$\star$-SUPER POTENT DOMAINS

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Abstract. For a finite-type star operation $\star$ on a domain R, we say that R is $\star$-super potent if each maximal $\star$-ideal of R contains a finitely generated ideal I such that (1) I is contained in no other maximal $\star$-ideal of R and (2) J is $\star$-invertible for every finitely generated ideal $J \supseteq I$. Examples of t-super potent domains include domains each of whose maximal t-ideals is t-invertible (e.g., Krull domains). We show that if the domain R is $\star$-super potent for some finite-type star operation $\star$, then R is t-super potent, we study t-super potency in polynomial rings and pullbacks, and we prove that a domain R is a generalized Krull domain if and only if it is t-super potent and has t-dimension one.

Introduction

Dedekind domains are characterized as those domains having all nonzero ideals invertible. On the other hand, if D is a Dedekind domain with quotient field k and x is an indeterminate, then the domain $R := D + (x^2, x^3)k[[x^2, x^3]]$ has invertibility strictly above $M := (x^2, x^3)R$, but M itself is not invertible in R. Similarly, it is well known that Krull domains are characterized as those domains having all nonzero ideals t-invertible (definitions reviewed below), while in the example above, one has t-invertibility only above a certain level. The goal of this paper is to explore one form of this kind of (t)-invertibility.

Now the t-operation is a particular example of a star operation, and it is useful to generalize to arbitrary finite-type star operations. Let R be a domain with quotient field K. Denoting by $\mathcal{F}(R)$ the set of nonzero fractional ideals of R, a map $\star : \mathcal{F}(R) \to \mathcal{F}(R)$ is a *star operation on* R if the following conditions hold for all $A, B \in \mathcal{F}(R)$ and all $c \in K \setminus (0)$:

(1) $(cA)^\star = cA^\star$ and $R^\star = R$;
(2) $A \subseteq A^\star$, and, if $A \subseteq B$, then $A^\star \subseteq B^\star$; and
(3) $A^{\star\star} = A^\star$.

An ideal I satisfying $I^\star = I$ is called a $\star$-ideal. Other than the d-operation ($I^d = I$ for all nonzero fractional ideals I), the best known star operation is the *v-operation*: for $I \in \mathcal{F}(R)$, put $I^{-1} = \{x \in K \mid xI \subseteq R\}$ and $I^v = (I^{-1})^{-1}$. For any star operation $\star$, we may define an associated star operation $\star_f$ merely by setting, for $I \in \mathcal{F}(R)$, $I^{\star_f} = \bigcup J^\star$, where the union is taken over all finitely generated subideals J of I, and we say that $\star$ has *finite type* if $\star = \star_f$. The *t-operation* is then given by $t = v_f$. It is well known that for a finite-type star operation $\star$ on

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a domain R, each $\star$-ideal is contained in a *maximal* $\star$-ideal, that is, a star ideal maximal in the set of $\star$-ideals; maximal $\star$-ideals are prime; and $R = \bigcap R_P$, where the intersection is taken over the set of maximal $\star$-ideals P. A nonzero ideal I is $\star$-*invertible* if $(II^{-1})^\star = R$. For star operations $\star_1, \star_2$, we say that $\star_1 \le \star_2$ if $I^{\star_1} \subseteq I^{\star_2}$ for each $I \in \mathcal{F}(R)$.

Generalizing notions from [5, 4], we call a nonzero finitely generated ideal I $\star$-*rigid* if it is contained in a unique maximal $\star$-ideal and $\star$-*super rigid* if, in addition, J is $\star$-invertible for each finitely generated ideal $J \supseteq I$. We say that a maximal $\star$-ideal M is $\star$-*potent* ($\star$-*super potent*) if M contains a $\star$-rigid ($\star$-super rigid) ideal and that the domain R is $\star$-*potent* ($\star$-*super potent*) if each maximal $\star$-ideal of R is $\star$-potent ($\star$-super potent). It is clear that any domain each of whose maximal ideals is invertible is d-super potent, as is any valuation domain. On the other hand, a Krull domain may not (even) be d-potent (e.g., a polynomial ring in two indeterminates over a field) but is t-super potent.

For the remainder of the introduction, we assume that *all star operations mentioned have finite type*. In Section 1 we lay out many of the basic properties of $\star$-super potency. In Corollary 1.6 we show that if $\star_1 \le \star_2$ and R is $\star_1$-super potent, then it is also $\star_2$-super potent; in particular, since $d \le \star \le t$ for all (finite-type) $\star$, we have that t-super potency is the weakest type of super potency. In Theorem 1.10 we obtain a local characterization: R is $\star$-super potent if and only if R is $\star$-potent and R_M is d-super potent for each maximal $\star$ ideal M of R. In Theorem 1.11 we establish, among other things, that if I is a $\star$-super rigid ideal, then $\bigcap_{n=1}^{\infty}(I^n)^\star$ is prime. In Section 2 we study local super potency and show that a (non-field) local domain (R, M) is d-super potent if and only there is a prime ideal $P \subsetneq M$ for which $P = PR_P$ and R/P is a valuation domain. In a brief Section 3, we show that t-potency and t-super potency extend from R to the polynomial ring $R[X]$. Section 4 is devoted to determining how t-potency and t-super potency behave in a commonly studied type of pullback diagram, and these results are used to provide several examples. In Section 5 the main result is a characterization of Ribenboim's *generalized Krull domains* [25], those domains that may be expressed as a locally finite intersection of essential rank-one valuation domains: the domain R is a generalized Krull domain if and only if it is t-super potent and every maximal t-ideal of R has height one.

1. Basic results on $\star$-super potency

From now on, we use R to denote a domain and K to denote its quotient field, and we use the convention that an ideal of R must be contained in R. We begin by repeating the definition of $\star$-(super) potency.

Definition 1.1. Let $\star$ be a finite-type star operation on the domain R. Call a finitely generated ideal I of R $\star$-*rigid* if it is contained in exactly one maximal $\star$-ideal of R and $\star$-*super rigid* if, in addition, each finitely generated ideal $J \supseteq I$ is $\star$-invertible. We then say that a maximal $\star$-ideal of R is $\star$-*potent* ($\star$-*super potent*) if it contains a $\star$-rigid ($\star$-super rigid) ideal and that R itself is $\star$-*potent* ($\star$-*super potent*) if each maximal $\star$-ideal of R is $\star$-potent ($\star$-super potent).

Remark 1.2. Recall that for a star operation $\star$ on R, a $\star$-ideal A is said to have *finite type* if $A = B^\star$ for some finitely generated ideal B of R. In [5] a finite type t-ideal J was dubbed *rigid* if it is contained in exactly one maximal t-ideal. For such a J, we have $J = I^t$ for some finitely generated subideal I of J, and, since it is more convenient to work with the subideal I, we apply the "rigid" terminology to I instead of J. Moreover, we want to consider finite-type star operations other than the t-operation (e.g., the d-operation!), and therefore prefer "t-rigid" in place of "rigid." Similarly, we replace "potent" with "t-potent."

