \neq \emptyset$, a maximal divisorial ideal is a prime t-ideal, but it might be a nonmaximal t-ideal (for explicit examples see [17], where the problem of when a maximal divisorial ideal is a maximal t-ideal is investigated).

Corollary 2.3. *Let D be a local domain with maximal ideal M. Then, D is t-local in each of the following situations.*

(1) *The maximal ideal M is minimal over (i.e., is the radical of) an integral t-ideal of D.*
(2) *The maximal ideal M is an associated prime over a principal ideal of D (i.e., there exist $a \in D$ and $b \in D \setminus aD$ such that M is minimal over $(aD :_D bD)$).*
(3) *The maximal ideal M is minimal over (i.e., is the radical of) a principal ideal of D.*
(4) *The maximal ideal M is principal.*
(5) *The integral domain D is* 1*-dimensional.*

Proof. (1) is a straightforward consequence of Lemma 2.1. (2) and (3) are obvious from (1), because a proper ideal of the type $(aD :_D bD)$ and a principal ideal are both t-ideals. (4) is trivial consequence of (3). Finally, (5) follows from the fact that, in this case, the maximal ideal is a minimal prime over every nonzero (principal) ideal contained in it. □

Proposition 2.4. *If (D, M) is a local domain and the prime ideals of D are comparable in pairs, i.e.,* $\mathrm{Spec}(D)$ *is linearly ordered under inclusion, then D is t-local.*

Proof. Let $I = (x_1, x_2, \ldots, x_n)$ be a nonzero proper finitely generated ideal of D and let P be a minimal prime of I. The prime spectrum $\mathrm{Spec}(D)$ being linearly ordered forces P to be unique. Now let, for each $i = 1, 2, \ldots, n$, $P(x_i)$ be the minimal prime of the principal ideal (x_i). Again, by the linearity of order of $\mathrm{Spec}(D)$, for some $1 \leq k \leq n$, $P(x_k) \subseteq P(x_j)$ for all $j \neq k$. So $P(x_k) \supseteq I$ and so $P(x_k) \supseteq P$. But as $x_k \in P$, $P(x_k) \subseteq P$. Whence every proper nonzero finitely generated ideal of D is contained in a prime ideal of D that is minimal over a principal ideal and, hence,

P is a t-ideal, by Corollary 2.3(1). Thus, $I^v = I^t \subseteq P \subseteq M$. Since I is arbitrary as a finitely generated proper ideal of D, M is a t-ideal. $\square$

Remark 2.5. Note that, *mutatis mutandis*, from the proof of the previous proposition, if $\mathrm{Spec}(D)$ is linearly ordered under inclusion, we do not deduce only that D is t-local, but also that every prime ideal of D is a t-ideal (see also [32, Theorem 3.19]).

It is known that if J is a t-ideal of a ring of fractions D_S of an integral domain D with respect to a multiplicative subset S of D, then $J \cap D$ is a t-ideal of D [32, Lemma 3.17(1)]. However, I being a t-ideal of the integral domain D does not imply, in general, that ID_S is a t-ideal of D_S, even though $ID_S \cap D$ is a t-ideal of D [32, Lemma 3.17(2)] In particular, as the following Example 2.6 will show, the prime t-ideals may have a "bad behaviour", that is if P is a prime t-ideal of D then PD_S may not be a prime t-ideal for some multiplicative set S disjoint with P.

The authors of [39] were led to this conclusion seeing an example given by W. Heinzer and J. Ohm [29] of an essential domain (i.e., an integral domain $D = \bigcap D_P$ where P ranges over prime ideals of D such that D_P is a valuation domain) that is not a PvMD. The reason for this conclusion came from the following observation. For each maximal ideal M of the Heinzer-Ohm example D, D_M is a unique factorization domain, meaning the Heinzer-Ohm example is a locally GCD domain. Now, if for each maximal t-ideal Q, QD_Q were a prime t-ideal of D_Q, then D_Q would be a t-local domain and a GCD domain. But, as we shall see in the following Proposition 5.2, a t-local GCD domain is a valuation domain. So, we would have D_Q a valuation domain, for every maximal t-ideal Q of D, making D a PvMD. Therefore, since in this example D is not a PvMD, QD_Q might not be a t-ideal, for some maximal t-ideal Q of D. Indeed, an integral domain D which is locally a PvMD is a PvMD if and only if QD_Q is a t-ideal for every maximal t-ideal Q of D.

In [52], a *prime (t-ideal) P* in an integral domain D was called *well behaved* if PD_P is a prime t-ideal of D_P. We say that an *integral domain D* is *well behaved* if every prime (t-ideal) of D is well behaved. In [52], M. Zafrullah characterized well behaved domains and showed that most of the known domains, including PvMDs, are well behaved. Furthermore, in the same paper, there is also an example of an integral domain D such that every $Q \in \mathrm{Max}^t(D)$ is well behaved, but D is not well behaved. This example is obtained by a pullback construction, as briefly recalled below (for the details of the proofs see [52]).

Example 2.6. Let (V, M) be a valuation domain with $\dim(V) \geq 2$ and let P be a nonzero nonmaximal prime ideal of V, set $D := V + XV_P[X]$. In [52, Lemma 2.3, 2.4, and Proposition 2.5], it is proved that

$$\mathrm{Max}^t(D) = \{fD \mid f \in D,\ f \text{ is a prime element of } D \text{ such that } f(0) \in V \setminus M\} \cup \{N\},$$

where $N := \{f \in D \mid f(0) \in M\} = M + XV_P[X]$ is a maximal ideal of D.

By the previous description of $\mathrm{Max}^t(D)$, it is not hard to see that, for each $Q \in \mathrm{Max}^t(D)$, QD_Q is a maximal t-ideal of D_Q. Now, we consider the prime ideal $\mathfrak{P} := P + XV_P[X]$ of D. Since $P = \bigcap\{aV \mid a \in M \setminus P\}$, a direct verification shows that $\mathfrak{P} = \bigcap\{aD \mid a \in M \setminus P\}$. Thus $\mathfrak{P}$ is a v-ideal and, in particular, a t-ideal of D. However, after observing that $\mathfrak{P} \cap (V \setminus P) = \emptyset$, and so $D_{\mathfrak{P}} = (V + XV_P[X])_{P+XV_P[X]} = (V_P[X])_{\mathfrak{P}V_P[X]}$ and $\mathfrak{P}V_P[X] = PV_P + XV_P[X]$, it can be shown that $\mathfrak{P}D_{\mathfrak{P}} = \mathfrak{P}V_P[X]_{\mathfrak{P}V_P[X]}$ is not a t-ideal of $D_{\mathfrak{P}}$.

By the previous observations and example, for each $P \in \mathrm{Spec}(D)$, if D_P is a t-local domain, then P is a t-prime ideal of D; on the other hand, if a prime ideal P is a t-ideal of D, it is not true, in general, that D_P is a t-local domain. We give next some sufficient conditions for the localizations of an integral domain to be t-local domains.

Proposition 2.7. *Let D be an integral domain.*

(1) *If Q is an associated prime ideal over a principal ideal of D, then D_Q is a t-local domain.*
(2) *If $Q \in \mathrm{Max}^t(D)$ and Q is a potent ideal (i.e., it contains a nonzero finitely generated ideal that is not contained in any other maximal t-ideal), then D_Q is a t-local domain.*
(3) *If D has the finite t-character (i.e., every nonzero nonunit element of D belongs to at most a finite number of maximal t-ideals), then D_Q is a t-local domain, for each $Q \in \mathrm{Max}^t(D)$.*

Proof. (1) Since Q is minimal over a t-ideal of D of the type $(aD :_D bD)$, QD_Q is minimal over the ideal $(aD :_D bD)D_Q = (aD_Q :_{D_Q} bD_Q)$, which is a t-ideal of D_Q, and thus $QD_Q$is a t-ideal of D_Q (Corollary 2.3(2)).

(2) was proven in [3, Theorem 1.1(1)] and (3) follows from (2), since each maximal t-ideal in an integral domain with finite t-character is potent [3, Theorem 1.1(2)]. □

Remark 2.8. Recall that a prime t-ideal P of an integral domain D is said to be a *t-sharp ideal* if $\bigcap\{D_Q \mid Q \in \mathrm{Max}^t(D), P \nsubseteq Q\} \nsubseteq D_P$ [31, Section 3]. For a PvMD, it is known that a prime t-ideal P is t-sharp if and only if it is potent [31, Proposition 3.1].

If D has the finite t-character, then every maximal t-ideal is well behaved (Proposition 2.7(3)). It was observed in [3, Example 3.9] that the integral domain D, described in Example 2.6, has the finite t-character and so even an integral domain with the finite t-character might not be well behaved. We provide next another example of an integral domain which happens to be t-local (and so, trivially, with the finite t-character) and it is not well behaved (see, also, [3, Remark 3.2(2)]).

Example 2.9. Let $D_1 := \mathbb{Z}_{(p)}$ and so D_1 is a rank 1 discrete valuation domain of the field of rational numbers $K_1 := \mathbb{Q}$, with maximal principal ideal $N_1 := p\mathbb{Z}_{(p)}$.

Let $D_2 := \mathbb{Q}[\![X, Y]\!]$ be the power series ring in two variables with coefficients in the field $\mathbb{Q}$. Clearly, D_2 is an integrally closed local Noetherian 2-dimensional integral domain with maximal ideal $N_2 := (X, Y)\mathbb{Q}[\![X, Y]\!]$ and field of quotients

$K_2 := \mathbb{Q}((X,Y))$. Let $D_3 = K_2[\![Z]\!] = \mathbb{Q}((X,Y))[\![Z]\!]$; D_3 is a rank 1 discrete valuation domain of the field $K_3 := K_2((Z))$, with maximal ideal $N_3 := ZK_2[\![Z]\!]$. Set

$$D := D_1 + N_2 + N_3 = \mathbb{Z}_{(p)} + (X,Y)\mathbb{Q}[\![X,Y]\!] + Z\mathbb{Q}((X,Y))[\![Z]\!]\,.$$

Clearly, $D \subset T := D_2 + N_3 = \mathbb{Q}[\![X,Y]\!] + Z\mathbb{Q}((X,Y))[\![Z]\!] \subset D_3 = K_2 + N_3 = \mathbb{Q}((X,Y))[\![Z]\!]$. By well known properties of rings arising from pullback constructions, it is not hard to see that the following hold.

