WELL-BEHAVED PRIME t-IDEALS AND ALMOST KRULL DOMAINS

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Abstract. Call a t-ideal I of an integral domain R *well-behaved* if IR_S is a t-ideal in R_S for every multiplicatively closed subset S of R that is disjoint from I. We show that the set of well-behaved t-ideals has maximal elements and use the induced star operation to study the almost Krull domains (domains whose localizations are Krull) introduced by E. Pirtle.

To the memory of Robert Gilmer

1. Introduction.

Let R be a domain. Recall that an ideal maximal in the set of t-ideals of R is called a *maximal t-ideal.* (Background on the t- and other star operations is reviewed below.) In [20], the second author called a prime t-ideal of a domain R *well-behaved* if PR_P is a t-ideal in R_P. It is easy to see that if P is well behaved in R, then PR_S is a t-ideal in R_S for each multiplicatively closed subset of R disjoint from P (Lemma 2.1). Hence we shall call a t-ideal I of R *well-behaved* if IR_S is a t-ideal for every multiplicatively closed subset of R that is disjoint from I. Also following [20], we say that R itself is well-behaved if each t-prime of R is well-behaved. We call a t-ideal P of R a *maximal well-behaved t-ideal* if P is well behaved and is not properly contained in a larger well-behaved t-ideal. Examples in [20] show that maximal well-behaved t-ideals need not be maximal t-ideals.

In Section 2, we show that maximal well-behaved t-ideals are prime (Lemma 2.2), that each well-behaved t-ideal of R is contained in a maximal one, and that $R = \bigcap_{P\in\mathcal{P}} R_P$, where $\mathcal{P}$ is the set of maximal well-behaved t-ideals (Theorem 2.3). By [1, Theorem 1] this induces a star operation $\blacklozenge$ on R, given by $I^{\blacklozenge} = \bigcap_{P\in\mathcal{P}} IR_P$ for each nonzero

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fractional ideal I of R. In the remainder of the paper we use this star operation to characterize certain types of behavior. For example, in Theorem 2.9 we prove that every maximal t-ideal of R is well-behaved if and only if $\blacklozenge \leq t$, that is, $I^{\blacklozenge} \subseteq I^t$ for each nonzero fractional ideal I of R.

In Section 3 we provide a condition which is necessary for well-behavedness of R to pass to the polynomial ring $R[x]$, and we give an example showing that the condition need not hold.

In Section 4 we study the almost Krull (meaning locally Krull) domains introduced by E. Pirtle [19]. Pirtle conjectured that an almost Krull domain whose height-one prime ideals are all divisorial is a Krull domain. Using examples of the type studied in [7], J. Arnold and R. Matsuda [5] showed that the conjecture is false. More precisely, they showed that, in order to force an almost Krull domain R to be Krull it is necessary not only that the height-one primes be divisorial but also that R be a Prüfer v-multiplication domain (PvMD) and that, moreover, this latter condition did not follow from the former. Thus our purpose in Section 4 is to study these conditions. For the PvMD-condition, we are able to carry out our study in the weaker class of domains whose localizations are PvMDs; we show that such a domain R is a PvMD if and only if the star operations $\blacklozenge, w, t$ coincide on R (Theorem 4.7). For divisoriality we show in Theorem 4.9 that, for a height-one prime Q in an almost Krull domain, Q is divisorial if and only if Q contains a divisorial ideal contained in no other height-one prime if and only if Q is $\blacklozenge$-invertible.

We close this introduction by recalling the basic facts about star operations that we shall need. Let R be a domain with quotient field K. Denote by $\mathbf{F}(R)$ the set of nonzero fractional ideals of R. A *star operation* on R is then a mapping $I \mapsto I^\star$ of $\mathbf{F}(R)$ into $\mathbf{F}(R)$ such that for all nonzero $a \in K$ and $I, J \in \mathbf{F}(R)$,

(1) $(aR)^\star = aR$ and $aI^\star = (aI)^\star$;
(2) $I \subseteq I^\star$, and $I \subseteq J$ implies $I^\star \subseteq J^\star$; and
(3) $(I^\star)^\star = I^\star$.

It is well known that if $\star$ is a star operation, then one may associate to $\star$ a star operation $\star_f$ by setting, for $I \in \mathbf{F}(R)$, $I^{\star_f} = \bigcup J^\star$, where the union is taken over the nonzero finitely generated subideals J of I. A star operation $\star$ has *finite type* if $\star = \star_f$. If $\star$ does have finite type, then (a) each nonzero $\star$ ideal is contained in a maximal one, (b) maximal $\star$-ideals are prime, and (c) a prime ideal minimal over a $\star$-ideal is also a $\star$-ideal (called a $\star$-prime). The most important non-trivial star operations, and, with the exception of the star operation

♦ defined above, the only ones we shall use here are the v, t, and w operations: For $I \in \mathbf{F}(R)$, put $I^{-1} = (R : I)$ $(= \{u \in K \mid uI \subseteq R\})$ and $I^v = (I^{-1})^{-1}$; $t = v_f$; and $I^w = \bigcap IR_P$, where the intersection is taken over the maximal t-ideals P of R.