In [28, 29] Wang and McCasland studied the w-operation in the context of strong Mori domains. Motivated by this, Anderson and Cook associated to any star operation $\star$ on a domain R a finite-type star operation $\star_w$, given by $A^{\star_w} = \{x \in K \mid xB \subseteq A$ for some finitely generated ideal B of R with $B^\star = R\}$ [3]. We always have $A^{\star_w} = \bigcap\{AR_P \mid P \in \star_f\text{-Max}(R)\}$ [3, Corollary 2.10], from which it follows that $A^{\star_w} R_P = AR_P$ for each $P \in \star_f\text{-Max}(R)$. (Recall that $\star_f$ is the finite-type star operation associated to $\star$ given by $A^{\star_f} = \bigcup B^\star$, where the union is taken over all finitely generated subideals B of A.) We also have $\star_f\text{-Max}(R) = \star_w\text{-Max}(R)$ [3, Theorem 2.16]. For the "original" w-operation, we have $w = v_w = t_w$ (hence the notation $\star_w$).

The following is an easy consequence of the definitions.

Proposition 1.3. *Let $\star_1, \star_2$ be finite-type star operations on a domain R for which $\star_1\text{-Max}(R) = \star_2\text{-Max}(R)$. Then $\star_1$-rigidity ($\star_1$-potency, $\star_1$-super potency) coincides with $\star_2$-rigidity ($\star_2$-potency, $\star_2$-super potency). In particular, R is $\star_1$-potent ($\star_1$-super potent) if and only if R is $\star_2$-potent ($\star_2$-super potent).* □

Observe that Proposition 1.3 may be applied to $\star$ and $\star_w$ for any finite-type star operation $\star$ on a domain R. In particular, the proposition may be applied to t- and w-operations.

Proposition 1.4. *Let $\star_1 \leq \star_2$ be finite-type star operations on a domain R. If $M \in \star_1\text{-Max}(R) \cap \star_2\text{-Max}(R)$ and M is $\star_1$-potent, then M is $\star_2$-potent.*

Proof. Let $M \in \star_1\text{-Max}(R) \cap \star_2\text{-Max}(R)$ with M $\star_1$-potent, and let I be a $\star_1$-rigid ideal contained in M. Suppose that $I \subseteq N$ for some maximal $\star_2$-ideal N of R. Since $N^{\star_1} \subseteq N^{\star_2} \neq R$, there is a maximal $\star_1$-ideal N' of R for which $N \subseteq N'$. Since $I \subseteq N'$, this forces $N' = M$ and hence $N = M$. Therefore, I is also $\star_2$-rigid, and hence M is $\star_2$-potent. □

With respect to Proposition 1.4, it is not true that for finite-type star operations $\star_1 \leq \star_2$ on a domain R and $M \in \star_1\text{-Max}(R) \cap \star_2\text{-Max}(R)$ with M $\star_2$-potent, we must also have M $\star_1$-potent–see Example 4.3 below. It is also not the case that for $\star_1 \leq \star_2$ and M a $\star_1$-potent maximal $\star_1$-ideal, we must have that M is a maximal $\star_2$-ideal. (Let $R = k[x, y]$, a polynomial ring in two variables over a field. Then $M = (x, y)$ is a d-potent maximal (d)-ideal but is not a t-ideal.) More interestingly, it is not the case that, for $\star_1 \leq \star_2$, R $\star_1$-potent implies R $\star_2$-potent, as we show in Example 5.4 below.

The situation is better for super potency:

Theorem 1.5. *Let $\star_1 \leq \star_2$ be finite-type star operations on a domain R. If M is a $\star_1$-super potent maximal $\star_1$-ideal of R, then M is also a $\star_2$-super potent maximal $\star_2$-ideal of R.*

Proof. Let M be a $\star_1$-super potent maximal $\star_1$-ideal of R, and let $A \subseteq M$ be a $\star_1$-super rigid ideal. We first show that $M^{\star_2} \neq R$. If, on the contrary, $M^{\star_2} = R$, then there is a finitely generated ideal $B \subseteq M$ with $B^{\star_2} = R$. Let $C := A + B$.

Then C is $\star_1$-invertible, whence $(C^{\star_2}C^{-1})^{\star_2} = (CC^{-1})^{\star_2} \supseteq (CC^{-1})^{\star_1} = R$. Since $C^{\star_2} = R$, this yields $C^{-1} = (C^{-1})^{\star_2} = R$. However, the equation $(CC^{-1})^{\star_1} = R$ then forces $C^{\star_1} = R$, the desired contradiction. Thus $M^{\star_2} \neq R$ and, since $M^{\star_2}$ is a $\star_1$-ideal, we must have $M^{\star_2} = M$. Then, again since $\star_2$ ideals are also $\star_1$-ideals, it must be the case that M is a maximal $\star_2$-ideal. That M must be $\star_2$-super potent now follows easily, since for any finitely generated ideal $I \supseteq A$, $\star_1$-invertibility of I implies $\star_2$-invertiblity. $\square$

As a consequence of the preceding result, we have that the weakest type of super potency is t-super potency:

Corollary 1.6. *Let $\star$ be a finite-type star operation on a domain R.*

1. *If M is a $\star$-super potent maximal $\star$-ideal of R, then M is a t-super potent maximal t-ideal of R.*
2. *If R is $\star$-super potent, then R is t-super potent.* $\square$

The converse of Corollary 1.6(2) is false: if k is a field, then the polynomial ring $k[X, Y]$, being a Krull domain, is t-super potent but is not d-super potent. However, we do not know whether one can have a maximal ideal M of a domain such that M is a t-super potent maximal t-ideal but is not d-super potent.

Now let R be a domain and T a flat overring of R. According to [27, Proposition 3.3], if $\star$ is a finite-type star operation on R, then the map $\star_T : IT \mapsto I^\star T$ is a well-defined finite-type star operation on T. In the following result, we study how (super) potency extends to flat overrings. We assume standard facts about flat overrings (including the fact, used above, that each fractional ideal of T is extended from a fractional ideal of R); these follow readily from [26].

Lemma 1.7. *Let R be a domain, T a flat overring of R, $\star$ a finite-type star operation on R, and $\mathcal{P}$ the set of $\star$-primes P of R maximal with respect to the property $PT \neq T$. Then:*

1. *$\star_T$-Max$(T) = \{PT \mid P \in \mathcal{P}\}$.*
2. *If M is a $\star$-(super) potent maximal $\star$-ideal of R for which $MT \neq T$, then MT is a $\star_T$-(super) potent maximal $\star_T$-ideal of T. (In fact, if M is as hypothesized and $I \subseteq M$ is $\star$-(super) rigid, then IT is $\star_T$-(super) rigid in T.)*

Proof. Let $P \in \mathcal{P}$. Then $(PT)^{\star_T} = P^\star T = PT$, that is, PT is a $\star_T$-ideal of T. Moreover, if Q is a prime of R for which QT is a maximal $\star_T$-ideal of T containing PT, then $Q^\star \subseteq Q^\star T \cap R = (QT)^{\star_T} \cap R = QT \cap R = Q$; that is, Q is a $\star$-ideal of R containing P. Since $P \in \mathcal{P}$, we have ($Q = P$ and hence) $QT = PT$. Therefore PT is a maximal $\star_T$-ideal of T. Conversely, let P be a prime of R for which PT is a maximal $\star_T$-ideal of T. Then $P^\star \subseteq P^\star T \cap R = (PT)^{\star_T} \cap R = PT \cap R = P$, and so P is a $\star$-ideal of R. Suppose that $P \subseteq Q$, where Q is a $\star$-prime of R and $QT \neq T$. Then QT is a $\star_T$-ideal of T (since, $(QT)^{\star_T} = Q^\star T = QT$) containing PT, whence ($QT = PT$ and hence) $Q = P$. This proves (1).