(1) T is a 3-dimensional local ring with maximal ideal $Q := N_2 + N_3$ and the localizations of T at each one of its infinitely many prime ideals of height 2 is a rank 2 discrete valuation domain.
(2) T has unique prime ideal of height 1, that is N_3. More precisely, N_3 is a common prime ideal of T and D_3 and $N_3 = (T : D_3)$, since N_3 is the maximal ideal of the local domain D_3; therefore, N_3 is a t-ideal (in fact, a v-ideal) of T. Furthermore, $T_{N_3} = D_3$ is a rank 1 discrete valuation domain.
(3) D is a 4-dimensional local domain, with maximal ideal $M := N_1 + N_2 + N_3$.
(4) M is a t-ideal (in fact, a v-ideal) of D, since $M = pD$, and so D is a t-local domain.
(5) $Q = N_2 + N_3 = \bigcap\{p^n D \mid n \geq 0\}$ is the unique prime of height 3 in D and it is a t-ideal (in fact, a v-ideal) of D, since Q is a common ideal of D and T and, since it is the maximal ideal of T, $Q = (D : T)$.
(6) For each one of the infinitely many height 2 prime ideals P of D, there exist a unique prime ideal P' of T such that $P' \cap D = P$ and the canonical embedding homomorphism $D_P \subseteq T_{P'}$ is an isomorphism; thus D_P is a rank 2 discrete valuation domain.
(7) Set $S := \{p^n \mid n \geq 0\}$, clearly S is a multiplicative set of D and $D_S = \mathbb{Q} + N_2 + N_3 = \mathbb{Q} + (X,Y)\mathbb{Q}[\![X,Y]\!] + Z\mathbb{Q}((X,Y))[\![Z]\!] = D_Q = T$.
(8) $QD_S = QD_Q = QT = Q$ is not a t-ideal of $D_Q = T$, since the elements $X, Y \in QD_Q = Q$ are v-coprime (note that, if F is a nonzero finitely generated ideal in a t-ideal I, then $F^v \subseteq I$).
(9) By the previous properties, it follows that T is a local, but not t-local, PvMD, since the localization at all its nonzero nonmaximal prime ideals is a valuation domain and its maximal ideal Q is not a t-ideal of T. Moreover, T is not completely integrally closed and so it is not a Krull domain, since its complete integral closure is D_3, because $N_3 = (T : D_3)$. T does not have the finite t-character, since each nonzero element inside its unique height 1 prime (t-)ideal N_3 is contained in all the infinitely many maximal t-ideals, which are all its prime ideals of height 2.
(10) Every nonzero prime ideal of D is a t-ideal and all of them are well behaved, except Q, its unique prime of height 3 (which is a t-ideal of D, but it is not a t-ideal in $D_Q = T$).

The following result was proved by D.D. Anderson, G. W. Chang, and M. Zafrullah in 2013 [3, Proposition 1.12(1)]:

Proposition 2.10. *Let D be a t-local domain, then the following hold.*

(1) *Every t-invertible ideal (i.e., an ideal I such that $(II^{-1})^t = D$) is principal.*
(2) *If I is an ideal of D such that $(I^n)^t = D$ for some $n \geq 2$, then I is principal.*

Proof. (1) If I be a t-invertible ideal of D then II^{-1} is in no maximal t-ideals of D and this implies that $II^{-1}D_Q = D_Q$ for every $Q \in \mathrm{Max}^t(D)$. In this special situation, $\mathrm{Max}^t(D) = \mathrm{Max}(D) = \{M\}$, where M is the only maximal ideal of the t-local domain D. Thus, I is invertible in a local domain and hence it is principal.

(2) In this situation, I is t-invertible, hence the conclusion follows from (1). $\square$

Note that the set $\mathtt{TI}(D)$ of all the fractional t-invertible t-ideals of an integral domain D is a group with respect to the operation $I \cdot_t J := (IJ)^t$, having as subgroup the set $\mathtt{Princ}(D)$ of all nonzero fractional principal ideals of D. The quotient group $\mathtt{Cl}^t(D) := \mathtt{TI}(D)/\mathtt{Princ}(D)$ is called the *t-class group of D*. The previous Proposition 2.10 can be also stated by saying that: *if D is a t-local domain then* $\mathtt{Cl}^t(D) = 0$.

3. t-local domains and local DW-domains

A nonzero ideal J of an integral domain D is called a *Glaz-Vasconcelos ideal* (for short, a *GV-ideal*) if J is finitely generated and $J^{-1} = D$. The set of Glaz-Vasconcelos ideals of D is denoted by $\mathtt{GV}(D)$ [21]. Given a nonzero fractional ideal E of D, the *w-closure of E* is the fractional ideal $E^w := \{x \in K \mid xJ \subseteq E, \text{ for some } J \in \mathtt{GV}(D)\}$. A nonzero fractional ideal E is called a *w-ideal* if $E = E^w$. The w-operation was introduced by Wang-McCasland in [47].

It is well known that w, like v, t, and the identity operation d are examples of star operations (respectively, w, like t, and d are examples of star operations of finite type) [25, Proposition 3.2] and also that $d \leq w \leq t \leq v$, this means that, for each $E \in \boldsymbol{F}(D)$, we have the following inclusions $E^d := E \subseteq E^w \subseteq E^t \subseteq E^v$. Furthermore, for each $E \in \boldsymbol{F}(D)$, $E^w = \bigcap\{ED_Q \mid Q \in \mathrm{Max}^t(D)\}$ and the set of maximal w-ideals of D, $\mathrm{Max}^w(D)$, coincide with the set of maximal t-ideals of D, $\mathrm{Max}^t(D)$ [45].

It is natural to ask what is the relation between a t-local domain and a *w-local domain*, i.e., a local domain such that its maximal ideal is a w-ideal. A t-local domain is necessarily a w-local domain, since $d \leq w \leq t$ and conversely, since as observed above, $\mathrm{Max}^w(D) = \mathrm{Max}^t(D)$. We will show that something more is true, that is, in a t-local domain, every nonzero ideal is a w-ideal. For showing this, we need some preliminaries.

Recall that a *DW-domain* is an integral domain D such that $d = w$, i.e., for each nonzero fractional ideal E of D, $E = E^w$; this is equivalent to requiring that every nonzero (integral) finitely generated ideal of D is a w-ideal. The following result is due to F. Wang [46, Proposition 1.3] (see also A. Mimouni [38, Proposition 2.2]).

Proposition 3.1. *Let D be an integral domain. The following are equivalent.*

(i) *D is a DW-domain.*
(ii) *Every nonzero prime ideal of D is a w-ideal*
(iii) *Every maximal ideal of D is a w-ideal*

(iv) *Every maximal ideal of D is a t-ideal*
(v) $\mathtt{GV}(D) = \{D\}$.

Proof. Obviously, (i)⇒(ii)⇒(iii).
(iii)⇒(iv) is a consequence of the fact that $\mathrm{Max}^w(D) = \mathrm{Max}^t(D)$.
(iv)⇒(v) Let $J \in \mathtt{GV}(D)$ and $J \subsetneq D$. Let $M \in \mathrm{Max}^t(D)$ such that $J \subseteq M$, then $D = J^v = J^t \subseteq M^t = M$, which is a contradiction.
(v)⇒(i) Let I be a nonzero ideal of D and let $0 \neq x \in I^w$ then, for some $J \in \mathtt{GV}(D)$, $xJ \subseteq I$. Since $\mathtt{GV}(D) = \{D\}$, $xD \subseteq I$ and so $I^w \subseteq I$. □

From the previous proposition we deduce immediately the following.

Corollary 3.2. *Let D be an integral domain. The following are equivalent.*
(i) *D is a t-local.*
(ii) *D is a w-local*
(iii) *D is a local DW-domain.*

Remark 3.3. Note that, for a t-local domain, it is not true that every nonzero ideal is a t-ideal, i.e., a domain such that $d = t$ or a *DT-domain*; even more, for a t-local domain, it may happen that every nonzero prime ideal is a t-ideal, without being a DT-domain (see the following Example 3.5). The DT-domains are also called *fgv-domains*, that is domains such that every nonzero finitely generated ideal is a v-ideal since, for each nonzero ideal I, $I = I^t$ if and only if, for each nonzero finitely generated ideal J, $J^v = J^t = J$. M. Zafrullah in [49] studied the fgv-domains and he proved that an integrally closed fgv-domain is a Prüfer domain. Note that, for a Noetherian domain, being a DT-domain is equivalent to being a domain such that each nonzero ideal is divisorial (i.e., a domain such that $d = v$). In particular, W. Heinzer has proven that, for a Noetherian domain D, if every nonzero ideal is divisorial, then $\dim(D) \leq 1$ [26, Corollary 4.3]; furthermore, for an integrally closed Noetherian domain (or, more generally, for any completely integraly cosed domain) D, every nonzero ideal is divisorial if and only if D is Dedekind domain [26, Proposition 5.5].
Finally, note that DT-domains are exactly the DW-domains that are at the same time *TW-domains*, i.e., domains such that $w = t$ [37].

Lemma 3.4. *Let (T, N) be a local domain, let $\boldsymbol{k}(T) := T/N$, let $\varphi : T \to \boldsymbol{k}(T)$ be the canonical projection, and let R be a subring of the field $\boldsymbol{k}(T)$. Set $D := \varphi^{-1}(R)$. then D is a t-local domain with maximal ideal M if and only if R is a t-local domain (with maximal ideal $\varphi(M)$).*

Proof. By the standard properties of the pullbacks constructions, D is a local domain with maximal ideal M if and only if R is a local domain (with maximal ideal $\varphi(M)$) [15, Corollary 1.5]. Moreover, for each $E \in \boldsymbol{F}(R)$, $\varphi^{-1}(E) \in \boldsymbol{F}(D)$ and $(\varphi^{-1}(E))^w = \varphi^{-1}(E^w)$ [37, Lemma 3.1]. Note that $M = \varphi^{-1}(\varphi(M))$, and thus $M = M^w$ if and only if $\varphi(M) = (\varphi(M))^w$. Therefore (D, M) is w-local if and only if $(R, \varphi(M))$ is w-local. The conclusion follows from Corollary 3.2. □

Example 3.5. *Example of a Noetherian t-local domain (hence, a local DW-domain) which is not a DT-domain, but each nonzero prime ideal is a t-ideal.*

Consider the 2-dimensional Noetherian integrally closed domain $T := \mathbb{C}[X,Y]_{(X,Y)}$, which is clearly not a t-local domain, since its (finitely generated maximal) ideal $M := (X,Y)\mathbb{C}[X,Y]_{(X,Y)}$ is not a divisorial ideal of T (the only divisorial ideals of T are its height 1 prime ideals). However, by the previous lemma, the local 2-dimensional Noetherian domain $D := \mathbb{R} + (X,Y)\mathbb{C}[X,Y]_{(X,Y)}$ ($= \varphi^{-1}(\mathbb{R})$, where $\varphi : T \to T/M \cong \mathbb{C}$ is the canonical projection) is a t-local domain, since its maximal ideal $M = (X,Y)\mathbb{C}[X,Y]_{(X,Y)}$ is divisorial as an ideal of D, being $M = (D : T)$. Moreover, every nonzero prime ideal of D is a t-ideal. Indeed, for the well known properties of the pullback constructions, every nonzero nonmaximal prime ideal P of D is such that $P = Q \cap D$, where Q is a nonzero nonmaximal prime ideal of T, and moreover D_P is canonically isomorphic to T_Q [15, Theorem 1.4 (part (c) of the proof)]. Since T_Q is a DVR, D_P is a DVR too and hence PD_P is a t-ideal of D_P and, in particular, P is a t-ideal of D.

Finally, D is not DT-domain or, equivalently for Noetherianity, D is not a divisorial domain, since $\dim(D) = 2$ (Remark 3.3). Explicitly, for instance, M^2 is not a divisorial ideal (or, equivalently, not a t-ideal) of D (and of T), since $(D : M^2) = ((D : M) : M) = (T : M) = T$ and so $(D : (D : M^2)) = (D : T) = M$.

Recall that an overring T of an integral domain D is is called *t-linked over D* if, for each nonzero finitely generated ideal J of D such that $J^t = D$, then $(JT)^t = T$. An integral domain is *t-linkative* if every overring is t-linked [13].