2. Well-behaved prime t-ideals and their induced star operation.

Lemma 2.1. *Let P be a prime ideal of a domain R such that PR_P is a t-ideal. Then PR_S is a t-ideal for each multiplicatively closed subset S of R that is disjoint from P.*

Proof. We have $PR_S = PR_P \cap R_S$. It is well-known that this implies that PR_S is a t-prime. □

It is possible to have t-primes $P \subset M$ with PR_M a t-prime of R_M but P not well-behaved–see Example 2.5.

Lemma 2.2. *Let R be a domain.*

1. *Let I be a well-behaved t-ideal in R, and suppose that P is a prime minimal over I. Then P is also a well-behaved t-ideal.*
2. *A maximal well-behaved t-ideal is prime.*
3. *Let $\{J_\alpha\}$ be a chain of well-behaved t-ideals. Then $\bigcup J_\alpha$ is also a well-behaved t-ideal.*

Proof. (1) PR_P is minimal over the t-ideal IR_P and hence is a t-ideal in R_P.

(2) This follows from (1).

(3) Let $J = \bigcup J_\alpha$, and let S be a multiplicatively closed set disjoint from J. Then $JR_S = \bigcup (J_\alpha R_S)$ is the union of a chain of t-ideals in R_S and is therefore a t-ideal. Hence J is well-behaved. □

Theorem 2.3. *Let R be a domain. Then*

1. *every well-behaved t-ideal of R is contained in a maximal well-behaved t-ideal, and*
2. *$R = \bigcap \{R_P \mid P$ is a maximal well-behaved t-ideal$\}$.*

Proof. For (1), apply Lemma 2.2 and Zorn's lemma. For (2), we proceed contrapositively. Thus let $x \in \mathrm{qf(R)} \setminus \mathrm{R}$ so that $(R :_R x)$ is a proper t-ideal of R. Let Q be a prime minimal over $(R :_R x)$. Then QR_Q is minimal over the t-ideal $(R :_R x)R_Q = (R_Q :_{R_Q} x)$, and hence QR_Q is a t-prime in R_Q. Thus Q is a well-behaved t-ideal, and, by Lemma 2.2, Q is contained in some maximal well-behaved t-ideal P. Since $(R :_R x) \subseteq P$, we then have $x \notin R_P$, as desired. □

Let I be a well-behaved t-ideal in a domain R, and let S be a multiplicatively closed subset of R disjoint from I. Then IR_S is a well-behaved t-ideal of R_S, and, applying Theorem 2.3 we have IR_S contained in a maximal well-behaved t-prime of R_S. Of course, this t-prime is of the form PR_S for some well-behaved t-prime P of R. However, although P is maximal among well-behaved t-primes disjoint from S, P need not be a maximal well-behaved t-ideal:

Example 2.4. Let x, y, z be indeterminates over a field k, and set $D = k[x, y, z]_{(y,z)}$, $V = k[x, y, z]_{(y)} = D_{yD}$, $W = k[x, z]_{(x)} + yV$, and $R = D \cap W$. Then W is a 2-dimensional valuation domain with maximal ideal xW and height-one prime yV. Let $M = (y, z)D \cap R$, $N = xW \cap R$, and $P = yV \cap R = yV \cap D = yD$. It is easy to see that $P \subseteq M \cap N$. Since $k[x, y, z] \subseteq R \subseteq D$, it is clear that $R_M = D$. We claim that $R_N = W$. Since $R_N = D_{R\setminus N} \cap W$, it suffices to show that $D_{R\setminus N} = V$. If $r \in R \setminus N$, then $r \notin P$ and hence $r \notin yV$. Thus $r^{-1} \in V$, and we have $D_{R\setminus N} \subseteq V$. For the reverse inclusion, it suffices to show that if $g \in k[x, y, z] \setminus yk[x, y, z]$, then $g^{-1} \in D_{R\setminus N}$. Since $g \notin yV = \bigcap_{n=1}^{\infty} x^n W$, we may write $g = x^k w$ for some $w \in W \setminus xW$ and $k \geq 0$. Since $w = gx^{-k} \in D$, we then have $w \in R \setminus N$ and then $g^{-1} = x^{-k} w^{-1} \in D_{R\setminus N}$, as desired.

Thus, taking $S = R \setminus M$ above, we have that PR_S is a maximal well behaved t-prime in the Krull domain $R_S = D$. (It is well-known that the only t-primes in a Krull domain are the height-one primes.) However, $P \subset N$, and N is a well-behaved t-prime in R since NR_N is automatically a t-ideal in the valuation domain $R_N = W$. □

As already mentioned, examples of non-well-behaved domains may be found in [20]. Here we give a particularly simple example and then tweak it to illustrate the comment following Lemma 2.1.

Example 2.5. Let x, y be indeterminates over $\mathbb{Q}$, $T = \mathbb{Q}[x, y]$, $P = (x, y)T$, and $R = \mathbb{Z} + P$. By standard properties of pullbacks, P is divisorial and therefore a t-prime of R. However, $PR_P = PT_P$ is not a t-ideal in the Krull domain $R_P = T_P$.