Let M be a $\star$-potent maximal $\star$-ideal of R such that $MT \neq T$. Then MT is a maximal $\star_T$-ideal of T by (1). Now let I be a $\star$-rigid ideal contained in M, and suppose that $IT \subseteq NT$, where N is a prime ideal of R for which NT is a maximal $\star_T$-ideal of T. Then $N^\star \neq R$, whence $N \subseteq N'$ for some maximal $\star$-ideal N' of R. Since I is contained in no maximal $\star$-ideal of R other than M, we must have

$N' = M$. However, this yields $N \subseteq M$ and hence $NT = MT$. It follows that IT is $\star_T$-rigid in T.

Now assume that M is $\star$-super potent and that $I \subseteq M$ is $\star$-super rigid. Let J be a finitely generated ideal of R for which $JT \supseteq IT$. Replacing J with $I + J$ if necessary, we may assume that $J \supseteq I$. Then J is $\star$-invertible, whence, in particular, $JJ^{-1} \nsubseteq M$. This, in turn, yields $(JT)(T : JT) \nsubseteq MT$. Since MT is the only maximal $\star_T$-ideal of T containing JT, JT is $\star$-invertible. Therefore, IT is $\star_T$-super rigid. This completes the proof of (2). □

Remark 1.8. Suppose that (R, M) is local and that $\star$ is a star operation on R for which M is a $\star$-ideal. Then if I is a $\star$-invertible ideal of R, we cannot have $II^{-1} \subseteq M$, and hence I is actually (invertible and hence) principal. In particular, if $\star$ is of finite-type and $I \subseteq M$ is $\star$-super rigid, then I is principal. We shall use this fact often in the sequel.

Lemma 1.9. *Let (R, M) be a local domain. The following statements are equivalent.*

(1) *M is a $\star$-super potent maximal $\star$-ideal for some finite-type star operation $\star$ on R.*
(2) *M is a t-super potent maximal t-ideal.*
(3) *M is d-super potent.*
(4) *M is a $\star$-super potent maximal $\star$-ideal for every finite-type star operation on R.*

Proof. The implications (1) ⇒ (2) and (3) ⇒ (4) follow from Theorem 1.5, and (4) ⇒ (1) is trivial. Assume (2), and let A be a t-super rigid ideal contained in M and B a finitely generated ideal containing A. Then B is t-invertible and hence principal (Remark 1.8). Therefore, A is d-super rigid, as desired. □

Since the extension (as defined above) of the d-operation on R to a flat overring T is the d-operation on T, we shall write "d" instead of "d_T" in this case.

It is now an easy matter to characterize $\star$-super potency locally:

Theorem 1.10. *Let $\star$ be a finite-type star operation on R and M a maximal $\star$-ideal of R. Then:*

(1) *M is $\star$-super potent if and only if M is $\star$-potent and MR_M is d-super potent.*
(2) *R is $\star$-super potent if and only if R is $\star$-potent and R_P is d-super potent for each maximal $\star$-ideal P of R.*

Proof. It suffices to prove (1). Assume that M is $\star$-potent and MR_M is d-super potent. Then there is a finitely generated ideal A of R such that AR_M is d-super rigid and a $\star$-rigid ideal B of R contained in M. Suppose that C is a finitely generated ideal of R with $C \supseteq A + B$. Then CR_M is principal, and $CR_N = R_N$ for each maximal $\star$-ideal $N \neq M$. It follows that C is $\star$-invertible. Thus ($A + B$ is $\star$-super rigid and hence) M is $\star$-super potent. For the converse, if M is $\star$-super potent, then M is certainly $\star$-potent. Moreover, MR_M is $\star_{R_M}$-super potent by Lemma 1.7 and hence d-super potent by Lemma 1.9. □

In spite of Theorem 1.10 (and Lemma 1.7), $\star$-super potency does not in general localize at non-maximal $\star$-primes–see Example 4.6 below. Also, observe that if R is a non-Dedekind almost Dedekind domain, then R_M is d-super potent for each maximal (t-)ideal M, but R is not t-potent. (Hence the $\star$-potency assumption is necessary in Theorem 1.10(1,2).)

Theorem 1.11. *Let $\star$ be a finite-type star operation on a domain R, M a $\star$-super potent maximal $\star$-ideal of R, and I a $\star$-super rigid ideal of R contained in M.*

(1) *If A is a finitely generated ideal for which $M \supseteq A^\star \supseteq I$, then A is $\star$-super rigid.*
(2) *If J is a $\star$-rigid ideal contained in M, then $I \subseteq J^\star$ or $J \subseteq I^\star$.*
(3) *If J is a $\star$-super rigid ideal contained in M, then IJ is also a $\star$-super rigid ideal.*
(4) *I^n is $\star$-super rigid for each positive integer n.*
(5) *If R is local with maximal ideal M, then I is comparable to each ideal of R, and $\bigcap_{n=1}^\infty I^n$ is prime.*
(6) *$I^\star = IR_M \cap R$.*
(7) *$\bigcap_{n=1}^\infty (I^n)^\star$ is prime.*
(8) *If P is a prime ideal of R with $P \subseteq M$ and $I \nsubseteq P$, then $P \subseteq \bigcap_{n=1}^\infty (I^n)^\star$.*

Proof. (1) Let A be a finitely generated ideal with $M \supseteq A^\star \supseteq I$. Then A is clearly $\star$-rigid. Let B be a finitely generated ideal with $B \supseteq A$. Set $C := I + B$. Then C is $\star$-invertible, and, since $C^\star = (I^\star + B^\star)^\star \subseteq (A^\star + B^\star)^\star = B^\star$, we have $C^\star = B^\star$, and hence B is $\star$-invertible. Therefore, A is $\star$-super rigid.

(2) Let J be a $\star$-rigid ideal contained in M, and set $C := I + J$. Then C is $\star$-invertible, and we have $(IC^{-1} + JC^{-1})^\star = R$. Note that $IC^{-1} \supseteq I$ and $JC^{-1} \supseteq J$, and hence $IC^{-1} \nsubseteq M$ or $JC^{-1} \nsubseteq M$. Since IC^{-1}, JC^{-1} can be contained in no maximal $\star$-ideal of R other than M, we must have $(IC^{-1})^\star = R$ or $(JC^{-1})^\star = R$, that is, $C^\star = I^\star$ or $C^\star = J^\star$. The conclusion follows easily.