Proposition 3.6. *Let D be an integral domain. Then, D is t-local domain if and only if D is a local t-linkative domain.*

The previous proposition is a straightforward consequence of the following theorem.

Theorem 3.7. (Dobbbs-Houston-Lucas-Zafrullah, 1989 [13, Theorem 2.6]) *Let D be an integral domain. The following are equivalent.*

(i) *Every overring of D is t-linked over D.*
(ii) *Every valuation overring of D is t-linked over D.*
(iii) *Every maximal ideal of D is a t-ideal.*
(iv) *For each nonzero proper ideal I of D, $I^t \neq D$.*
(v) *For each nonzero proper finitely generated ideal J of D, $J^t \neq D$.*
(vi) *Each t-invertible ideal of D is invertible.*

Finally, we introduce a construction for building new examples of t-local domains.

We recall that, given an integral domain D, the *Nagata ring of D* (see, for instance, [18, Section 33]) is defined as follows:

$$D(X) := \{f/g \mid f, g \in D[X],\ g \neq 0,\ \text{with } \boldsymbol{c}(g) = D\},$$

(where $\boldsymbol{c}(h)$ is the content of a polynomial $h \in D[X]$).

First in [32] and then in [16], the construction of the Nagata ring was extended to the case of an arbitrary chosen star (or, even semistar) operation. Given a star operation $*$ on D, set:

$$\mathtt{Na}(D,*) := \{f/g \mid f,g \in D[X],\ g \neq 0,\ \text{with } \boldsymbol{c}(g)^* = D\}.$$

With this notation $\mathtt{Na}(D,d) = D(X)$. Moreover, it is clear that

$$\mathtt{Na}(D,v) = \mathtt{Na}(D,t) = \mathtt{Na}(D,w)$$

since, for each nonzero finitely generated ideal F of D, $F^v = F^t$ and, moreover, $F^t = D$ if and only if $F^w = D$, because $\mathrm{Max}^t(D) = \mathrm{Max}^w(D)$.

Proposition 3.8. *Let D be an integral domain.*

(1) *The Nagata ring $\mathtt{Na}(D,v)$ is a DW-domain; in particular, if $\mathrm{Max}^t(D) = \{Q\}$ is a singleton, then $\mathtt{Na}(D,v)$ is a t-local-domain with maximal t-ideal $Q\mathtt{Na}(D,v)$.*

(2) *The following are equivalent.*
 (i) *D is a t-local domain.*
 (ii) *$\mathtt{Na}(D,v) = D(X)$ and $D(X)$ is local.*
 (iii) *$D(X)$ is a t-local domain.*

Proof. (1) Recall that $\boldsymbol{\mathcal{N}} := \{g \in D[X] \mid g \neq 0 \text{ and } \boldsymbol{c}(g)^* = D\}$ is a saturated multiplicatively closed subset of $D[X]$, $\boldsymbol{\mathcal{N}} = D[X] \setminus \left(\bigcup\{QD[X] \mid Q \in \mathrm{Max}^{\star_f}(D)\}\right)$, $\mathtt{Na}(D,v) = D[X]_{\boldsymbol{\mathcal{N}}}$, and $\mathrm{Max}(\mathtt{Na}(D,v)) = \{Q\mathtt{Na}(D,v) \mid Q \in \mathrm{Max}^t(D)\}$ (see [16, Proposition 3.1] or [32, Proposition 2.1]). Then, it is easy to see that $\mathtt{Na}(D,v)_{Q\mathtt{Na}(D,v)} = D[X]_{QD[X]} = D_Q(X)$ and $Q\mathtt{Na}(D,v) = QD_Q(X) \cap \mathtt{Na}(D,v)$, for each $Q \in \mathrm{Max}^t(D)$, and so:

$$\mathtt{Na}(D,v) = \bigcap\{D_Q(X) \mid Q \in \mathrm{Max}^t(D)\}.$$

Moreover, for each ideal I of D, $(I\mathtt{Na}(D,v))^t = I^t\mathtt{Na}(D,v)$ [32, Corollary 2.3]. Therefore, in particular, $Q\mathtt{Na}(D,v)$ is a t-ideal of $\mathtt{Na}(D,v)$ for each $Q \in \mathrm{Max}^t(D)$, i.e., $\mathrm{Max}(\mathtt{Na}(D,v)) = \mathrm{Max}^t(\mathtt{Na}(D,v))$.

(2) (i)$\Rightarrow$(ii). We already observed that $\mathtt{Na}(D,v) = \mathtt{Na}(D,t) = \mathtt{Na}(D,w)$. In the present situation $d = w$ and so $\mathtt{Na}(D,w) = \mathtt{Na}(D,d) = D(X)$.

(ii)$\Rightarrow$(iii). Obvious, since we have shown in (1) that, when D is t-local, $\mathtt{Na}(D,v)$ is t-local too.

(iii)$\Rightarrow$(i) Since the maximal ideals of $D(X)$ are exactly the ideals $M(X) := MD(X)$, with $M \in \mathrm{Max}(D)$ [18, Proposition 33.1], and since $M(X)^t = M^t(X)$ [32, Corollary 2.3], the conclusion is straightforward. □

By the previous proposition, the Nagata ring can be used to give new examples of DW-domains and, in particular, of t-local domains. For instance, it is known that $D(X)$ is treed (i.e., the prime spectrum is a tree under the set theoretic inclusion $\subseteq$) if and only if D is treed and the integral closure $\overline{D}$ of D is a Prüfer domain [4, Theorem 2.10]. Thus, if we take a treed domain D such that $\overline{D}$ is not Prüfer, in this case $D(X)$ is a DW-domain, but not treed. For an explicit example, take $D := \mathbb{Q} + U\mathbb{Q}(V)[[U]]$, where U and V are two indeterminates, then $D = \overline{D}$ [4, Remark 2.11], D is a t-local (treed) integrally closed domain but not a valuation domain, and thus $D(X)$ is a t-local non treed integrally closed domain, since the integral closure $\overline{D(X)} = \overline{D}(X) = D(X)$ [4, Proposition 2.6].

4. Comparable elements and t-local domains

A *nonzero* element $c \in D$ is called *comparable in D* if, for all $x \in D$, we have $cD \subseteq xD$ or $xD \subseteq cD$. It is easy to see that $c \in D$ is comparable if cD is comparable (under inclusion) with each ideal I of D. The following result is essentially Lemma 3.2 of [8].

Lemma 4.1. *Let α be a nonzero nonunit element of a local domain (D, M). If, for each $x \in D$, $\alpha D + xD = yD \subseteq M$, then α is a comparable element.*

Proof. By the assumption, it follows that $(\alpha/y)D + (x/y)D = D$ and, since D is local, α/y or x/y is a unit of D. Thus, the element y is an associate of α or of x. In the first case, $y|x$ (or, equivalently, $\alpha|x$) and, in the second case, $y|\alpha$ (or, equivalently, $x|\alpha$). Therefore, α is a comparable element of D. □

Lemma 4.2. *Let c be a comparable element in an integral domain D. If h is a nonunit factor of c, then h is also a comparable element of D.*

Proof. Let $c = hy$ and let $x \in D$. Then $cD + xyD = hyD + xyD = y(hD + xD)$ coincides with cD or xyD, since c is comparable. In the first case, $y(hD + xD) = cD = yhD$, thus $hD + xD = hD$, i.e., $x|h$. In the second case, $y(hD + xD) = xyD$ and thus $hD + xD = xD$, i.e., $h|x$. □

The comparable elements were introduced and studied in [5] to prove, in case of valuation domains, a Kaplansky-type theorem (recall that Kaplansky proved that an integral domain D is a UFD if and only if every nonzero prime ideal of D contains a prime element [33, Theorem 5]).

Lemma 4.3. (D.D. Anderson and M. Zafrullah [5, Theorem 3]) *An integral domain D is a valuation domain if and only if every nonzero prime ideal of D contains a comparable element.*

An important part of the result was the proof of the fact that the set of all comparable elements of D is a saturated multiplicative set.

We recall in the next lemma some of the consequences of the existence of a nonzero nonunit comparable element in an integral domain.

Lemma 4.4. (Gilmer-Mott-Zafrullah [20, Theorem 2.3]) *Suppose the integral domain D contains a nonzero nonunit comparable element and let $\mathscr{C}$ be the (nonempty) set of nonzero comparable elements of D. Then:*

1. *$P := \bigcap\{cD \mid c \in \mathscr{C}\}$ is a prime ideal of D and $D \setminus P = \mathscr{C}$ (in particular, $\mathscr{C}$ is a saturated multiplicative set of D).*
2. *D/P is a valuation domain.*
3. *$P = PD_P$.*

(4) *D is local, P compares with every other ideal of D under inclusion, and* $\dim(D) = \dim(D/P) + \dim(D_P)$.
(5) *If T is any integral domain such that there is a nonmaximal prime ideal Q of T such that* (a) *T/Q is a valuation domain, and* (b) *$Q = QT_Q$, then each element of $T \setminus Q$ is comparable.*
(6) *If, in addition, Q is minimal in T with respect to properties* (5, a) *and* (5, b) *above, then $T \setminus Q$ is precisely the set of nonzero comparable elements of T.*

Of course, an integral domain D is a valuation domain if and only if every nonzero element of D is comparable. As an easy consequence of the previous lemma we obtain immediately the following.

Corollary 4.5. *Suppose the integral domain D contains a nonzero nonunit comparable element and let $\mathscr{C}$ be the (nonempty) set of nonzero comparable elements of D. Then, D is a valuation domain if and only if $\cap\{cD \mid c \in \mathscr{C}\} = (0)$.*

Proof. The statement follows from (1) and (2) of Lemma 4.4. □

Recall that E.D. Davis proved that, given a ring S and a subring R, if R is local then (R, S) is a normal pair (i.e., every ring T, $R \subseteq T \subseteq S$, is integrally closed in S) if and only if there is a prime ideal Q in R such that $S = R_Q$, $Q = QR_Q$, and R/Q is a valuation domain [12, Theorem 1]. From the previous remark and Lemma 4.4, we deduce immediately the following.

Corollary 4.6. *Suppose the integral domain D contains a nonzero nonunit comparable element. Let $\mathscr{C}$ be the set of nonzero comparable elements of D and $P := \bigcap\{cD \mid c \in \mathscr{C}\}$, as in Lemma 4.4(1). In this situation, (D, D_P) is a normal pair.*

In [20], a part of the following result was proved as a consequence of Lemma 4.4. We next prove, directly, that the existence of a nonzero nonunit comparable element in an integral domain is a sufficient but not necessary condition for being a t-local domain.

Proposition 4.7. *An integral domain D that contains a nonzero nonunit comparable element is a t-local domain, while a t-local domain may not contain a nonzero nonunit comparable element.*

Proof. Let D be an integral domain and let c be a nonzero nonunit comparable element in D. We first show that D is local. Suppose, by way of contradiction, that there exist two co-maximal nonunit elements x, y in D, i.e., $rx + sy = 1$ for some $r, s \in D$. Now, as c is comparable, $c|rx$ or $rx|c$. So rx has a nonzero nonunit comparable factor c or, being a factor of c, rx is a nonzero nonunit comparable element. Thus, in both cases, rx has a nonzero nonunit comparable factor h. Similarly sy has a nonzero nonunit comparable factor k. Since h, k are comparable, $h|k$ or $k|h$, say $h|k$. Thus, assuming that $rx + sy = 1$, we get the contradictory conclusion that a nonunit divides a unit. So, D is local. We denote by M its maximal ideal.