To illustrate the statement following Lemma 2.1, we localize. Thus let p be prime in $\mathbb{Z}$ and $M = (p) + P$. Then PR_M is divisorial in $R_M = \mathbb{Z}_{p\mathbb{Z}} + MT_P$, and hence $PR_M = PT_P$ is a t-ideal of R_M, but $P(R_M)_{PR_P} = PR_P = PT_P$ is not a t-ideal of $R_P = T_P$. □

In (both parts of) Example 2.5, the maximal t-ideals are (principal and hence) well-behaved. It is more difficult to give examples of maximal t-ideals that are not well-behaved. Again, these are discussed in [20]. For our purposes we cite the Heinzer-Ohm example of an essential domain that is not a PvMD [13]. It is not difficult to show that, in fact, this example is an almost Krull domain R with a height-two maximal t-ideal M, and hence MR_M cannot be a t-ideal in R_M. Thus M is not well-behaved.

For convenience, we recall the star operation $\blacklozenge$ from the introduction.

Notation 2.6. Let R be a domain, and denote by $\mathcal{P}$ the set of maximal well-behaved t-ideals of R. Then, for a nonzero fractional ideal I of R, set $I^{\blacklozenge} = \bigcap_{P\in\mathcal{P}} IR_P$.

The next three results give some simple facts about $\blacklozenge$.

Lemma 2.7. *Let R be a domain. Then:*

(1) *$P^{\blacklozenge} = P$ for each $P \in \mathcal{P}$.*
(2) *If Q is a is a nonzero prime of R, then $Q \subseteq P$ for some $P \in \mathcal{P}$ if and only if $Q^{\blacklozenge} = Q$.*
(3) *A maximal t-ideal M of R is well-behaved if and only if $M^{\blacklozenge} = M$.*
(4) *$I^{\blacklozenge}R_P = IR_P$ for each nonzero ideal I of R and $P \in \mathcal{P}$.*

Proof. (1) This follows from [1, Theorem 1].

(2) If $Q \subseteq P \in \mathcal{P}$, then $Q^{\blacklozenge} \subseteq QR_P \cap R = Q$. (This also follows from the proof of [3, Theorem 2.15].) The converse is covered by Theorem 2.3.

(3) Let M be a maximal t-ideal. If M is also well-behaved, then $M \in \mathcal{P}$ and $M^{\blacklozenge} = M$ by (1). Conversely, if $M^{\blacklozenge} = M$, then $M \subseteq P$ for some $P \in \mathcal{P}$ by (2), and we must have $M = P$, that is, M is well-behaved.

(4) This follows from [1, Theorem 1(1)]. □

Proposition 2.8. *The set $\mathcal{P}$ of maximal well-behaved t-ideals coincides with the set of maximal $\blacklozenge$-ideals.*

Proof. Let I be a nonzero ideal of R, and consider the representation $I^{\blacklozenge} = \bigcap_{P\in\mathcal{P}} IR_P$. If $I^{\blacklozenge} \neq R$, then we must have $I \subseteq P$ for some $P \in \mathcal{P}$. In particular, if Q is a maximal $\blacklozenge$-ideal, then $Q \subseteq P$ for some $P \in \mathcal{P}$. However, $P^{\blacklozenge} = P$ (Lemma 2.7), whence $Q = P$, that is, $Q \in \mathcal{P}$. Conversely, suppose that $L \in \mathcal{P}$. Then $L = L^{\blacklozenge}$. If $L \subseteq L_1$ with $L_1 = (L_1)^{\blacklozenge} \neq R$, then, by the argument above, L_1 is contained in a maximal well-behaved t-ideal L_2, whence $L = L_1 = L_2$. Hence L is a maximal $\blacklozenge$-ideal. □

Theorem 2.9. *Let R be a domain. The following statements are equavialent.*

(1) *Every maximal t-ideal of R is well-behaved.*

(2) *$\blacklozenge = w$.*
(3) *$\blacklozenge$ has finite type.*
(4) *$\blacklozenge \leq t$.*

Proof. (1) $\Rightarrow$ (2). (1) yields $\mathcal{P} = t\text{-Max}(R)$. Hence for a nonzero ideal I, we have $I^{\blacklozenge} = \bigcap_{M \in t\text{-Max}(R)} IR_M = I^w$.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (4). This follows from the fact that t is the largest finite-type star operation.

(4) $\Rightarrow$ (1). Let M be a maximal t-ideal of R . Assuming (4), we have $M^{\blacklozenge} \subseteq M^t = M$, so that $M^{\blacklozenge} = M$. Hence M is well-behaved by Lemma 2.7(3). □

Following convention, we say that a domain R has *finite $\blacklozenge$-character* if each nonzero element of R lies in only finitely many maximal $\blacklozenge$-ideals (that is, in only finitely many maximal well-behaved ideals).

Theorem 2.10. *A domain R has finite $\blacklozenge$-character if and only if it has finite t-character. If R does have finite $\blacklozenge$-character, then the equivalent conditions in Theorem 2.9 hold.*

Proof. Assume that the domain R has finite $\blacklozenge$-character. Then $\blacklozenge$ must have finite type [1, Theorem 1(6)]. Hence the conditions of Theorem 2.9 hold. In particular, $\blacklozenge = w$, and since maximal w-ideals and maximal t-ideals coincide, this yields that R has finite t-character. Conversely, if R has finite t-character, then every maximal t-ideal is well-behaved by [2, Theorem 1.1(2)]. It follows that maximal $\blacklozenge$-ideals and maximal t-ideals coincide, and hence R has finite $\blacklozenge$-character. □

3. Polynomial rings

In Theorem 3.3 below, we give a condition that characterizes when well-behavedness extends from a domain R to its polynomial ring $R[x]$, and we give an example showing that this condition may not hold.