(3) Let J be a $\star$-super rigid ideal contained in M, and let C be a finitely generated ideal with $M \supseteq C \supseteq IJ$. Then C is $\star$-rigid. If $C^\star \supseteq I$, then C is $\star$-invertible by (1). Thus by (2) we may as well assume that $C \subseteq I^\star$, so that $CI^{-1} \subseteq R$. Since I is $\star$-invertible, $I^{-1} = A^\star$ for some finitely generated fractional ideal A, and this yields $(CA)^\star \supseteq (IJA)^\star = J^\star \supseteq J$. Hence CA is $\star$-invertible, from which it follows that C is $\star$-invertible.

(4) This follows from (3).

(5) Assume that R is local with maximal ideal M. By Lemma 1.9 (and its proof) I is d-super rigid and therefore principal (Remark 1.8), say $I = (c)$. Choose $r \in M \setminus (c)$. Then (c, r) is principal, and, since R is local, $(c, r) = (r)$, i.e. $c \in (r)$. It follows that I is comparable to each ideal of R. Now suppose, by way of contradiction, that $a, b \in R$ with $ab \in \bigcap (c^n)$ and $a, b \notin \bigcap (c^n)$. Choose n, m with $a \in (c^n) \setminus (c^{n+1})$ and $b \in (c^m) \setminus (c^{m+1})$. Then $a/c^n, b/c^m \notin (c)$, whence, by the claim, $c \in (a/c^n) \cap (b/c^m)$. Hence $c^{n+m+2} \in (ab) \subseteq (c^{n+m+3})$, yielding the contradiction that $1 \in (c)$. Hence $\bigcap_{n=1}^\infty I^n$ is prime.

(6) We have $I^\star \subseteq I^\star R_M \cap R = (IR_M)^{\star_{R_M}} \cap R = IR_M \cap R$ (since IR_M is principal). On the other hand, $IR_N = R_N$ for $N \in \mathcal{N} := \star\text{-Max}(R) \setminus \{M\}$, and hence $I^\star \supseteq I^{\star_w} = IR_M \cap (\bigcap_{N \in \mathcal{N}} IR_N) = IR_M \cap R$.

(7) By (4), Lemma 1.7, and (the proof of) Lemma 1.9, $I^n R_M$ is d-super rigid for each n. Using (6), we have $\bigcap_{n=1}^\infty (I^n)^\star = \bigcap_{n=1}^\infty (I^n R_M \cap R) = (\bigcap_{n=1}^\infty I^n R_M) \cap R$, which is prime by (5).

(8) Let P be as described. Since $IR_M \nsubseteq PR_M$, we have by (5) and (6) that $P \subseteq \bigcap_{n=1}^\infty I^n R_M \cap R = \bigcap_{n=1}^\infty (I^n)^\star$. $\square$

We record the following useful consequence of Theorem 1.11.

Corollary 1.12. *If M is a t-super potent ideal of height one in a domain R, then R_M is a valuation domain. In particular, a one-dimensional local d-super potent domain is a valuation domain.*

Proof. We begin with the "in particular" statement. Let R be a one-dimensional local d-super potent domain, I a d-super rigid ideal of R, and J a finitely generated ideal of R. Then $J \supseteq I^n$ for some positive integer n. Since I^n is d-super rigid by Theorem 1.11, J must be (invertible and hence) principal. It follows that R is a valuation domain. Now assume that M is t-super potent of height one in a domain R. By Theorem 1.10, R_M is d-super potent and is therefore a valuation domain by what has just been proved. □

It is easy to see that the requirement on the height of M in Corollary 1.12 is necessary–take R to be any local non-valuation domain having principal maximal ideal and dimension at least two.

2. THE LOCAL CASE

Let (R, M) be a local domain and $\star$ a finite-type star operation on R. Recall from Lemma 1.9 that R is $\star$-super potent if and only if R is d-super potent. We shall characterize and study local d-super potency.

As in [7] we say that a prime ideal P of a domain R is *divided* if $P = PR_P$. Domains in which each prime ideal is divided were introduced and briefly studied in [1], apparently motivated by considerations from [15]. Recall that if P is a prime ideal of a domain R, then $R + PR_P$ is called the *CPI-extension of R with respect to P* [6]. ("CPI" is short for "complete pre-image.") The next lemma follows easily from arguments in [1, 7, 6].

Lemma 2.1. *Let P be a prime ideal of a domain R. Then the following statements are equivalent.*

1. *P is divided.*
2. *P is comparable to each principal ideal of R.*
3. *P is comparable to each ideal of R.*
4. *R is the CPI-extension of R with respect to P.*

Theorem 2.2. *Let (R, M) be a local domain, not a field. Then R is d-super potent if and only if there is a divided prime $P \subsetneq M$ such that R/P is a valuation domain.*

Proof. Suppose that R is d-super potent, and let $I \subseteq M$ be d-super rigid. Then $I = (c)$ for some $c \in M$. Moreover, by Theorem 1.11(4,5), $P := \bigcap (c^n)$ is prime, and, for each positive integer m, (c^m) is d-super rigid and hence comparable to each ideal of R . Let $a \in M \setminus P$. Then $a \notin (c^k)$ for some k, whence $P \subseteq (c^k) \subseteq (a)$. Hence (a) is d-super rigid. This shows both that P is divided (Lemma 2.1) and that any two principal ideals generated by elements of $M \setminus P$ must be comparable (since each is a d-super rigid ideal). It follows that R/P is a valuation domain.

Now assume that P is a divided prime properly contained in M and that R/P is a valuation domain. Let $a \in M \setminus P$. Since P is divided, we have $P \subsetneq (a)$ (Lemma 2.1). Suppose that $I = (a_1, \ldots, a_n)$ is a finitely generated ideal containing (a). Then $I/P \supseteq (a)/P$ in the valuation domain R/P, and it follows that (I/P and hence) I is principal. Therefore, (a) is super rigid. □

Recall from Corollary 1.12 that a one-dimensional d-super potent domain is a valuation domain. Of course, this is also an immediate corollary of Theorem 2.2, as is the following result in the two-dimensional case.

Corollary 2.3. *If R is a two-dimensional local d-super potent domain, then R has exactly two nonzero prime ideals.* □

It is trivial that a Noetherian domain R is $\star$-potent for any star operation $\star$ on R. As another consequence of Theorem 2.2, we have a characterization of Noetherian t-super potent domains:

Corollary 2.4. *Let R be a Noetherian domain.*

(1) *If M is a t-super potent maximal t-ideal of R, then* $\mathrm{ht}(M) = 1$.
(2) *If R is t-super potent, then R is a Krull domain.*

Proof. (1) Let M be a t-super potent maximal t-ideal of R. Then R_M is a d-super potent Noetherian domain, and hence we may as well assume that R is local with d-super potent maximal ideal M. By Theorem 2.2, there is a divided prime $P \subsetneq M$ such that R/P is a Noetherian valuation domain. Moreover, if we choose $a \in M \setminus P$ and shrink M to a prime Q minimal over a, then $Q \supseteq (a) \supsetneq P$. By the principal ideal theorem, we must have $\mathrm{ht}(Q) = 1$, and hence $P = (0)$. But then R is a Noetherian valuation domain, and we must have $\mathrm{ht}(M) = 1$. For (2), suppose that R is t-super potent. By (1) R_M is a Noetherian valuation domain for each $M \in t$-$\mathrm{Max}(R)$, and hence the representation $R = \bigcap\{R_M \mid M \in t\text{-}\mathrm{Max}(R)\}$ shows that R is (completely) integrally closed and therefore a Krull domain. □

Remark 2.5. (1) Recall from [16] that a nonzero element a of a domain R is said to be *comparable* if (a) compares to each ideal of R under inclusion. By [16, Theorem 2.3] and Theorem 2.2, non-field local d-super potent domains coincide with domains that admit nonzero, nonunit comparable elements. Moreover, again by [16, Theorem 2.3], for such a domain R, the ideal $P_0 := \bigcap\{(c) \mid c \text{ is a nonzero comparable element of } R\}$ is a divided prime and is such that R/P_0 is a valuation domain, and P_0 is the (unique) smallest prime L of R such that L is divided and R/L is a valuation domain.