Next, let $x_1, x_2, \ldots, x_n \in M$ and note that, as above, each of the x_i has a nonzero nonunit comparable factor h_i. Thus, $(x_1, x_2, \ldots, x_n) \subseteq (h_1, h_2, \ldots, h_n)$.

Now, consider h_1, h_2. They must have a nonzero nonunit common factor k_1 (which is equal to h_1 or h_2). So, $(x_1, x_2, \ldots, x_n) \subseteq (h_1, h_2, \ldots, h_n) \subseteq (k_1, h_3, \ldots, h_n)$. Continuing this process, we eventually get a nonzero nonunit comparable element k such that $(x_1, x_2, \ldots, x_n) \subseteq (h_1, h_2, \ldots, h_n) \subseteq (k) \subseteq M$. But, as $(x_1, x_2, \ldots, x_n) \subseteq (k)$ implies $(x_1, x_2, \ldots, x_n)^v \subseteq (k)$, we conclude that, for each finitely generated ideal $(x_1, x_2, \ldots, x_n) \subseteq M$, $(x_1, x_2, \ldots, x_n)^v \subseteq M$. Thus, D is a t-local domain.

For the converse, note that a one dimensional local domain has only one nonzero prime (=maximal) ideal and so it is a valuation ring if and only if it contains a nonunit comparable element, by the Kaplansky-type theorem mentioned above (Lemma 4.3). The proof is complete once we note that there do exist one-dimensional, (Noetherian t-)local domains that are not valuation domains (in fact, non integrally closed domains) (e.g., $\mathbb{R} + X\mathbb{C}[\![X]\!]$).

Note also that there even exist 1-dimensional t-local integrally closed domains that are not valuation domains (e.g., $\overline{\mathbb{Q}} + X\mathbb{C}[\![X]\!]$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$). $\square$

Remark 4.8. Note that the previous example shows that a local domain with divisorial maximal ideal may not contain a nonzero nonunit comparable element. On the other hand, a valuation domain V with nonprincipal maximal ideal (in particular, $\dim(V) \geq 2$) is a domain containing a nonzero nonunit comparable element and so it is a t-local domain with nondivisorial maximal ideal.

Recall that an integral domain D with quotient field K is called a *pseudo-valuation domain* (for short, PVD) if D is local and the maximal ideal M of D is strongly prime (i.e., whenever elements x and y of K satisfy $xy \in M$, then either $x \in M$ or $y \in M$). From the proof of the previous Proposition 4.7, we give now a general class of t-local domains that do not contain nonzero nonunit comparable elements.

Example 4.9. Let (T, M) be any local domain, let $\boldsymbol{k}(T) := T/M$, let $\varphi : T \to \boldsymbol{k}(T)$ be the canonical projection, and let F be a proper subfied of $\boldsymbol{k}(T)$. Set $D := \varphi^{-1}(F)$. It is known that D is a local domain with maximal ideal M and $(M : M) = (D : M) = T$. Since $M = (D : T)$, it is easy to see that M is a divisorial ideal in D and, in particular, a t-ideal. Thus, (D, M) is a t-local domain. In particular, any PVD is a t-local domain [24, Theorem 2.10].

Remark 4.10. Note that the argument used in the previous example can be used to construct a more general class of t-local domains. Start from a (not necessarily local) integral domain T such that its Jacobson ideal $J(T)$ is nonzero and suppose that the ring $T/J(T)$ contains properly a field F. Let $\varphi : T \to T/J(T)$ be the canonical projection and let $D := \varphi^{-1}(F)$, then D is a t-local domain.

A fractional ideal $E \in \boldsymbol{F}(D)$ is said to be *v-invertible* (respectively, *t-invertible*) if there is $G \in \boldsymbol{F}(D)$ such that $((EG)^v = D$ (respectively, $(EG)^t = D$). Obviously, every invertible ideal is t-invertible.

Recall that a *GCD domain* is an integral domain D such that, for each $a, b \in D$, $aD \cap bD$ is principal or, equivalently, $(a,b)^v$ is principal. Therefore, a GCD domain (e.g., a Bézout domain) is a PvMD.

Corollary 4.11. *Let D be a PvMD, not a field. Then, D is a valuation domain if and only if D contains a nonzero nonunit comparable element.*

Proof. The statement follows from Proposition 4.7, from the fact that a t-local PvMD is a valuation domain anyway and from the fact that a valuation domain that is not a field must contain many nonunit comparable elements (in fact, all nonunit elements are comparable). □

From the previous corollary it follows that every Krull domain (e.g., UFD) containing a nonzero nonunit comparable element is a DVR and that every GCD domain containing a nonzero nonunit comparable element is a valuation domain.

Now, here comes something more general and a tad surprising. Call an integral domain D *atomic* if every nonzero nonunit of D is expressible as a finite product irreducible elements. An irreducible element is called also *atom*. For instance, every Noetherian domain and every UFD is atomic.

Corollary 4.12. *An atomic domain that contains a nonzero nonunit comparable element is a DVR.*

Proof. Let D be an atomic domain and let c be a nonzero nonunit comparable element in D. Then, by Proposition 4.7, D is t-local domain; denote by M its maximal ideal. Let h be an irreducible factor of c. Then h is a comparable element, being a factor of a comparable element (Lemma 4.2). So, for every x in D, either $h|x$ or $x|h$. Now, as h is irreducible, $x|h$ means that x is a unit or $x = h$. Thus, for all nonunits $x \in D$, necessarily $h|x$. That is $M = hD$ and so h is a prime element in D. Next, as $h|x$ for each nonzero nonunit $x \in D$, we have $x = x_1h$ and if x_1 is a nonunit then $x_1 = x_2h$ and so $x = h^2x_2$. Continuing this way, since D is atomic, for each nonzero nonunit $x \in D$ there is an integer $n = n(x)$ (depending on x) such that $x = h^nx_n$ where x_n is a unit. But then we can conclude that D is a DVR and h is a uniformizing parameter of D. □

Corollary 4.12 was first proved for Noetherian domains; we thank Tiberiu Dumitrescu for suggesting the atomic domain assumption. With hindsight we can prove a more precise result.

Corollary 4.13. *Let D be a domain that contains a nonzero nonunit comparable element.*

(1) *In this situation, D is local (Proposition 4.7) and the maximal ideal of D is generated by the nonunit comparable elements of D.*
(2) *The integral domain D contains an atom α if and only if α is the generator of the (unique) maximal ideal of D and, hence, α is a prime and comparable element.*

Proof. (1) By Proposition 4.7, D is t-local; let M denote the maximal ideal of D. With the notation of Lemma 4.4, M properly contains the comparable prime ideal P of D. If $(x_1, x_2, \ldots, x_n)$ is a finitely generated ideal and $P \subseteq (x_1, x_2, \ldots, x_n) \subseteq M$, since D/P is a valuation domain, then $(x_1, x_2, \ldots, x_n) = (x)$ for some $x \in \{x_1, x_2, \ldots, x_n\}$. Therefore, since $M = M^t$, M is generated by the nonunit comparable elements of D.

(2) Let α be an atom of D and let c be a nonzero nonunit comparable element of D. Then, either $c|\alpha$ or $\alpha|c$. If $c|\alpha$ then, as α is an atom and c a nonunit, c and α must be associate, so α is a comparable element. If, on the other hand, $\alpha|c$ then α is a comparable element, being a factor of a comparable element (Lemma 4.2). Thus, as above, $\alpha D = M$.

The converse is obvious, indeed if the maximal ideal M of a local domain D is principal and $M = \alpha D$ then, up to associates, α is the only atom in D. □

Note that if, instead considering atoms (=irreducible elements), we consider prime elements, we can state a result analogous to the previous corollary in a more general setting, with a different proof.

Proposition 4.14. *Let D be a domain.*

(1) *If a maximal t-ideal M of D contains a prime element p, then $M = pD$.*
(2) *If (D, M) is a t-local domain (e.g., if D contains a nonzero nonunit comparable element), then D contains a prime element p if and only if p is the generator of the maximal ideal of D and, hence, p is a comparable element.*

Proof. (1) Let p be a prime element of a domain D then, for each x in D, $pD \cap xD = xD$ or $pD \cap xD = pxD$.

So,

$$((p, x)D)^{-1} = \frac{pD \cap xD}{px} = \left(\frac{1}{p}\right) D \quad \text{or} \quad ((p, x)D)^{-1} = D.$$

But then $((p, x)D)^v = pD$ or $((p, x)D)^v = D$. So, if a prime element p belongs to a maximal t-ideal M then $M = pD$.

(2) If a prime element p belongs to a t-local ring (D, M) then $M = pD$, by (1) and consequently p is a comparable element of D. □

It is well known that, if p is a prime element in an integral domain D, then $\bigcap_{n \geq 0} p^n D$ is a prime ideal too (see, for instance, Kaplansky [33, Exercise 5, pages 7-8]).

Theorem 4.15. *If a domain D contains a nonzero nonunit comparable element then, for every nonzero nonunit comparable element x of D, we have that $Q := \bigcap_{n \geq 0} x^n D$ is a prime ideal such that D/Q is a valuation domain and $Q = QD_Q$.*

Conversely, if there is a nonzero element x in a domain D such that $Q := \bigcap_{n \geq 0} x^n D$ is a prime ideal, D/Q is a valuation domain, and $Q = QD_Q$, then D is t-local and x is a comparable element of D.

Proof. Indeed Q is an ideal, being an intersection of ideals. Now, consider $S := D\backslash Q$ and let $a, b \in S$. Then $a \notin x^m D$ for some positive integer m and $b \notin x^n D$ for some positive integer n. Since x and hence x^m, x^n are comparable, we conclude that $aD \supsetneq x^m D$ and $bD \supsetneq x^n D$. Therefore, $abD \supsetneq ax^n D \supsetneq x^{n+m} D$ and so $ab \in S$ and Q is a prime ideal.

From the above proof it follows that S consists of factors of powers of the comparable element x and so every element of S is comparable; this implies that D/Q is a valuation domain. Next, let $\alpha/\tau \in QD_Q$ where $\alpha \in Q$ and $\tau \in D\backslash Q$. In particular, τ divides some power of x and so τ is comparable. Hence, $\alpha D \subseteq Q \subsetneq \tau D$ which means that for some nonunit y we have $\alpha = \tau y$. As $\tau \notin Q$, then necessarily $y \in Q$. So $\alpha/\tau = y \in Q$. Thus $QD_Q \subseteq Q$, i.e. $Q = QD_Q$.

The converse follows from Lemma 4.4(5) and Proposition 4.7 (see also [20, Theorem 2.3]). $\square$

Note that there are integral domains that may or may not be local, but have elements x such that $\cap x^n D =: Q$ is a prime ideal such that $Q = QD_Q$, but D/Q is not a valuation domain. Here are some examples using the $D + M$ construction studied by Gilmer [18, page 202].