Proposition 3.1. *Let R be a domain with quotient field K. A nonzero prime ideal P of R is a well-behaved t-prime if and only if $P[x]$ is a well-behaved t-prime of $R[x]$.*

Proof. Let P be a nonzero prime ideal of the domain R, and let I be a finitely generated ideal of R. It is well-known, and easy to verify, that P is a t-prime in R if and only if $P[x]$ is a t-prime in $R[x]$ and that $I^{-1}R[x]_{P[x]} = (IR[x]_P[x])^{-1}$, where the inverse on the right is taken with respect to $R[x]_{P[x]}$. It is also easy to verify that $I^{-1}R[x]_{P[x]} \cap K = I^{-1}R_P$. Using these facts, it is not difficult to prove the proposition. Suppose that P is a well-behaved t-prime of

R, and let J be a finitely generated subideal of $PR[x]_{P[x]}$. Then $J = AR[x]_{P[x]}$ for a finitely generated ideal A of $R[x]$ with $A \subseteq P[x]$. Let $I = c(A)$ (the ideal generated by the coefficients of the polynomials in A). Then $IR[x] \subseteq P[x]$. Since P is well-behaved, $(IR_P)^t \subseteq PR_P$, whence $I^{-1}R_P = (IR_P)^{-1} \neq R_P$. Thus $I^{-1} \nsubseteq R[x]_{P[x]}$ (since, from above, $I^{-1}R[x]_{P[x]} \cap K = I^{-1}R_P$). If follows easily that for $u \in I^{-1} \setminus R_P$, we also have $u \in (IR[x]_{P[x]})^{-1} \setminus R[x]_{P[x]}$. In turn, this yields $J^t \subseteq (IR[x]_{P[x]})^t \subseteq PR[x]_{P[x]}$. Hence $P[x]$ is well-behaved in $R[x]$. For the converse, assume that $P[x]$ is well-behaved, and let I be a finitely generated subideal of P. Then $(IR[x]_{P[x]})^t \subseteq PR[x]_{P[x]}$, whence $I^{-1}R[x]_{P[x]} \neq R[x]_{P[x]}$. It follows easily that $I^{-1} \nsubseteq R_P$ and hence that $I^t \subseteq P$. Thus P is well-behaved. □

Recall that a nonzero prime ideal Q of $R[x]$ is an *upper to zero* if $Q \cap R = (0)$, equivalently, if Q is contracted from a nonzero prime of $K[x]$. If Q is an upper to zero, then $R[x]_Q$ is a valuation domain, and hence Q is automatically a well-behaved t-prime.

Corollary 3.2. *Let R be a domain. Then each maximal t-ideal of $R[x]$ is well-behaved if and only if each maximal t-ideal of R is well-behaved.*

Proof. By [15, Proposition 1.1] maximal t-ideals of $R[x]$ are either extended from maximal t-ideals of R or are uppers to zero. The conclusion now follows easily from Proposition 3.1. □

Theorem 3.3. *Let R be a domain. Then $R[x]$ is well-behaved if and only if R is well-behaved and each t-prime of $R[x]$ is either extended from a prime of R or is an upper to zero.*

Proof. If R is well-behaved and the t-primes of $R[x]$ are as described in the statement of the theorem, then $R[x]$ is well-behaved by Proposition 3.1. Now, suppose that $R[x]$ is well-behaved. It is clear from Proposition 3.1 that R must also be well-behaved. Let Q be a t-prime of $R[x]$ such that, for $P = Q \cap R$, we have $P \neq (0)$ and $Q \neq P[x]$. Since $QR_P[x]$ is maximal ideal of $R_P[x]$ and is neither an upper to zero nor extended from a prime of R_P, it cannot be a t-ideal by [15, Proposition 1.1]. Therefore, since $R_P[x]$ is a quotient ring of $R[x]$, Q is not well-behaved. □

The next result gives several large classes of domains in which well-behavedness extends from R to $R[x]$. Recall that R is a UMt-domain if uppers to zero in $R[x]$ are maximal t-ideals. For $f \in R[x]$ we denote by $c(f)$ the ideal generated by the coefficients of f.

Corollary 3.4. *Let R be a well-behaved domain. Then the polynomial ring $R[x]$ is well-behaved in each of the following cases:*

(1) *R is integrally closed.*
(2) *R is Noetherian.*
(3) *R is a UMt-domain.*

Proof. Let Q be a prime ideal of $R[x]$ with $P = Q \cap R \neq (0)$ and $Q \neq P[x]$. By Theorem 3.3 it suffices to prove that Q is not a t-ideal. To this end, choose $a \in P$, $a \neq 0$, and $f \in Q \setminus P[x]$. If $k \in (a, f)^{-1}$, then, since $ak \in R[x]$, we must have $k \in K[x]$ (where, as usual, K is the quotient field of R).

In the integrally closed case, since $fk \in R[x]$, we have $c(f)c(k) \subseteq (c(f)c(k))^v = c(fk)^v \subseteq R$ [12, Proposition 34.8]. Thus $c(f)k \subseteq R[x]$, and we have $c(f)(a, f)^{-1} \subseteq R[x]$, that is, $c(f) \subseteq (a, f)^v$. Since $f \notin P[x]$, $c(f) \nsubseteq Q$, and hence Q is not a t-ideal.