(2) With the notation above, the following statements are equivalent: (a) R is a valuation domain, (b) R_{P_0} is a valuation domain, and (c) $P_0 = (0)$: the implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) are clear, and (b) $\Rightarrow$ (c) by the remark following Theorem 2.3 of [16].

(3) As explained in the just-mentioned remark in [16], every local domain (R, M) that admits a nonzero, nonunit comparable element arises as a pullback

$$\begin{array}{ccc} R & \longrightarrow & V \\ \big\downarrow & & \big\downarrow \\ T & \xrightarrow{\varphi} & T/M = k, \end{array}$$

where (T, M) is a local domain and V is a valuation domain with quotient field k (in which case we have $T = R_M$). In particular, if T is a two-dimensional Noetherian domain, it must have infinitely many height-one primes and hence so must R. Thus Corollary 2.3 does not extend to higher dimensions; indeed, the primes of a local d-super potent domain need not even be linearly ordered (e.g. $R = \mathbb{Z}_{(p)} + (x, y)\mathbb{Q}[[x, y]]$, where p is prime and x, y are indeterminates).

We end this section with an attempt to globalize local $\star$-super potency.

Lemma 2.6. *Let M, N, P be primes in a domain R with $P \subseteq M \cap N$, and assume that PR_M is divided in R_M and that R_M/PR_M and R_N are valuation domains. Then R_M is a valuation domain.*

Proof. Since R_N is a valuation domain, so is $R_P = (R_N)_{PR_N}$. However, we also have $R_P = (R_M)_{PR_M}$, and since $R_M = \varphi^{-1}(R_M/PR_M)$, where $\varphi : R_P \to R_P/PR_P$ is the canonical projection, R_M is a valuation domain by [14, Proposition 18.2(3)]. □

Definition 2.7. Let R be a domain, and let $P \subsetneq M$ be prime ideals of R. We say that P *belongs* to M if $PR_M = PR_P$ and R_M/PR_M is a valuation domain.

Note that, by Theorem 2.2, a prime ideal M of a domain R contains a belonging prime if and only if R_M is d-super potent. Moreover, if M contains a belonging prime, then it contains a smallest one by Remark 2.5(1).

Lemma 2.8. *Let R be a domain, let M, N be prime ideals of R, and suppose that there is a prime belonging to both M and N. Then the smallest prime of R that belongs to N also belongs to M (and vice versa).*

Proof. Let P belong to both M and N, and let Q be the smallest prime belonging to N. We have $Q \subseteq P$. Applying Lemma 2.6 to R/Q yields that $R_M/QR_M = (R/Q)_{M/Q}$ is a valuation domain. Also, since QR_N is divided in R_N,

$$QR_Q = QR_N \subseteq QR_P \subseteq PR_P = PR_M \subseteq R_M.$$

Hence $QR_Q = QR_Q \cap R_M = QR_M$. Therefore, Q belongs to M. □

Remark 2.9. Let $\star$ be a finite-type star operation on a domain R, and assume that R is $\star$-super potent. Then each maximal $\star$-ideal of R contains a belonging prime by Theorems 1.10 and 2.2. Define $\sim$ on $\star$-Max(R) by $M \sim N$ if M and N contain a common belonging prime. It is perhaps interesting that $\sim$ is an equivalence relation: it is clearly reflexive and symmetric, and transitivity follows easily from Lemma 2.8.

Observe that the relation described above forces a certain amount of "independence" in $\star$-Max(R): if M, N are two maximal $\star$-ideals in the $\star$-super potent domain R with $M \not\sim N$, P belongs to M, Q belongs to N, and $Q \subseteq P$, then $P \nsubseteq N$. We give a simple example illustrating this.

Example 2.10. Let F be a field, and x, y indeterminates. Set $V = F(x)[y]_{yF(x)[y]}$, $T = F(y)[x^2, x^3]_{(x^2,x^3)F(y)[x^2,x^3]}$, $R_1 = V + P$, where P is the maximal ideal of T, $R_2 = F(x)[y]_{(y+1)F(x)[y]}$, and $R = R_1 \cap R_2$. Then R_1 and R_2 are d-super potent (both have principal maximal ideals). Denote the maximal ideal of R_1 by M_1. Then $M := M_1 \cap R$ and $N := (y+1)R_2 \cap R$ are the maximal ideals of R, and by [23, Theorem 3], we have $R_M = R_1$ and $R_N = R_2$. The domain R is therefore d-super potent by Theorem 1.10, and it is clear that (0) belongs to N and that P (but not (0)) belongs to M.

3. Polynomial rings over t-super potent domains

We begin with some well-known facts about t-ideals in polynomial rings. Recall that if R is a domain and Q is a nonzero prime of $R[X]$ for which $Q \cap R = (0)$, then Q is called an *upper to zero*.

Lemma 3.1. *Let R be a domain.*

(1) *An ideal A of R is a t-ideal if and only if $A[X]$ is a t-ideal of $R[X]$.*
(2) *If Q is maximal t-ideal of $R[X]$, then $Q = P[X]$ for some maximal t-ideal of R or Q is an upper to zero in $R[X]$.*
(3) *An ideal M of R is a maximal t-ideal if and only if $M[X]$ is a maximal t-ideal of $R[X]$.*
(4) *If Q is an upper to zero in $R[X]$ and is also a maximal t-ideal, then Q is t-super potent.*

Proof. For (1) see [18, Proposition 4.3]. Let Q be a maximal t-ideal of $R[X]$. By [19, Proposition 1.1], $Q = (Q \cap R)[X]$ or Q is an upper to zero. It then follows from (1), that if $Q = (Q \cap R)[X]$, then $P := Q \cap R$ must be a maximal t-ideal of R. This gives (2), and (3) follows from (1) and (2). Now suppose that Q is an upper to zero and also a maximal t-ideal in $R[X]$. Then $Q = fK[X] \cap R[X]$ for some polynomial $f \in Q$ such that f is irreducible in $K[X]$. By [19, Theorem 1.4] there is an element $g \in Q$ such that $c(g)^v = R$ (where $c(g)$, the content of g, is the ideal of R generated by the coefficients of g), and it is easy to see via (1) and (2) that the ideal (f, g) of $R[X]$ is contained in no maximal t-ideal of $R[X]$ other than Q. Hence Q is t-potent and therefore by Theorem 1.10 also t-super potent since $R[X]_Q$ is a valuation domain. Hence (4) holds. □

Theorem 3.2. *Let R be a domain. Then R is t-(super) potent if and only if $R[X]$ is t-(super) potent.*

Proof. Suppose that R is t-potent, and let Q be a maximal t-ideal of $R[X]$. By Lemma 3.1(2), Q is either an upper to zero or $Q = P[X]$ with P a maximal t-ideal of R. If Q is an upper to zero, it is t-super potent by Lemma 3.1(4). If $Q = P[X]$ with $P \in t\text{-Max}(R)$, then there is a t-rigid ideal I of R contained in P, and it is easy to see that $I[X]$ is t-rigid in $R[X]$. Hence $R[X]$ is t-potent.