We start from a valuation domain V, with quotient field K, expressible as $V = \boldsymbol{k} + M$, where $\boldsymbol{k}$ is a subfield of V (and K) and M is the maximal ideal of V; thus, in the present situation, the residue field V/M is canonically isomorphic to $\boldsymbol{k}$. Let D be a subring of $\boldsymbol{k}$. The ring $R := D + M$ (subring of V) with quotient field K (the same as V) has some interesting properties due to the mode of this construction, as indicated for instance in [7] (see also [15, Theorem 1.4]). Our concrete model for these examples would be $V := \boldsymbol{k}[\![X]\!] = \boldsymbol{k} + X\boldsymbol{k}[\![X]\!]$.

Example 4.16. Given a field $\boldsymbol{k}$, let D be a 1-dimensional local domain contained in $\boldsymbol{k}$, with quotient field F ($\subseteq \boldsymbol{k}$) and suppose that D is not a valuation domain. Then $R := D + X\boldsymbol{k}[\![X]\!]$ is a (local) 2-dimensional domain such that, for each nonzero nonunit x in D, we have $\bigcap_{n\geq 0} x^n R = X\boldsymbol{k}[\![X]\!]$. Indeed, for a nonunit x in a 1-dimensional local domain D, we have $\bigcap_{n\geq 0} x^n D = (0)$ and so $\bigcap_{n\geq 0} x^n R = X\boldsymbol{k}[\![X]\!]$. Moreover, since $R_{X\boldsymbol{k}[\![X]\!]} = F + X\boldsymbol{k}[\![X]\!]$, then $X\boldsymbol{k}[\![X]\!]R_{X\boldsymbol{k}[\![X]\!]} = X\boldsymbol{k}[\![X]\!](F + X\boldsymbol{k}[\![X]\!]) = X\boldsymbol{k}[\![X]\!]$. In this situation, $R/X\boldsymbol{k}[\![X]\!] = D$.

What makes the above example work is the fact that, for a nonunit x in a one dimensional local domain D, we have $\bigcap_{n\geq 0} x^n D = (0)$. Call an integral domain D an *Archimedean domain* if, for all nonunit elements x in D, we have $\bigcap_{n\geq 0} x^n D = (0)$ [43, Definition 3.6] (this class of domains was previously considered in [41] without naming them). By the Krull intersection theorem, every Noetherian domain is Archimedean. Since Mori domains satisfy the ascending chain condition on principal ideals, they are Archimedean; in particular, Krull domains are Archimedean. The class of Archimedean domains includes also completely integrally closed domains [19, Corollary 5] and 1-dimensional integral domains [41, Corollary 1.4].

An Archimedean (possibly non local or any dimensional) version of the previous Example 4.16 is given next.

Example 4.17. Given a field $\boldsymbol{k}$, let D be an Archimedean domain contained in $\boldsymbol{k}$, with quotient field F ($\subseteq \boldsymbol{k}$) and suppose that D is not a valuation domain. Then, as above, $R := D + X\boldsymbol{k}[\![X]\!]$ is such that, for each nonzero nonunit x in D, we have $\bigcap_{n\geq 0} x^n R = X\boldsymbol{k}[\![X]\!]$, $X\boldsymbol{k}[\![X]\!] = X\boldsymbol{k}[\![X]\!]R_{X\boldsymbol{k}[\![X]\!]}$ and $R/X\boldsymbol{k}[\![X]\!] = D$. In the present situation, $\mathrm{Max}(R)$ has the same cardinality of $\mathrm{Max}(D)$ and $\dim(R) = \dim(D) + 1$.

Example 4.18. Let D be an integral domain and S a multiplicative subset of D. Following the construction $R := D + XD_S[X]$ of [11], if s is a nonunit element in S such that $\bigcap_{n\geq 0} s^n D = (0)$ then $\bigcap_{n\geq 0} s^n R = XD_S[X]$ a prime ideal of R. Also in this case $R/XD_S[X] = D$, which might not be a valuation domain. However, in the present situation, $XD_S[X] \subsetneq XD_S[X](R_{XD_S[X]}) = XD_S[X]_{(X)}$.

5. From t-local domains to valuation domains

Because in a valuation domain (V, M) every finitely generated ideal is principal, the maximal ideal M is obviously a t-ideal. So t-local domains are "cousins" of valuation domains, but sort of far removed. For instance, a localization of a t-local domain is not necessarily t-local (see, for instance, Example 2.9 or [52]), but of course a localization of a valuation domain is a valuation domain.

Explicitly, a more simple example is given by $R := \mathbb{Z}_{(p)} + (X, Y)\mathbb{Q}[\![X, Y]\!]$. The integral domain R is local with maximal ideal $M := p\mathbb{Z}_{(p)} + (X, Y)\mathbb{Q}[\![X, Y]\!] = pR$, and so it is obviously a t-local domain. However, $R[1/p] = R_Q = \mathbb{Q}[\![X, Y]\!]$, where $Q := (X, Y)\mathbb{Q}[\![X, Y]\!]$, is a 2-dimensional local Noetherian Krull domain, and so it is far away from being t-local.

So it is legitimate to ask: Under what conditions is a t-local domain a valuation domain? Here we address this question.

The following is a simple result that hinges on the fact that if F is a nonzero finitely generated ideal in a t-ideal I then $F^v \subseteq I$.

Proposition 5.1. *For a finite set of elements $x_1, x_2, \ldots, x_n$, in a t-local domain (D, M), the following are equivalent.*

(i) $(x_1, x_2, \ldots, x_n)^v = D$.
(ii) *At least one x_i is a unit.*
(iii) $(x_1, x_2, \ldots, x_n) = D$.

Proof. Clearly, (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii) By the previous observation $(x_1, x_2, \ldots, x_n) \nsubseteq M$, and so at least one $x_i \notin M$. $\square$

Proposition 5.2. *For an integral domain D the following are equivalent.*

(i) *D is a valuation domain*
(ii) *D is a t-local GCD domain (or, equivalently, a t-local Bézout domain).*
(iii) *D is a t-local PvMD (or, equivalently, a t-local Prüfer domain).*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are straightforward.

For (iii) $\Rightarrow$ (i) note for instance that, in a PvMD, every nonzero finitely generated ideal $(x_1, x_2, ..., x_n)$ is t-invertible. But, by [3, Proposition 1.12(1)], $(x_1, x_2, ..., x_n)$ is a principal ideal. □

Recall that a ring is *coherent* if every finitely generated ideal is finitely presented. It is well known that a commutative integral domain D is coherent if and only if the intersection of every pair of finitely generated ideals is finitely generated [9, Theorem 2.2].

Call a domain D *a finite conductor domain* (for short, *FC domain*; this name was used for the first time in [48]) if the intersection of every pair of principal ideals of D is finitely generated. Indeed, "finite conductor domain" is a generalization of "coherent domain".

Proposition 5.3. *For an integral domain D the following are equivalent.*

(i) *D is a valuation domain.*
(ii) *D is an integrally closed coherent t-local domain.*
(iii) *D is an integrally closed finite conductor t-local domain.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are all straightforward.

For (iii) $\Rightarrow$ (i) note that an integrally closed FC domain is a PvMD [48, Theorem 2] (or, [18, Exercise 21, page 432]) and we already observed that a t-local PvMD is a valuation domain (Proposition 5.2((iii)$\Rightarrow$(i))). □

As an application of the previous proposition, we easily obtain the following result due to S. McAdam.

Corollary 5.4. (S. McAdam [35, Theorem 1]) *Let D be an integrally closed local domain whose primes are linearly ordered by inclusion. Assume that D is a FC domain, then D is a valuation domain.*

Proof. By Proposition 2.4, D is t-local. The conclusion follows from Proposition 5.3((iii)$\Rightarrow$(i)). □

A nonzero element r of a domain D is called a *primal element* if for all $x, y \in D\backslash\{0\}$ $r|xy$ implies that $r = st$ where $s|x$ and $t|y$. A domain whose nonzero elements are all primal is called a *pre-Schreier domain*. An integrally closed pre-Schreier domain was called a *Schreier domain* by P.M. Cohn in his paper [10, page 254]. There, he showed that a GCD domain is a Schreier domain [10, Theorem 2.4].

Based on considerations initiated by McAdam and Rush [36], a module M is said to be *locally cyclic* if every finitely generated submodule of M is contained in a cyclic submodule of M. Thus, in particular, an ideal I of D is locally cyclic if, for any finite set of elements $x_1, x_2, \ldots, x_n \in I$, there is an element $d \in I$ such that $d|x_k$ for each k, $1 \leq k \leq n$.

In [51, Theorem 1.1], M. Zafrullah has shown that *an integral domain D is pre-Schreier if and only if for all $a, b \in D\backslash(0)$ and $x_1, x_2, \ldots, x_n \in (a) \cap (b)$ there is $d \in (a) \cap (b)$ such that $d|x_k$, for each k, $1 \leq k \leq n$.*

Based on this, we easily obtain the following.

Lemma 5.5. *If D is a pre-Schreier domain and $a, b \in D\backslash\{(0)\}$, then the following are equivalent:*

(i) *$(a) \cap (b)$ is principal.*
(ii) *$(a) \cap (b)$ is finitely generated.*
(iii) *$(a) \cap (b)$ is a v-ideal of finite type.*

Proof. Indeed (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are all straightforward. All we need is show (iii) $\Rightarrow$ (i). For this note that if $(a) \cap (b) = (x_1, x_2, \ldots, x_n)^v$, then, $x_1, x_2, \ldots, x_n \in (a) \cap (b)$. Since D is pre-Schreier, there is an element $d \in (a) \cap (b)$ such that $d|x_k$, for each k, $1 \leq k \leq n$, i.e., $(x_1, x_2, \ldots, x_n) \subseteq (d)$. But then $(x_1, x_2, ...x_n)^v \subseteq (d)$, and so $(d) \subseteq (a) \cap (b) = (x_1, x_2, ...x_n)^v \subseteq (d)$. □

Call a domain D a *v-finite conductor* (for short, *v-FC*) *domain* if, for each pair $0 \neq a, b \in D$, the ideal $(a) \cap (b)$ is a v-ideal of finite type. Then, recalling that a GCD domain is integrally closed, from Lemma 5.5, we easily deduce:

Corollary 5.6. *Let D be an integral domain. The following are equivalent.*

(i) *D is a GCD domain*
(ii) *D is a Schreier and a v-FC domain.*
(iii) *D is a pre-Schreier and a v-FC domain.*

With this preparation, we have the following result.

Corollary 5.7. *For an integral domain D, the following are equivalent:*

(i) *D is a valuation domain,*
(ii) *D is a pre-Schreier t-local coherent domain,*
(iii) *D is a pre-Schreier t-local FC domain,*
(iv) *D is a pre-Schreier t-local v-FC domain,*
(v) *D is a GCD t-local domain.*

Proof. It is obvious that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv); (iv) $\Leftrightarrow$ (v) by Corollary 5.6 and (v) $\Leftrightarrow$ (i) by Proposition 5.2. □

Obviously, the above are not the only situations in which a t-local integral domain becomes a valuation domain. We describe next another interesting situation of this phenomenon, in case of existence of a comparable element.

Proposition 5.8. *Suppose that an integral domain D contains a nonzero nonunit comparable element x and let $Q := \bigcap_{n \geq 0} x^n D$. Then, D is a valuation domain if and only if D_Q is a valuation domain.*

Proof. Indeed, if D is a valuation domain, since Q is a prime ideal (Theorem 4.15), D_Q is also a valuation domain and so we have only to take care of its converse.