In the Noetherian case replace the use of [12, Proposition 34.8] by the content formula [12, Theorem 28.1] to get $c(f)^{m+1}c(k) = c(f)^m c(fk) \subseteq R$ for some positive integer m. Although the exponent m depends on k, the fact that $(a, f)^{-1}$ is finitely generated yields a positive integer r with $c(f)^r(a, f)^{-1} \subseteq R[x]$. Thus $c(f)^r \subseteq (a, f)^v$ but $c(f)^r \nsubseteq Q$.

In the UMT case, ignore the element a; instead apply [9, Theorem A] to produce an upper to zero U with $f \in U \subseteq Q$. Then, since U is a maximal t-ideal, Q cannot be a t-ideal. □

We end this section with the promised example of a well-behaved domain R for which the polynomial ring $R[x]$ is not well-behaved.

Example 3.5. Let k be a field, and set $D = k[y, \{yz^{2^n}\}_{n=0}^{\infty}]$ and $S = k[y, z]$, where y, z are indeterminates. Then $M = (y, \{yz^{2^n}\})$ is a maximal ideal of D. Finally, put $T = k(y, z)[[w]]$ (w an indeterminate), $N = wT$, and $R = D + N$. We shall show that R is well-behaved, but the polynomial ring $R[x]$ is not. We begin by discussing some facts about D.

Claim 1. The ideal M is the only prime ideal of D containing y.

Proof. Let P be a prime of D, and suppose that $y \in P$. Then for each n, we have $(yz^{2^n})^2 = y \cdot yz^{2^{n+1}} \in P$. Thus each generator of M lies in P; therefore, since M is maximal, we have $P = M$.

Claim 2. For each prime P of D with $P \neq M$, there is a unique prime Q of S with $Q \cap D = P$, and for this Q we have $D_P = S_Q$.

Proof. Let $s \in S$. For sufficiently large i, $y^i s \in D$. Then $s = y^i s/y^i \in D_P$ since $y^i \notin P$ (Claim 1). Hence $S \subseteq D_P$. Then for $Q = PD_P \cap S$, we have $Q \cap D = P$. It is clear that for any prime L of S with $L \cap D = P$, we must have $S_L = D_P$, and the claim follows.

Claim 3. Each maximal ideal of D has height 2.

Proof. Note that $M = yS \cap D = (y,z)S \cap D \supsetneq zS \cap D$. The conclusion (for M as well as the other maximal ideals of D) now follows from Claim 2 (and well-known facts about $\operatorname{Spec} k[y,z]$).

Claim 4. The only maximal ideal of D that is also a t-ideal is M. Moreover, M is well-behaved.

Proof. Fix a maximal ideal $P \neq M$, and let Q denote the maximal ideal of S that contracts to P. Then Q is not a t-ideal of S, whence $Q^{v_S} = S$ (where v_S is the v-operation on S). Choose i sufficiently large that $y^i A \subseteq P$, where A is a finite generating set for Q, and choose $b \in P \setminus M$. Then $(y^i A, b)D_M = D_M$, whence, using the fact that $(y^i A, b)$ is a finitely generated ideal of D, we have $(y^i A, b)^{-1}D_M = (D_M : (y^i A, b)D_M) = D_M$. For a maximal ideal $L \neq M, P$, denote the unique prime of S contracting to L by L'. Then, since $y \notin L$, $(y^i A, b)D_L = (y^i A, b)S_{L'} = QS_{L'} = S_{L'} = D_L$, from which it follows that $(y^i A, b)^{-1}D_L = D_L$. As for the maximal ideal P itself, using the fact that $y \notin P$, we have

$$(y^i A, b)^{-1}D_P \subseteq (D_P : AD_P) = (S_Q : QS_Q) = (S : Q)S_Q = S_Q = D_P,$$

whence $(y^i A, b)^{-1}D_P = D_P$. It follows that $(y^i A, b)^{-1} = D$, so that P is not a t-ideal. Finally, since $(yz^{2^n})^2 z = yz^{2^{n+1}} \cdot yz \in D$, we have $M = \operatorname{rad}(D :_D z)$, whence M is a well-behaved t-ideal (Lemma 2.2).

Claim 5. D is well-behaved.

Proof. The maximal ideal M is well-behaved by Claim 4. The other height-two primes are not t-ideals, and the height-one primes are automatically well-behaved t-primes.

It is now relatively easy to see that R is well-behaved. According to [10, Proposition 1.8], $M + N$ and $Q + N$, where Q is a height-one prime of D, are t-primes of R, while $P + N$, where P is a height-two non-t-prime of D, is not a t-prime of R. Since $R_{M+N} = D_M + N$ and $R_{Q+N} = R_Q + N$ for each Q, we may again invoke [10, Proposition 1.8] to see that $M + N$ and each $Q + N$ localize to t-primes. The only other t-prime of R is N itself, and $NR_N = NT = N$ is a (principal prime and hence a) t-prime of T. Therefore, R is well-behaved. Next, if we denote the quotient field of D by K and put $U = (x - z)K[x] \cap D[x]$, then by [14, Example 2.5] $U + N[x]$ is a t-prime of $R[x]$ that contracts to N in R (but is not equal to $N[x]$). By Theorem 3.3, $R[x]$ is not well-behaved. □

4. Almost Krull domains

Our primary goal in this section is to study the PvMD condition and divisoriality in the class of almost Krull domains introduced by Pirtle [19]. It is convenient to begin with a characterization of Krull domains.