Now assume that R is t-super potent. Then $R[X]$ is t-potent by what has already been proved. Hence, by Theorem 1.10, it suffices to show that $R[X]_Q$ is d-super potent for each maximal t-ideal Q of $R[X]$. To this end, let Q be a maximal t-ideal of $R[X]$. Again by Lemma 3.1(4), we may as well assume that $Q = P[X]$ with P a maximal t-ideal of R. We shall show that $R[X]_Q$ satisfies the requirements of Theorem 2.2. Since $R[X]_Q = R_P[X]_{PR_P[X]}$ and R_P is d-super potent, we change notation and assume that R is local with d-super potent maximal ideal P, and we wish to show that $R[X]_{P[X]}$ is d-super potent. By Theorem 2.2 there is a prime L of R such that $L \subsetneq P$, R/L is a valuation domain, and $L = LR_L$. Then $R[X]_{P[X]}/LR[X]_{P[X]} = (R/L)[X]_{(P/L)[X]}$, which is a valuation domain. Finally, we must show that $LR[X]_{L[X]} = LR[X]_{P[X]}$. Let $f, g \in R[X]$ with $c(g) \subseteq L$ and $f \in R[X] \setminus L[X]$. If $f \notin P[X]$, then $g/f \in LR[X]_{P[X]}$, as desired. Suppose that $f \in P[X]$. Since $L = LR_P$ and $f \notin L[X]$, $c(f) \supsetneq c(g)$, and, since R/L is a valuation domain, $c(f) = (b)$ for some $b \in P \setminus L$. Note that $b^{-1}f \in R[X] \setminus P[X]$. Also, since $b^{-1}g \cdot b \in L[X]$ and $b \notin L$, $b^{-1}g \in L[X]$. Thus $g/f = b^{-1}g/(b^{-1}f) \in LR[X]_{P[X]}$, as desired.

For the converse, first assume that $R[X]$ is t-potent, and let P be a maximal t-ideal of R. Then $P[X]$ is a maximal t-ideal of $R[X]$, and we may find a t-rigid ideal $A \subseteq P[X]$. Let I denote the ideal of R generated by the coefficients of the polynomials in a finite generating set of A. Then I is a finitely generated ideal of R contained in P, and since $A \subseteq I[X] \subseteq P[X]$ yields that $I[X]$ is t-rigid in $R[X]$, it is clear that I is t-rigid in R. Hence R is t-potent. Finally, suppose that $R[X]$ is t-super potent. Using the notation above, we may assume that A is t-super rigid, whence $I[X]$ is also t-super rigid. If J is a finitely generated ideal of R containing I, then $J[X]$ is a finitely generated ideal of $R[X]$ containing $I[X]$; this yields that $J[X]$ is t-invertible in $R[X]$, from which it follows easily that J is t-invertible in R. Hence t-super potency of $R[X]$ implies t-super potency of R. □

Remark 3.3. It is interesting to note that in the proof above, it was easy to show that $R[X]_{PR[X]}/LR[X]_{P[X]}$ is a valuation domain using only the fact that R/L is a valuation domain, but the proof that $LR[X]_{L[X]} = LR[X]_{P[X]}$ used not only the assumption that $L = LR_L$ but also the assumption that R/L is a valuation domain. Here is an example that shows the necessity of the latter assumption. Let F be a field, $k = F(u)$, u an indeterminate, V a 2-dimensional valuation domain of the form $k + P$ with height-one prime L, and $R = F + P$. According to [14, Theorem 19.15 and its proof], denoting the common quotient field of R and V by K, $Q := (X - u)K[X] \cap R[X]$ is an upper to zero in $R[X]$ satisfying $Q \subseteq P[X]$. We have $L = LV_L = LR_L$. However, R/L is not a valuation domain, and we claim that we do not have $LR[X]_{P[X]} = LR[X]_{L[X]}$. To see this choose $a \in L$, $a \neq 0$, and $c \in P \setminus L$. Then $a/(cX - cu) \in LR[X]_{L[X]}$. Suppose that we can write $a/(cX - cu) = g/f$ with $g \in L[X]$ and $f \in R[X] \setminus P[X]$. We have $af = g(cX - cu)$, so that $f = a^{-1}g(cX - cu) \in (X - u)K[X] \cap R[X] = Q \subseteq P[X]$, a contradiction. This verifies the claim.

4. Pullbacks

Let T be a domain, M a maximal ideal of T, $\varphi : T \to k := T/M$ the natural projection, and D a proper subring of k. Then let $R = \varphi^{-1}(D)$ be the integral domain arising from the following pullback of canonical homomorphisms.

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & T/M = k. \end{array}$$

We list some properties that we shall need.

Lemma 4.1. *Consider the pullback diagram above.*

1. *T is a flat R-module if and only if k is the quotient field of D.*
2. *If I is a nonzero finitely generated ideal of D, then $\varphi^{-1}(I)$ is a finitely generated ideal of R.*
3. t-Max$(R) = \{N \cap R \mid N \in t$-Max$(T),\ N \nsubseteq M\} \bigcup \{\varphi^{-1}(P') \mid P' \in t$-Max$(D)\}$. *(By convention, if D is a field, then (0) is a maximal t-ideal of D, in which case M is a maximal t-ideal of R).*
4. *If N is a prime ideal of T that is incomparable to M, then $R_{N \cap R} = T_N$.*

(5) *Assume that D is not a field. If I is a t-invertible ideal of R with $I \supsetneq M$, then $\varphi(I)$ is a t-invertible ideal of D. Conversely, if I' is a t-invertible ideal of D, then $\varphi^{-1}(I')$ is a t-invertible ideal of R.*

Proof. Statement (1) is well-known (see [11, Proposition 1.11]), (2) is part of [9, Corollary 1.7]), and (3) follows from [11, Theorems 2.6, 2.18] (but the ideas are from [9]). For (4), see, e.g. [11, Theorem 1.9], and for (5), see [11, Theorem 2.18 and Proposition 2.20]. □

Theorem 4.2. *Consider the pullback diagram above. Then R is t-potent if and only if each of the following conditions holds:*

(1) *D is t-potent (or a field).*
(2) *N is t-potent for each $N \in t\text{-Max}(T)$ with $N \nsubseteq M$.*
(3) *If D is a field and M is a t-ideal of T, then M is t-potent in T.*