The presence of a nonzero nonunit comparable element makes D a t-local domain (Proposition 4.7). In order to prove that D is a valuation domains, we consider the finitely generated ideals of D. We split the proper finitely generated ideals into two types: **(a)** ones that contain a nonunit factor of a power of x and **(b)** ones that do not contain a nonunit factor of a power of x.

Ones in part **(a)** are principal by [20, Theorem 2.4] and ones in part **(b)** are contained in Q and are principal proper ideals of the valuation domain D_Q and hence are in QD_Q. By Proposition 4.15 above, $QD_Q = Q$, so, for each y in Q, yD_Q is (also) an ideal of D, i.e., $yD_Q = yD$. Now, let $x_1, x_2, \dots, x_n \in Q$ and consider the ideal $(x_1, x_2, \dots, x_n)$. Since D_Q is a valuation domain, $(x_1, x_2, \dots, x_n)D_Q = dD_Q$ and we can assume that d is in D. So, for some $r_i \in D$ and $s_i \in D \backslash Q$ we have $x_i = \frac{r_i}{s_i} d$, for each i.

So $(x_1, x_2, \dots, x_n) = (\frac{r_1}{s_1} d, \frac{r_2}{s_2} d, \dots, \frac{r_n}{s_n} d)$. Removing the denominators, we get $s(x_1, x_2, \dots, x_n) = (t_1 d, t_2 d, \dots, t_n d) = (t_1, t_2, \dots, t_n)d$, for some $s \in D \setminus Q$, where $s_i | s$ and $t_i := \frac{s}{s_i} r_i$, for each i. As $dD_Q = (x_1, x_2, \dots, x_n)D_Q = s(x_1, x_2, \dots, x_n)D_Q$ $= (t_1, t_2, \dots, t_n)dD_Q$, we conclude that $(t_1, t_2, \dots, t_n)D_Q = D_Q$. But that means that at least one of the t_i is in $D \backslash Q$ and hence is a comparable element (Lemma 4.4(5)). But then, by [20, Theorem 2.4], $(t_1, t_2, \dots, t_n)$ is principal generated by a comparable element t. Thus, $s(x_1, x_2, \dots, x_n) = (t_1, t_2, \dots, t_n)d = tdD$. Since s and t are comparable, we have two possibilities: **($\boldsymbol{\alpha}$)** $u(x_1, x_2, \dots, x_n) = dD$ or **($\boldsymbol{\beta}$)** $(x_1, x_2, \dots, x_n) = vdD$, for some $u, v \in D$. In both cases $(x_1, x_2, ...x_n)$ turns out to be a principal ideal of D (in case **($\boldsymbol{\alpha}$)** because $d \in u(x_1, x_2, \dots, x_n)$ and so $u | d$ in D). $\square$

6. Applications: Shannon's quadratic extension

A domain D is a *treed domain* if it has a treed spectrum, i.e., $\mathrm{Spec}(D)$ is a tree as a poset with respect to the set inclusion. Note that D is a treed domain if and only if any two incomparable primes of D are co-maximal. Indeed, if D is a treed then D_P is also a treed (more precisely, $\mathrm{Spec}(D_P)$ is linearly ordered) for every nonzero prime ideal P of D. So, by Proposition 2.4, D_P is a t-local domain and thus $P = PD_P \cap D$ is a t-ideal of D. Indeed, if F is a finitely generated ideal of D contained in P, then $F^t D_P = F^v D_P \subseteq (FD_P)^v = (FD_P)^t \subseteq (PD_P)^t = PD_P$ and so $F^t \subseteq (FD_P)^t \cap D \subseteq PD_P \cap D = P$ (see also [53, page 436]). Therefore, in a treed domain, every nonzero prime ideal is a t-ideal (Proposition 2.4), in particular every maximal ideal is a t-ideal, and moreover it is well behaved. However, a general t-local domain D may not have $\mathrm{Spec}(D)$ a tree as, for instance, Examples 2.9 and 4.17 indicate. So the class of treed domains is strictly contained in the class of domains whose maximal ideals are t-ideals. But, in the presence of some extra conditions, this distinction may disappear.

Proposition 6.1. *For a Prüfer v-multiplication domain D, the following conditions are equivalent.*

(i) *Every maximal ideal of D is a t-ideal.*
(ii) *Every prime ideal of D is a t-ideal.*
(iii) $\mathrm{Spec}(D)$ *is a tree.*
(iv) *D is a Prüfer domain.*

Proof. (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) hold in general (without the PvMD assumption). More precisely, (iv) $\Rightarrow$ (iii) is clear because in a Prüfer domain D, D_P is a valuation domain for every nonzero prime ideal P and so $\mathrm{Spec}(D)$ is a tree. (iii) $\Rightarrow$ (ii) has been explained above.

(i) $\Rightarrow$ (iv) For every prime t-ideal P of a PvMD D, we have D_P a valuation domain (see, for instance, [39, Corollary 4.3]) and if we assume that D_M is a valuation domain, for every maximal ideal M of D, then D is well known to be a Prüfer domain. □

The previous proposition leads to the following result for FC domains.

Corollary 6.2. *Let D be an integral domain. The following are equivalent.*

(i) *$D$is an integrally closed finite conductor treed domain.*
(ii) *D is a treed PvMD;*
(iii) *D is Prüfer.*

Proof. (i) $\Rightarrow$ (ii), since an integrally closed finite conductor domain is a PvMD by Proposition 5.3 and [39, Corollary 4.3]. (ii) $\Leftrightarrow$ (iii) by Proposition 6.1 and (iii) $\Rightarrow$ (i) because a Prüfer domain is a FC domain [48, Corollary 10]. □

Indeed, it is worth noting that a nonzero proper ideal I in an integral domain D is said to be an *ideal of grade* 1 if I does not contain a set of elements forming a regular sequence of length ≥ 2. Recall that, if an ideal I of an integral domain D contains a regular sequence of length 2, then $I^{-1} = D$ [33, Exercise 1, page 102]. So, every t-ideal of an integral domain is a grade 1 ideal and every nonzero prime ideal in a treed domain is a grade 1 ideal. With this background, for the next application we need a little bit of preparation.

Let $(R, \mathfrak{m})$ be a regular local integral domain with quotient field F and $\mathfrak{p}$ a prime ideal of R so that $R/\mathfrak{p}$ is a regular local domain. A *monoidal transform of R with nonsingular center* $\mathfrak{p}$ is a local domain of the type $T := R[\mathfrak{p}x^{-1}]_Q$, where $0 \neq x \in \mathfrak{p}$ and Q is a prime ideal in $R[\mathfrak{p}x^{-1}]$ such that $\mathfrak{m} \subseteq Q$. In particular, assume that $\dim(R) = n$, and $\mathfrak{p} = \mathfrak{m} = (x_1, x_2, \ldots, x_n)R$, where $\{x_1, x_2, \ldots, x_n\}$ form a regular sequence in R. Choose $i \in \{1, 2, \ldots, n\}$, and consider the overring $R[x_1/x_i, x_2/x_i, \ldots, x_n/x_i]$ of R. Take any prime ideal Q of $R[x_1/x_i, x_2/x_i, \ldots, x_n/x_i]$ such that $Q \supseteq \mathfrak{m}$. The ring $R_1 := R[x_1/x_i, x_2/x_i, \ldots, x_n/x_i]_Q$ is called a *local quadratic transform* (for short, *LQT*) of R, and, again, R_1 is a regular local integral domain with maximal ideal $\mathfrak{m}_1 := QR[x_1/x_i, x_2/x_i, \ldots, x_n/x_i]_Q$ [40, Corollary 38.2]. Assume that $\dim(R) \geq 2$ in order to have that $R \neq R_1$. By Cohen's dimension inequality formula $\dim(R_1) \leq n$ [34, Theorem 15.5] (and, more precisely, $\dim(R_1) = n$ if and only if $R_1/\mathfrak{m}_1$ is an algebraic extension of $R/\mathfrak{m}$) [2, (1.4)].

If we iterate the process, we obtain a sequence $R =: R_0 \subseteq R_1 \subseteq R_2 \subseteq ...$ of regular local overrings of R such that for each $j \geq 0$, R_{j+1} is a LQT of R_j. After a finite number of iterations, the sequence of nonincreasing integers $\dim(R_j)$ is necessarily bound to stabilize, and this process of iterating LQTs of the same Krull dimension (definitively, after a certain point) and ascending unions of the resulting regular sequences are of interest in algebraic geometry. For a description the reader may consult a couple of recent papers [23] and [27]. So, let $R =: R_0 \subseteq R_1 \subseteq R_2 \subseteq$

... be a sequence of LQTs from a regular local integral domain R with $\dim(R) \geq 2$ and $\dim(R_j) \geq 2$, for each $j \geq 1$, as described above. The ring $S := \bigcup_{j\geq 0} R_j$, dubbed in recent work as *Shannon's Quadratic Extension of* R, to honor David Shannon [43] for his interesting contribution, has drawn particular attention.

Briefly, before Shannon, Abhyankar [1, Lemma 12] had shown that, if the regular local ring R has dimension 2, then S is a valuation overring of R such that the maximal ideal $\mathfrak{m}_S$ of S contains the maximal ideal $\mathfrak{m}$ of R. David Shannon, one of Abhyankar's students, showed that if $\dim(R) > 2$, S need not be a valuation ring [43, Examples 4.7 and 4.17].

Our purpose here is to look at S from a simple star-operation theoretic perspective, to provide some direct straightforward and brief proofs of some known results and point to known results that could simplify some of the considerations in recent work.

We start by gathering some information about the Shannon's Quadratic Extension S. Next two properties can be easily proved.

(1) $S(:= \bigcup_{j\geq 0} R_j)$*, as described above, is a local ring and, if* $\mathfrak{m}_S$ *denotes the maximal ideal of* S*,* $\mathfrak{m}_S = \bigcup_{j\geq 0} \mathfrak{m}_j$ *where* $\mathfrak{m}_j$ *is the maximal ideal of the LQT* R_j*.*

(2) S *is integrally closed, as being integrally closed a first order property which is preserved by directed unions and hence, in particular, by ascending unions.*

Since S is directed union of regular local integral domains and, by the Auslander-Buchsbaum theorem [34, Theorem 20.3], each regular local integral domain is a UFD and hence, in particular, a GCD domain and so, *a fortiori*, a Schreier domain. This observation gives us the next property of S.

(3) S *is (at least) a Schreier domain.*

This follows from a direct verification that a direct union of (pre-)Schreier domains is a (pre-)Schreier domain.

Remark 6.3. Note that it is not true that a direct union of GCD-domains is a GCD-domain. An example can be given by an integral domain of the type $D^{(\Sigma)} := D + XD_\Sigma[X] = \bigcup\{D[X/s] \mid s \in \Sigma\}$, where D is a GCD domain and Σ is a saturated multiplicative subset D, since it is known that $D^{(\Sigma)}$ is not a GCD if Σ is not a splitting set, i.e., if Σ does not verify the condition that, for each $0 \neq d \in D$, $d = sa$ for some $s \in \Sigma$ and $a \in D$ with $aD \cap s'D = as'D$ for all $s' \in \Sigma$ [50, Corollary 1.5].