Theorem 4.1. *The following statements are equivalent for a domain R.*

1. *Each nonzero ideal of R is t-invertible.*
2. *Each associated prime of a principal ideal of R is t-invertible.*
3. *Each minimal prime of a principal ideal of R is t-invertible.*
4. *R is a Krull domain.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear. (3) $\Rightarrow$ (4) is part of [17, Theorem 3.6], and (4) $\Rightarrow$ (1) is due to Jaffard [16]. (The equivalence of (2) and (4) first appeared in [18]). □

In Theorem 4.1 one cannot weaken (3) to "each height-one prime of R is t-invertible." (For example, if V is a valuation domain with no height-one prime, then V satisfies this condition vacuously.) However, if R has t-dimension one (all t-primes have height one), or R is an almost Krull domain, then this weaker condition suffices, as we now show.

Theorem 4.2. *The following statements are equivalent for a domain R.*

1. *R has t-dimension one, and each height-one prime of R is t-invertible.*
2. *R is an almost Krull domain, and each height-one prime of R is t-invertible.*
3. *R is a Krull domain.*

Proof. It is clear that condition (3) implies (1) and (2). Let P be a prime of R that is minimal over a principal ideal. In particular, P is a t-ideal. If R has t-dimension one, then P must have height one. On the other hand, if R is almost Krull, then PR_P is minimal over a principal ideal in the Krull domain R_P, and, again, P must have height one. Thus, assuming either (1) or (2), we have that minimal primes of principal ideals are t-invertible, whence R is a Krull domain by Theorem 4.1 □

Our next result is a characterization of almost Krull domains corresponding to Theorem 4.1.

Theorem 4.3. *The following statements are equivalent for a domain R.*

(1) *Each nonzero ideal of R is locally t-invertible.*
(2) *Each associated prime of a principal ideal of R is locally t-invertible.*
(3) *Each minimal prime of a principal ideal of R is locally t-invertible.*
(4) *R is an almost Krull domain.*

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial. Now let M be a maximal ideal of R, and let Q be a minimal prime of a principal ideal of R_M. Then $Q = PR_M$, where P is a minimal prime of a principal ideal in R. Assuming statement (3) then forces Q to be t-invertible. By Theorem 4.1, this yields that R_M is Krull, and the implication (3) $\Rightarrow$ (4) follows. Finally, the implication (4) $\Rightarrow$ (1) follows from an even easier localization argument. □

We now turn to the PvMD condition. It turns out that, for most of this study, we can relax "almost Krull" to "locally PvMD." It is not difficult to give examples of almost PvMDs that are neither almost Krull nor PvMDs. For example, let D be an almost Krull domain with quotient field K such that D is not a PvMD, e.g., [13], let x be an indeterminate, and set $R = D + xK[x]$. Since D is not a PvMD, neither is R [6, Theorem 4.43]. Moreover, since R shares an ideal with $K[x]$, it is not completely integrally closed and is therefore not almost Krull. To see that R is locally a PvMD, let N be a maximal ideal of R; then, by [6, Theorem 4.21], $N = fR$ for some irreducible $f \in K[x]$ with $f(0) = 1$ or $N = M + xK[x]$ for some maximal ideal M of R. In the first case, R_N is a quotient ring of $K[x]$ and is therefore a PvMD (in fact, $R[x]_N$ is the valuation domain $K[x]_{fK[x]}$). In the second case, we have $R_N \supseteq D_M$, whence R_N is a quotient ring of $D_M + xK[x]$. Since D_M is a Krull domain and hence a PvMD, $D_M + xK[x]$ is also a PvMD, again by [6, Theorem 4.21]. Therefore, R_N is a PvMD.

Recall the following well-known characterization of PvMDs: a domain R is a PvMD if and only if R_P is a valuation domain for each t-prime P of R if and only if each nonzero finitely generated ideal of R is t-invertible. The next result is a first step towards a similar characterization for domains whose localizations are PvMDs.

Proposition 4.4. *The following conditions are equivalent for a domain R.*

(1) *R_P is a valuation domain for each well-behaved t-ideal P of R.*
(2) *R_P is a valuation domain for each $P \in \mathcal{P}$.*
(3) *Each nonzero finitely generated ideal I of R is $\blacklozenge$-invertible.*

Moreover, if these conditions hold, then $\blacklozenge \geq t$.

Proof. The implication (1) $\Rightarrow$ (2) is trivial. Let P be a well-behaved t-prime and I a nonzero finitely generated ideal. We have $II^{-1}R_P = (IR_P)(R_P : IR_P)$. We use this equality to prove that (2) $\Rightarrow$ (3) $\Rightarrow$ (1). Assuming (2), the equality yields $II^{-1} \nsubseteq P$. Hence $(II^{-1})^{\blacklozenge} = \bigcap_{Q \in \mathcal{P}} II^{-1}R_Q = R$, that is, (3) holds. Assuming (3), we have $R = (II^{-1})^{\blacklozenge} \nsubseteq P$. Hence $II^{-1} \nsubseteq P$, and the same equality ensures that IR_P is (invertible and hence) principal in R_P. It follows that R_P is a valuation domain, and we have (1).