Proof. Suppose that R is t-potent. If D is not a field and $P' \in t\text{-Max}(D)$, then by Lemma 4.1(3), $P := \varphi^{-1}(P') \in t\text{-Max}(R)$, whence there is a t-rigid ideal I contained in P. Then, again using Lemma 4.1(3), it is easy to see that $\varphi(I)$ is a t-rigid ideal of D contained in P'. Hence D is t-potent. Now let $N \in t\text{-Max}(T), N \nsubseteq M$. Then (Lemma 4.1(3)) $N \cap R \in t\text{-Max(R)}$ and hence there is a t-rigid ideal J contained in $N \cap R$. (In particular, $J \nsubseteq M$.) Then JT is a t-rigid ideal of T contained in N (Lemma 4.1(3)), and N is t-potent. This holds whether D is a field or not. If D is a field, then M is a maximal t-ideal and hence t-potent in R, and there is a t-rigid ideal I of R contained in M. If M is a (maximal) t-ideal of T, then IT is a t-rigid ideal of T contained in M, and hence T is t-potent in this case.

For the converse, let $P \in t\text{-Max}(R)$. If $P \supsetneq M$, then $P = \varphi^{-1}(P')$ for some $P' \in t\text{-Max}(D)$. By assumption, there is a t-rigid ideal C of D contained in P', and Lemma 4.1(2,3) then implies that $\varphi^{-1}(C)$ is a t-rigid ideal of R contained in P.

Next, suppose that $P = M$. Then D is a field. By assumption M is t-potent in T or M is not a t-ideal of T. In the first case, let A_1 be a t-rigid ideal of T contained in M. In the second case, we have $M^{t_T} = T$ (where t_T is the t-operation on T), and there is a finitely generated subideal A_2 of M with $(A_2)^{t_T} = T$. In either case, there is a finitely generated subideal A of M with $A \nsubseteq N$ for each $N \in t\text{-Max}(T) \setminus \{M\}$. For such an A we have $A = IT$ for some finitely generated ideal I of R, and it is clear from the conditions satisfied by A that I is t-rigid in R.

Finally, suppose that P is incomparable to M. Then $P = N \cap R$ for some $N \in t\text{-Max}(T)$ with $N \nsubseteq M$. By assumption there is a t-rigid ideal B of T contained in N, and we may assume that B contains an element $t \in T \setminus M$. Now $\varphi(t) \neq 0$, whence there is an element $t' \in T$ with $\varphi(tt') = 1$. This implies that $tt' \in R$, and, since $1 - tt' \in M$, it is clear that $tt' \notin Q$ for each ideal Q of R such that $Q \supseteq M$. We consider three cases:

Case 1. Suppose that k is the quotient field of D. Then T is flat over R (Lemma 4.1), and hence $B = JT$ for some finitely generated ideal J of R. By construction, J is a t-rigid ideal of R contained in $P = N \cap R$ in this case.

Case 2. Suppose that D is a field. Then, arguing as in the "$P = M$" situation above, there is a finitely generated subideal A of M with $A \nsubseteq L$ for each $L \in t\text{-Max}(T) \setminus \{M\}$, and $A = IT$ for some finitely generated ideal I of R. Write $I = \sum_{i=1}^m Ra_i$ and $B = \sum_{j=1}^n Tb_j$, and let $J = \sum Ra_ib_j$. Then $JT = IB \nsubseteq L$ for $L \in t\text{-Max}(T) \setminus \{M, N\}$. It then follows easily that $J + Rtt'$ is a t-rigid ideal of R contained in $P = N \cap R$.

Case 3. Suppose that k is not the quotient field of D and that D is not a field, and put $S := \varphi^{-1}(F)$, where F is the quotient field of D. By what has already been proved, S is t-potent. It then follows that $P = N \cap R$ is t-potent by Case 1 above. This completes the proof. $\square$

We next give an example, promised immediately after Proposition 1.4, of finite-type star operations $\star_1 \leq \star_2$ on a domain R and a $\star_2$-potent maximal $\star_2$-ideal that is $\star_1$-maximal but not $\star_1$-potent.

Example 4.3. Let $F \subsetneq k$ be fields, $T = k[x_1, x_2, \ldots]$ a polynomial ring in countably many variables, and $R = F + M$, where M is the maximal ideal of T generated by the x_i. In R, M is both a maximal (d-)ideal and a maximal t-ideal. Since T is a Krull domain, it is t-potent, whence so is R by Theorem 4.2. In particular, M is t-potent in R. However, it is clear that each finitely generated subideal of M is contained in infinitely many maximal ideals of (T and hence of) R, and so M is not d-potent.

Theorem 4.4. *Consider the pullback diagram at the beginning of this section. Then R is t-super potent if and only if D is t-super potent and not a field, and each maximal t-ideal of T not contained in M is t-super potent.*

Proof. Assume that R is t-super potent, and suppose, by way of contradiction, that D is a field. Then we have the following associated pullback diagram

$$\begin{array}{ccc} R_M & \longrightarrow & D \\ \downarrow & & \downarrow \\ T_M & \longrightarrow & k \end{array}$$

Choose $a \in M$, $a \neq 0$, and let $t \in T_M \setminus R_M$. Then $at \notin aR_M$ since $t \notin R_M$, and $a \notin atR_M$, since $t^{-1} \notin R_M$. However, this implies that $aR_M + atR_M$ is not principal, and hence that aR_M is not d-super rigid. It follows that MR_M is not d-super potent, and then, by Theorem 1.10, that M is not t-super potent, the desired contradiction. Thus D is not a field.

In the rest of the proof, we freely use Lemma 4.1. Let P' be a maximal t-ideal of D. Then $P := \varphi^{-1}(P')$ is a maximal t-ideal of R properly containing M and therefore contains a t-super rigid ideal I. It is clear that $I' := \varphi(I)$ is contained in P' and in no other maximal t-ideal of D. Let $J' \supseteq I'$ be a finitely generated ideal of D. Then, since P is t-super potent, $J := \varphi^{-1}(J')$ is a t-invertible ideal of R, and hence $\varphi(J) = J'$ is t-invertible in D. Therefore, D is t-super potent.

Now let $N \nsubseteq M$ be a maximal t-ideal of T. Then $N \cap R$ is a maximal t-ideal of R, and hence $T_N = R_{N \cap R}$ is d-super potent by Theorem 1.10. Therefore, since N is t-potent by Theorem 4.2, N is t-super potent by Theorem 1.10.