We give now an explicit example. Let $\boldsymbol{\mathcal{E}}$ be the ring of entire functions. It is well known that $\boldsymbol{\mathcal{E}}$ is a Bézout domain [18, Exercise 18, page 147] and that every nonzero nonunit x of $\boldsymbol{\mathcal{E}}$ can be written uniquely as a countable product of finite powers of non associate primes, i.e., $x = u\prod_{\alpha\in A} p_\alpha^{n_\alpha}$ where A is a countable set, n_α are natural numbers and p_α are mutually non associated primes elements of $\boldsymbol{\mathcal{E}}$ and u is a unit in $\boldsymbol{\mathcal{E}}$. The last property follows from the fact that the set of zeros of a nontrivial entire function is discrete, including multiplicities, the multiplicity of a zero of an entire function is a positive integer and a zero of an entire function determines a principal prime in $\boldsymbol{\mathcal{E}}$ [30, Theorem 6]. Clearly, each of these primes generate a height one maximal ideal of $\boldsymbol{\mathcal{E}}$ [18, Exercise 19, page 147].

Let Σ be the multiplicative set generated by all of these principal, height one primes and let X be an indeterminate. Then, the ring $\boldsymbol{\mathcal{E}}^{(\Sigma)} := \boldsymbol{\mathcal{E}} + X\boldsymbol{\mathcal{E}}_{\Sigma}[X] = \bigcup\{\boldsymbol{\mathcal{E}}[X/s] \mid s \in \Sigma\}$ is not a GCD domain, even though $\boldsymbol{\mathcal{E}}[X/s]$ is a GCD domain for each $s \in \Sigma$.

Indeed, if $x \in \boldsymbol{\mathcal{E}}$ is an infinite product of primes then it is not possible to write $x = sx_1$ where $s \in \Sigma$ and x_1 is not divisible by any of the nonunits in Σ, since each s is a finite product of primes and x is a product of infinitely many primes from Σ. Thus, Σ is not a splitting set and so $\boldsymbol{\mathcal{E}}^{(\Sigma)}$ cannot be a GCD domain.

However, we claim that $\boldsymbol{\mathcal{E}}^{(\Sigma)}$ is a locally GCD domain. For proving the claim, we need some preliminaries. A prime ideal P of an integral domain D is said *to intersect in detail a multiplicative set* Σ of D if every nonzero prime ideal Q contained in P intersects Σ. It was shown [50, Proposition 4.1] that if D is a locally GCD domain and Σ is a multiplicative set of D such that every maximal ideal of D that intersects Σ, intersects Σ in detail, then $D^{(\Sigma)}$ is a locally GCD domain.

Indeed, clearly the Bézout domain $\boldsymbol{\mathcal{E}}$ is a locally GCD domain. Moreover, as every maximal ideal of $\boldsymbol{\mathcal{E}}$ that intersects Σ contains a finite product of principal primes and so must be a principal ideal. Thus, every maximal ideal of $\boldsymbol{\mathcal{E}}$ that intersects Σ, intersects it in detail. Consequently $\boldsymbol{\mathcal{E}}^{(\Sigma)}$ is a locally GCD domain; however, $\boldsymbol{\mathcal{E}}^{(\Sigma)}$ is not a PvMD, since $\boldsymbol{\mathcal{E}}^{(\Sigma)}$ is a Schreier domain and a PvMD which also is a Schreier domain is a GCD domain [6, Proposition 2.3].

As a final remark, we recall from [50, Proposition 4.3] that in a locally GCD non-PvMD D there always exists a maximal t-ideal Q of D such that QD_Q is not a t-deal of D_Q. More precisely, it can be shown that an integral domain D is a PvMD if and only if D is locally PvMD and, for every t-prime ideal P of D, PD_P is a (maximal) t-ideal of D_P [50, Corollary 4.4].

We now resume our study of Shannon's Quadratic Extension S.

(4) *There exists an element $x \in \mathfrak{m}_S$ such that $\mathfrak{m}_S = \sqrt{xS}$* [27, Proposition 3.8].

The last property gives us, in light of Corollary 2.3(1), the following property that is of interest to us.

(5) *S is a t-local integral domain.*

This is enough information to provide very naturally the statements and easy new proof(s) of [23, Theorem 6.2].

Theorem 6.4. (L. Guerrieri, W. Heinzer, B. Olberding and M. Toeniskoetter [23, Theorem 6.2]) *Let S be a quadratic Shannon extension of a regular local integral domain R. Then, the following are equivalent.*

(i) *S is a valuation domain*
(ii) *S is coherent.*
(iii) *S is a finite conductor domain.*
(iv) *S is a GCD domain.*
(v) *S is a PvMD.*
(vi) *S is a v-finite conductor domain.*

Proof. The equivalence of (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) comes from Corollary 5.3. Now (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) follow from Proposition 5.2 and, as S is Schreier (by (3)), (i) $\Leftrightarrow$ (vi) by Corollary 5.7. □

From Lemma 5.5, Corollary 5.7, and Theorem 6.4 we easily deduce the following.

Corollary 6.5. *Let S be a quadratic Shannon extension of a regular local integral domain R. If S is not a valuation domain, then S contains a pair of elements a, b such that $aS \cap bS$ is not a v-ideal of finite type.*

Proof. If, for each pair of elements $a, b \in S$, we had that $aS \cap bS$ is a v-ideal of finite type, then S would be a GCD domain by Corollary 5.6, since S is a Schreier domain (by point (3) above). Therefore, S would be a valuation domain by Theorem 6.4, which is not the case. □

This corollary is significant with reference to the proof of the previous theorem (Theorem 6.4) in that there are PvMDs D, such as Krull domains, that contain elements a, b such that $aS \cap bS$ a v-ideal of finite type, which may not be finitely generated.

From [27, Proposition 4.1], we conclude that S has another property of interest.

(5) *For each element $x \in \mathfrak{m}_S$ such that $\mathfrak{m}_S = \sqrt{xS}$, the integral domain $T := S[1/x]$ is a regular local ring with* $\dim(T) = \dim(S) - 1$.

So, if $\dim(S) = 2$ and $\mathfrak{m}_S$ contains a nonzero comparable element then we know that S is a valuation domain (Theorem 4.15 and (5)).

If $\dim(S) > 2$ then S cannot be a valuation domain, whether S contains a comparable element or not, because a regular local ring T, constructed from S as in (5), has $\dim(T) > 1$, and thus T may not be a valuation domain. However, if $\mathfrak{m}_S = pS$ is principal then, S is a non-valuation t-local domain that contains a comparable element, by Proposition 4.14(2). This fact, together with Proposition 5.8, provides a definitive criterion that can be used to construct examples of non-valuation t-local domains containing a comparable element, even in dimension two.

Example 6.6. Let $\mathbb{Z}$ be the ring of integers, $\mathbb{Q}$ (resp., $\mathbb{R}$) the field of rational numbers (resp. real numbers) and p a prime element in $\mathbb{Z}$. Let P be the maximal ideal of the DVR $\mathbb{R}[\![X]\!]$ and set $D := \mathbb{Z}_{(p)} + X\mathbb{R}[\![X]\!] = \mathbb{Z}_{(p)} + P$. The integral domain D is local with principal maximal ideal $M := pD$ and $\bigcap_{n\geq 0} p^n D = X\mathbb{R}[\![X]\!] = P$. Clearly, p is a proper comparable element in D. Since $D_P = \mathbb{Q} + X\mathbb{R}[\![X]\!]$ is not a valuation domain, D is a 2-dimensional non-Noetherian non-valuation t-local integral domain with prime spectrum linearly ordered given by $\{M \supset P \supset (0)\}$.

In the same vein, and this is suggested by Tiberiu Dumitrescu, we have another example.

Example 6.7. Let $\mathbb{Z}$ be the ring of integers, $\mathbb{Q}$ the field of rational numbers and p a nonzero prime element in $\mathbb{Z}$. Let $D := \mathbb{Z}_{(p)} + P$ where P is the maximal ideal (X^2, X^3) of $\mathbb{Q}[\![X^2, X^3]\!]$. As above, D is a local domain with maximal ideal $M = p\mathbb{Z}_{(p)} + P = pD$ and $\bigcap_{n\geq 0} p^n D = P$. In this case, $D_P = \mathbb{Q}[\![X^2, X^3]\!]$ which is a well known 1-dimensional Noetherian domain that is not a valuation domain (in fact, it is non integrally closed). Thus, D is a 2-dimensional non-Noetherian non-valuation t-local integral domain, having a proper comparable element and prime spectrum linearly ordered given by $\{M := p\mathbb{Z}_{(p)} + (X^2, X^3)\mathbb{Q}[\![X^2, X^3]\!] \supset P \supset (0)\}$.

We can provide examples in any dimension. Let P be the maximal ideal of the n-dimensional regular local ring $\mathbb{Q}[\![X_1, X_2, \ldots, X_n]\!]$. Then $D := \mathbb{Z}_{(p)} + P$ is local with maximal ideal $M := pD$. In particular, D contains a proper comparable element, e.g., p, and, of course, D_P is far from being a valuation domain. Thus, D is an $(n+1)$-dimensional non-valuation t-local integral domain.

Note that a 1-dimensional domain that contains a nonzero nonunit comparable element is a valuation domain. This follows from the following two facts (1) the presence of a comparable element forces the domain to be (1-dimensional) t-local and (2) a domain is a valuation domain if and only if every nonzero prime ideal contains a nonzero comparable element (Lemma 4.3).

From (5), we deduce another interesting property of S.

(6) *Let S be as above (i.e., a quadratic Shannon extension of a regular local integral domain), for each element $x \in \mathfrak{m}_S$ such that $\mathfrak{m}_S = \sqrt{xS}$, call the saturation of the multiplicative set $\{x^n \mid n \in \mathbb{N}\}$,* span of x *and denote it by $\mathtt{span}(x)$. Then,*

(6a) *for every nonunit h in $\mathtt{span}(x)$ we have $\mathfrak{m}_S = \sqrt{hS}$* and

(6b) *$\mathfrak{m}_S$ is generated by nonunits in $\mathtt{span}(x)$.*

The saturated multiplicative set $\mathtt{span}(x)$ has been used before, by Dumitrescu, Lequain, Mott, and Zafrullah in [14], to determine the number of distinct maximal t-ideals that the element x belongs to. Here, the statement that the ideal $\mathfrak{m}_S$ is generated by nonunit members of $\mathtt{span}(x)$ is caused by the fact that there is only one maximal t-ideal (i.e., $\mathfrak{m}_S$) involved.

Note that, before introducing quadratic Shannon extensions of local regular rings, all examples of t-local domains that we have considered in the present paper were valuation domains or rings obtained by some pullback construction. At this point, it is natural to ask if the quadratic Shannon extensions, that are not valuation domains, could as well be obtained by some appropriate pullback construction. For this purpose, we start by recalling some other properties of the quadratic Shannon extensions.

(7) *Let S be as above (i.e., a quadratic Shannon extension of a regular local integral domain of dimension > 2). If S is Archimedean, then its complete integral closure S^* coincides with $(\mathfrak{m}_S : \mathfrak{m}_S) = T \cap W$, where $\mathfrak{m}_S$ is the maximal ideal of S, $T = S[1/x]$ is the local regular overring of S introduced in* (5) *and W is a uniquely determined valuation overring of S and if $S \neq S^*$, S^* is a generalized Krull domain* [27, Theorem 6.2].