Finally, suppose that (2) holds, and let I be a nonzero ideal of R and P a well-behaved t-prime. Then R_P is a valuation domain, whence $I^tR_P \subseteq (IR_P)^{t_{R_P}} = IR_P$. We then have $I^t \subseteq \bigcap_{Q \in \mathcal{P}} IR_Q = I^{\blacklozenge}$, as desired. □

We now add a condition needed to characterize domains that are locally PvMDs. The condition loosely states that if P is "partially" well-behaved, then it is well-behaved.

Lemma 4.5. *Let R be locally a PvMD. Then R satisfies the conditions of Proposition 4.4 as well as the following condition: if $P \subseteq M$ are primes in R with M maximal and PR_M a t-prime in R_M, then P is a well-behaved t-prime of R.*

Proof. Suppose that R is locally a PvMD. Then for $P \in \mathcal{P}$, R_P is a PvMD whose maximal ideal PR_P is a t-ideal, whence, in fact, R_P is a valuation domain.

With $P \subseteq M$ as stated, we have $R_P = (R_M)_{PR_M}$. Since R is locally a PvMD, this yields that R_P is a valuation domain, whence PR_P is automatically a t-ideal in R_P. Therefore, P is a well-behaved t-prime of R. □

Theorem 4.6. *The following statements are equivalent for a domain R.*

(1) *R is locally a PvMD.*
(2) *Each nonzero finitely generated ideal of R is locally t-invertible.*
(3) *Each nonzero two-generated ideal of R is locally t-invertible.*
(4) *$(a) \cap (b)$ is locally t-invertible for all nonzero $a, b \in R$.*
(5) *If $P \subseteq M$ are primes in R with M maximal and PR_M a t-ideal in R_M, then R_P is a valuation domain.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from the well-known characterization of PvMDs as domains each of whose nonzero finitely generated ideals is t-invertible, and the equivalence of (1), (3), and (4) follows from [18, Lemma 1.7 and Corollary 1.8]. The implication (1)

$\Rightarrow$ (5) is taken care of by Lemma 4.5. For the converse, let M be a maximal ideal of R and P a prime ideal contained in M such that PR_M is a t-prime in R_M. Then (5) implies that $(R_M)_{PR_M} = R_P$ is a valuation domain. Therefore, R_M is a PvMD. □

We now determine when a domain that is locally a PvMD is in fact a PvMD.

Theorem 4.7. *Let R be locally a PvMD. Then:*
The following statements are equivalent.

(1) *R is a PvMD.*
(2) *R is well-behaved.*
(3) *Each maximal t-ideal of R is well-behaved.*
(4) $\blacklozenge = w$.
(5) $\blacklozenge = t$.

Proof. (1) $\Rightarrow$ (2). If R is a PvMD and P is a t-prime of R, then PR_P is automatically a t-prime in the valuation domain R_P.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (4). Apply Theorem 2.9.

(4) $\Rightarrow$ (5). This follows from Proposition 4.4 and the fact that we always have $w \leq t$.

(5) $\Rightarrow$ (1) Assume that $\blacklozenge = t$, and let M be a maximal t-ideal of R. Then $M = M^t = M^{\blacklozenge}$, whence M is well-behaved by Lemma 2.7(3). Thus MR_M is a t-prime in the PvMD R_M, whence R_M is a valuation domain. Therefore, R is a PvMD. □

In [8] S. El Baghdadi, L. Izelgue, and A. Tamoussitt define a star operation $*$ on a domain R by $I^* = \bigcap_{M \in \mathrm{Max}(R)} (IR_M)^{t_{R_M}}$ for $I \in \mathbf{F}(R)$. Although this star operation is quite different from $\blacklozenge$ (for example, $* \geq t$ in general but this is not true for $\blacklozenge$), they coincide for almost Krull domains. Thus our next result, which specializes Theorem 4.7 to almost Krull domains, is essentially a restatement of (part of) [8, Proposition 1.7].

Corollary 4.8. *Let R be an almost Krull domain. Then:*

(1) *A prime t-ideal P of R is well-behaved if and only if* $\mathrm{ht}(P) = 1$.
(2) $I^{\blacklozenge} = \bigcap_{\mathrm{ht}\, P=1} IR_P$ *for each nonzero ideal I of R.*
(3) $\blacklozenge \geq t$ *with equality holding if and only if R is a PvMD.*

Proof. (1) If P is a well-behaved t-prime in R, then PR_P is a t-prime in the Krull domain R_P, and hence $\mathrm{ht}\, P = 1$. The converse is trivial.

(2) This follows from (1).