In the previous situation, if $S \neq S^*$, $\mathfrak{m}_S$ is a height 1 prime ideal of S^*, since it is the center of the maximal ideal of the valuation overring W of S^* (see [27, Corollary 6.3] and [28, Theorem 7.4]). Therefore, S is the pullback of the residue field $S/\mathfrak{m}_S$ with respect to the canonical projection $S^* \to S^*/\mathfrak{m}_S$.

On the other hand, in the non-Archimedean case, we know the following fact.

(8) *Let S be as above (i.e., a quadratic Shannon extension of a regular local integral domain of dimension > 2). If S is non-Archimedean, then its complete integral closure S^* coincides with the overring $T = S[1/x]$, $\bigcap\{x^n S \mid n \geq 0\} =: \mathfrak{p}$ is a proper prime ideal of S and $T = (\mathfrak{p} : \mathfrak{p})$* [27, Threorem 6.9 and Corollary 6.10].

In the previous situation, the integral domain $S/\mathfrak{p}$ is a DVR [27, Lemma 3.4], and $T = S_{\mathfrak{p}}$, since $T = S[1/x]$ is a ring of fractions of S and $\mathfrak{p}$ is disjoint from the multiplicative set $\{x^n \mid n \geq 0\}$. Therefore, S is the pullback of $S/\mathfrak{p}$ with respect to the canonical projection $T \to T/\mathfrak{p}$, where $T/\mathfrak{p}$ is a field, coinciding with the residue field $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$ (isomorphic to the field of quotients of the integral domain $S/\mathfrak{p}$).

The last remaining case is when the quadratic Shannon extension S is (Archimedean and) completely integrally closed. An example is given in [28, Corollary 7.7]. In this situation S may not be obtained by a pullback construction of some of its overrings, since, if an integral domain A shares a nonzero ideal with one of its proper overrings B then A and B must have the same complete integral closure [19, Lemma 5].

We end with a classification of the t-local domains, which could be useful for detecting t-local domains that are not issued from a pullback construction. The following proposition is a consequence of more general results concerning DT-domains, proved by G. Picozza and F. Tartarone in [42].

Proposition 6.8. *Let (D, M) be a local domain.*

(1) *If $D \neq (M : M)$, then D is a t-local domain.*
(2) *If $D = (M : M)$ and M is finitely generated, then D is a t-local domain if and only if M is principal.*
(3) *If $D = (M : M)$, and M is not finitely generated, then D is a t-local domain if and only if M is not t-invertible.*

Proof. (1) If $D \neq (M : M)$, then necessarily the maximal ideal M is the conductor of the inclusion $D \hookrightarrow (M : M)$ and so M is a divisorial ideal of D.

(2) Assume that $D = (M : M)$, and M is finitely generated, clearly M is divisorial if and only if $(M : M) = D \neq M^{-1} = (D : M)$ and this happens if and only if $M \neq MM^{-1} (\subseteq D)$ or, equivalently, if and only if $MM^{-1} = D$. In a local domain, a nonzero ideal is invertible if and only if it is a principal ideal.

(3) Assume that $D = (M : M)$, M is not finitely generated and, moreover, M is not a t-invertible ideal. If M is not a t-ideal, then $M^t = D$ and thus $(MM^{-1})^t = M^t = D$, which is a contradiction.

Conversely, since M is not finitely generated, M is not invertible and, since D is t-local, M is not even t-invertible (Theorem 3.7 ((iii)$\Rightarrow$(vi)). $\square$

Any pseudo-valuation non-valuation domain provides an example of case (1); a discrete valuation domain (for short, DVR) is an example of case (2) and a rank 1 non-DVR valuation domain is an example of case (3).

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References

[1] S. Abhyankar, On the valuations centered in a local domain, Amer. J. Math. **78** (1956), 321–348.

[2] S. Abhyankar, *Resolution of singularities of embedded algebraic surfaces,* Academic Press, New York, 1966.

[3] D.D. Anderson, G.W. Chang and M. Zafrullah, Integral domains of finite t-character, J. Algebra **396** (2013) 169–183.

[4] D.F. Anderson, D.E. Dobbs, and M. Fontana, On treed Nagata Rings, J. Pure Appl. Alg. **261** (1989), 107–122.

[5] D.D. Anderson and M. Zafrullah, On a theorem of Kaplansky, Boll. Un. Mat. Ital. A(7), **8** (1994), 397–402.

[6] D.D. Anderson and M. Zafrullah, The Schreier property and Gauss' Lemma, Boll. Un. Mat. Ital. **10-B** (2007), 43–62.

[7] E. Bastida and R. Gilmer, Overrings and divisorial idels of rings of the form $D+M$, Michigan Math. J. **20** (1973), 79–95.

[8] G. Chang, T. Dumitrescu and M. Zafrullah, t-Splitting sets in integral domains, J. Pure Appl. Algebra **187** (2004), 71–86.

[9] S. U. Chase, Direct product of modules, Trans. Am. Math. Soc. **97** (1960), 457–493.

[10] P.M. Cohn, Bezout rings and their subrings, Proc. Cambridge Phil. Soc. **64** (1968) 251–264.

[11] D. Costa, J. Mott and M. Zafrullah, The construction $D + XD_S[X]$, J. Algebra **53** (1978), 423–439.

[12] E.D. Davis, Overrings of commutative rings, III, Trans. Am. Math. Soc. **182** (1973), 175–185.

[13] D. Dobbs, E. Houston, T. Lucas and M. Zafrullah, t-linked overrings and PVMDs, Comm. Algebra **17** (1989), 2835–2852.

[14] T. Dumitrescu, Y. Lequain, J. Mott, and M. Zafrullah, Almost GCD domains of finite t-character, J. Algebra **245** (2001), 161–181.

[15] M. Fontana, Topologically defined classes of commutative rings, Ann. Mat. Pura Appl. **123** (1980), 331–355.

[16] M. Fontana and K.A. Loper, Nagata rings, Kronecker function rings, and related semistar operations, Comm. Algebra **31** (2003), 4775–4805.

[17] S. Gabelli and M. Roitman, Maximal divisorial ideals and t-maximal ideals, JP J. Algebra, Number Theory and Applications **4** (2004), 323–336.

[18] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, NewYork, 1972.

[19] R.W. Gilmer and W.J. Heinzer, On the complete integral closure of an integral domain, J. Australian Math. Soc. **6** (1966), 351–361.

[20] R. Gilmer, J. Mott and M. Zafrullah, t-invertibility and comparability, in *"Commutative Ring Theory"*, P.-J. Cahen, D. Costa, M. Fontana and S.-E. Kabbaj (Editors), Marcel Dekker, New York, 1994, 141–150.

[21] S. Glaz and W.V. Vasconcelos, Flat ideals, II, Manuscripta Math. **22** (1977), 325–341.

[22] M. Griffin, Some results on v-multiplication rings, Canad. J. Math. **19** (1967), 710–722.

[23] L. Guerrieri, W. Heinzer, B. Olberding and M. Toeniskoetter, Directed unions of local quadratic transforms of regular local rings and pullbacks, in *"Rings, Polynomials, and Modules"* Fontana, Marco; Frisch, Sophie; Glaz, Sarah; Tartarone, Francesca; Zanardo, Paolo (Editors), Springer International Publishing 2017.

[24] J.R. Hedstrom and E. Houston, Pseudo-valuation domains, Pacific J. Math. **75** (1978), 137–147.

[25] J.R. Hedstrom and E. Houston, Some remarks on star-operations, J. Pure Appl. Algebra **18** (1980), 37–44.

[26] W. Heinzer, Integral domains in which each non-zero ideal is divisorial, Mathematika **15** (1968),164–170.

[27] W. Heinzer, K. A. Loper, B. Olberding, H. Schoutens and M. Toeniskoetter, Ideal theory of infinite directed unions of local quadratic transforms, J. Algebra **474** (2017), 213–239.

[28] W. Heinzer, Bruce Olberding, and M.Toeniskoetter, Asymptotic properties of infinite directed unions of local quadratic transforms, J. Algebra **479** (2017), 216–243.

[29] W. Heinzer and J. Ohm, An essential ring which is not a v-multiplication ring, Canad. J. Math. **25** (1973), 856–861.

[30] O. Helmer, Divisibility properties of integral functions, Duke J. Math. **6** (1940) 345–356.

[31] E. Houston, M. Fontana, and M.H. Park, Sharpness and semistar operations in Prüfer-like domains, Submitted.

[32] B.G. Kang, Prüfer v-multiplication domains and the ring $R[X]_{N_v}$, J. Algebra **123** (1989), 151–170.

[33] I. Kaplansky, *Commutative Rings*, Allyn and Bacon, 1970.

[34] H. Matsumura, *Commutative rings*, Cambridge University Press, Cambridge, 1986.

[35] S. McAdam, Two conductor theorems, J. Algebra **23** (1972), 239–240.

[36] S. McAdam and D. Rush, Schreier rings, Bull. London Math. Soc. **10** (1978), 77–80.

[37] A. Mimouni, TW-domains and strong Mori domains, J. Pure Appl. Algebra **177** (2003), 79–93.

[38] A. Mimouni, Integral domains in which each ideal is a w-ideal, Comm. Algebra **33** (2005), 1345–1355.

[39] J. Mott and M. Zafrullah, On Prüfer v-multiplication domains, Manuscripta Math. **35** (1981), 1–26.

[40] M. Nagata, *Local rings,* Interscience, New York, 1962

[41] J. Ohm, Some counterexamples related to integral closure in $D[\![x]\!]$, Trans. Amer. Math. Soc. **122** (1966), 321–333.

[42] G. Picozza and F. Tartarone, When the semistar operation $\widetilde{\star}$ is the identity, Comm. Algebra. **36** (2008), 1954–1975.

[43] D. Shannon, Monoidal transforms of regular local rings, Amer. J. Math. **95** (1973), 294–320.

[44] Philip B. Sheldon, How changing $D[\![x]\!]$ changes its quotient field, Trans. Amer. Math. Soc. **159** (1971), 223–244.

[45] Fanggui Wang, w-modules over a PVMD, *in:* Proceedings of the International Symposium on Teaching and Applications of Engineering Mathematics, Hong Kong, 2001, pp. 11–120.

[46] Fanggui Wang, w-dimension of domains, II, Comm. Algebra **29** (2001), 224–28.

[47] Fanggui Wang and R.L. McCasland, On w-modules over strong Mori domains, Comm. Algebra **25** (1997), 1285–1306.

[48] M. Zafrullah, On finite conductor domains, Manuscripta Math. **24** (1978), 191–203.

[49] M. Zafrullah, The v-operation and intersections of quotient rings of integral domains, Comm. Algebra **13** (1985), 1699–1712.

[50] M. Zafrullah, The $D + XD_S[X]$ construction from GCD domains, J. Pure Appl. Algebra **50** (1988), 93–107.

[51] M. Zafrullah, On a property of pre-Schreier domains, Comm. Algebra **15** (1987), 1895–1920.

[52] M. Zafrullah, Well behaved prime t-ideals, J. Pure Appl. Algebra **65** (1990), 199–207.

[53] M. Zafrullah, Putting t-Invertibility to use, *in "Non-Noetherian Commutative Ring Theory"*, S.T. Chapman and S. Glaz (Editors), 429–457, Kluwer, Dordrecht, 2000.

[54] http://www.lohar.com/mithelpdesk/hd0504.pdf

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