(3) This follows from Proposition 4.4 and Theorem 4.7. □

Following Gilmer [11], it is natural to call a maximal $\blacklozenge$-ideal Q $\blacklozenge$-*sharp* if R_Q does not contain $\bigcap R_P$, where the intersection is taken over all maximal $\blacklozenge$-ideals distinct from Q. We then say that R itself is $\blacklozenge$-sharp if each maximal $\blacklozenge$-ideal of R is $\blacklozenge$-sharp. In an almost Krull domain, this restricts to the height-one primes. We have:

Theorem 4.9. *The following statements are equivalent for a height-one prime Q in an almost Krull domain R.*

1. *Q is divisorial.*
2. *$Q^{-1} \neq R$.*
3. *$Q = (R :_R u)$ for some $u \in Q^{-1} \setminus R$.*
4. *Each Q-primary ideal of R is divisorial.*
5. *Q is $\blacklozenge$-sharp.*
6. *Q contains a divisorial ideal contained in no other height-one prime.*
7. *$QQ^{-1} \nsubseteq Q$.*
8. *Q is $\blacklozenge$-invertible.*

Proof. (1) $\Rightarrow$ (2), (4) $\Rightarrow$ (1). Trivial.

(2) $\Rightarrow$ (3). Let $u \in Q^{-1} \setminus R$. Then $Q \subseteq (R :_R u)$. Let P be a prime minimal over $(R :_R u)$. Then P is well-behaved, whence $\operatorname{ht} P = 1$ by Corollary 4.8. This forces $P = Q$, and we have $Q = (R :_R u)$.

(3) $\Rightarrow$ (4). Write $Q = (R :_R u) = (1, u)^{-1}$. According to [4, Theorem 1.6], $(1, u^k)^{-1} = (((1, u)^{-1})^k)^{\blacklozenge}$ for each positive integer k. (It suffices, and is not difficult, to verify this locally at each height-one prime.) We then have

$$(1, u^k)^{-1} = (Q^k)^{\blacklozenge} = \bigcap_{\operatorname{ht} P=1} Q^k R_P = Q^k R_Q \cap R.$$

Since $(1, u^k)^{-1}$ is divisorial, we have that the symbolic powers of Q are divisorial. However, if I is Q-primary, then, since R_Q is a DVR, $IR_Q = Q^k R_Q$ for some k, whence $I = Q^k R_Q \cap R$, which is divisorial.

(3) $\Rightarrow$ (5). If $Q = (R :_R u)$, it is clear that $u \notin R_Q$. Let P be a height-one prime different from Q. Then, since $uQ \subseteq R$, we have $uR_P = uQR_P \subseteq R_P$, and hence $u \in R_P$. Therefore, Q is $\blacklozenge$-sharp.

(5) $\Rightarrow$ (6). Let $u \in \bigcap_{\operatorname{ht} P=1, P\neq Q} R_P \setminus R_Q$. Then the divisorial ideal $(R :_R u)$ works.

(6) $\Rightarrow$ (2). Let I be a divisorial ideal contained in Q but contained in no other height-one prime, and choose $w \in I^{-1} \setminus R$. We have $IR_Q = Q^k R_Q$ for some positive integer k. Then $wQ^k \subseteq wQ^k R_Q = wIR_Q \subseteq R_Q$. For P a height-one prime not equal to Q, $(R :_R w) \nsubseteq P$ (since $I \nsubseteq P$). Then $w \in R_P$, and hence $wQ^k \subseteq R_P$. It follows that $wQ^k \subseteq R$ (Theorem 2.3). We may as well assume that k is minimal with this

property, so that $wQ^{k-1} \nsubseteq R$. Pick $u \in wQ^{k-1} \setminus R$. Then $u \in Q^{-1} \setminus R$, as desired.

(2) $\Rightarrow$ (7). Proceeding contrapositively, suppose that $QQ^{-1} \subseteq Q$. Since R is completely integrally closed, this implies that $Q^{-1} = R$.

(7) $\Rightarrow$ (8). Suppose $QQ^{-1} \nsubseteq Q$. Since it is clear that $QQ^{-1} \nsubseteq P$ for P a height-one prime different from Q, we must have $(QQ^{-1})^{\blacklozenge} = R$.

(8) $\Rightarrow$ (2). This is clear. □

Globalizing the preceding theorem, we obtain the following characterization of divisoriality in almost Krull domains. However, we note that the equivalence of (1) and (4) follows directly from [8, Theorem 1.9].

Corollary 4.10. *The following statements are equivalent for an almost Krull domain R.*

(1) *The height-one primes of R are divisorial.*
(2) *R is $\blacklozenge$-sharp.*
(3) *Each nonzero ideal of R is $\blacklozenge$-invertible.*
(4) *$\blacklozenge = v$.*

Proof. The equivalence of statements (1) and (2) follows from Theorem 4.9. Assume (1). If I is a nonzero ideal, then $I^{\blacklozenge} = \bigcap_{\mathrm{ht}\, P=1} IR_P = \bigcap_{\mathrm{ht}\, P=1}(IR_P \cap R)$. For each P, $IR_P \cap R$ is either P-primary, in which case it is divisorial by Theorem 4.9, or is equal to R. Thus, as an intersection of divisorial ideals, $I^{\blacklozenge}$ is divisorial, whence $I^v \subseteq (I^{\blacklozenge})^v = I^{\blacklozenge}$. Hence (1) $\Rightarrow$ (4). Since R is completely integrally closed, each nonzero ideal is v-invertible, and the implication (4) $\Rightarrow$ (3) follows. Assume (3), and let Q be a height-one prime. By assumption Q is $\blacklozenge$-invertible, whence $Q^{-1} \neq R$ (lest $Q = Q^{\blacklozenge} = (QQ^{-1})^{\blacklozenge} = R$). By Theorem 4.9, Q is divisorial, and we have (3) $\Rightarrow$ (1). □

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