For the converse, let $P \in t\text{-Max}(R)$. If $P \supsetneq M$, then $\varphi(P)$ is t-super potent in D, and we can argue more or less as above to see that P is t-super potent in R. Since D is not a field, the only other possibility is $P = N \cap R$, where $N \in t\text{-Max}(T)$, $N \nsubseteq M$. In this case, t-super potency of N in T yields d-super potency of $NT_N = (N \cap R)R_N$ (Theorem 1.10). Since $N \cap R$ is t-potent by Theorem 4.2, we may again apply Theorem 1.10 to conclude that $P = N \cap R$ is t-super potent. $\square$

From Theorems 4.2 and 4.4, we can determine t-(super) potency in a large class of domains that appear frequently in the literature:

Corollary 4.5. *Let D be a subdomain of the field k and x an indeterminate. Let $R = D + xk[x]$ or $D + xk[[x]]$. Then*

1. *R is t-potent if and only if D is t-potent (or a field).*
2. *R is t-super potent if and only if D is t-super potent and not a field.*

Using Theorem 4.4, it is easy to give examples of t-super potent domains with non-t-super potent localizations:

Example 4.6. In the notation of Theorem 4.4, assume that R is t-super potent.

1. If the quotient field of D is $F \neq k$, then R_M is not t-super potent. We may take R integrally closed or not.
2. If T is a one-dimensional local non-valuation domain, then R_M is not t-super potent.

Proof. (1) In this case, let $S = \varphi^{-1}(F)$. Then we have the pullback diagram

$$\begin{array}{ccc} R_M & \longrightarrow & F \\ \big\downarrow & & \big\downarrow \\ T_M & \longrightarrow & k, \end{array}$$

whence R_M is not t-super potent by Theorem 4.4. Let x be an indeterminate, and z an element of a field $k \supseteq \mathbb{Q}$. Let $D = \mathbb{Z}$ and $T = \mathbb{Q}(z)[[x]]$ (so that $R = \mathbb{Z} + x\mathbb{Q}(z)[[x]]$). In this case, if z is an indeterminate, then R is integrally closed. On the other hand, if $z = \sqrt{2}$, then R is not integrally closed.

(2) If k is not the quotient field of D, this follows from (1) . If k is the quotient field of D, then $R_M = T$ is not t-super potent by Corollary 1.12. $\square$

5. t-DIMENSION ONE

The primary goal of this section is to characterize generalized Krull domains using t-super potency. We recall some definitions. First, a set $\mathcal{P}$ of prime ideals in a domain R is a *defining family* if $R = \bigcap_{P \in \mathcal{P}} R_P$. A defining family has *finite character* (or is *locally finite*) if each nonzero element $a \in R$ lies in at most finitely many elements of $\mathcal{P}$. (Thus, in this terminology, if $\star$ is a finite-type star operation on R, then R has finite $\star$-character if the defining family of maximal $\star$-ideals of R has finite character.) A prime P of R is *essential* if R_P is a valuation domain, and R itself is an *essential domain* if it possess a defining family of essential primes. Finally, R is a *generalized Krull domain* if R possesses a finite character defining family of height-one essential primes. For convenience we begin with a lemma, much of which comes from [2] (and no doubt all of which is well known).

Lemma 5.1. *Let R be a domain and $\mathcal{P}$ a defining family for R. Define $\star$ by $A^\star = \bigcap_{P \in \mathcal{P}} AR_P$ for each nonzero fractional ideal A of R. Then:*

1. *$\star$ is a star operation on R.*
2. *If I is an integral ideal of R for which $I^\star \neq R$, then $I \subseteq P$ for some $P \in \mathcal{P}$.*
3. *$P^\star = P$ for each $P \in \mathcal{P}$.*
4. *If $\mathcal{P}$ has finite character, then $\star$ has finite type.*
5. *If $\star$ has finite type, then:*

(a) *For each $P \in \mathcal{P}$, there is a maximal element Q of $\mathcal{P}$ such that $P \subseteq Q$. Hence if $\mathcal{P}'$ denotes the set of maximal elements in $\mathcal{P}$, then $A^\star = \bigcap_{P \in \mathcal{P}'} AR_P$ for each nonzero fractional ideal A of R.*
(b) *Each proper t-ideal of R is contained in some $P \in \mathcal{P}$.*
(c) $\star$-$\mathrm{Max}(R) = \mathcal{P}'$.
(d) *If $\mathrm{ht}(P) = 1$ for each $P \in \mathcal{P}$ and $\mathcal{Q}$ denotes the set of height-one primes of R, then $\mathcal{P} = t$-$\mathrm{Max}(R) = \star$-$\mathrm{Max}(R) = \mathcal{Q}$.*

Proof. Statements (1, 2, 3, 4) are in [2]. For (5a), Zorn's lemma applies since the union P of a chain of elements of $\mathcal{P}$ satisfies $P^\star = P$ and by (2) $P \subseteq Q$ for some $Q \in \mathcal{P}$. The "hence" statement follows easily. Statement (5b) follows from (2) in view of the fact that a proper t-ideal is also a proper $\star$-ideal. For (5c), if $Q \in \star$-$\mathrm{Max}(R)$, then $Q \subseteq P$ for some $P \in \mathcal{P}'$ by (2). But then $Q = P'$ by (3). Hence $\star$-$\mathrm{Max}(R) \subseteq P'$. The reverse inclusion is trivial. Finally, (5d) follows easily from (5a,b,c) and the fact that height-one primes are t-primes. $\square$

Remark 5.2. With the notation of Lemma 5.1, let R be an almost Dedekind domain with exactly one non-invertible maximal ideal M, and let $\mathcal{P}$ denote the set of maximal ideals other than M. Then, as is well known, $\mathcal{P}$ is a defining family for R, but the associated star operation does not have finite type. Indeed, conclusions (5b,d) fail to hold in this case: M is a t-ideal but $M \nsubseteq P$ for all $P \in \mathcal{P}$.

For our next result, recall that if $\star$ is a finite-type star operation on a domain R, then R is said to have *$\star$-dimension one* if each maximal $\star$-ideal of R has height one.

Theorem 5.3. *Let R be a domain and $\star$ a finite-type star operation on R. Assume that R has $\star$-dimension one and that R is $\star$-potent. Then R has finite $\star$-character.*

Proof. Denote the set of maximal $\star$-ideals of R by $\{M_\gamma\}_{\gamma \in \Gamma}$. For each γ, choose a $\star$-rigid ideal I_γ contained in M_γ. Now suppose, by way of contradiction, that a is a nonzero element of R and Λ is an infinite subset of Γ with $a \in M_\lambda$ for $\lambda \in \Lambda$ and $a \notin M_\gamma$ for $\gamma \in \Gamma \setminus \Lambda$. For $\lambda \in \Lambda$, R_{M_λ} is one-dimensional, and hence there is an element $s_\lambda \in R \setminus M_\lambda$ and a positive integer n_λ for which $s_\lambda I_\lambda^{n_\lambda} \subseteq (a)$. By construction, $(a, \{s_\lambda\})^\star = R$, whence $(a, s_1, \ldots, s_k)^\star = R$ for some finite subset $\{1, \ldots, k\}$ of Λ. On the other hand, $(a, s_1, \ldots, s_k)I_1^{n_1} \cdots I_k^{n_k} \subseteq (a)$, and since Λ is infinite, there is a maximal $\star$-ideal $M \in \{M_\lambda\}_{\lambda \in \Lambda} \setminus \{M_1, \ldots, M_k\}$ with $a \in M$. However, $(a, s_1, \ldots, s_k) \nsubseteq M$ (since $(a, s_1, \ldots, s_k)^\star = R$ and) $I_j \nsubseteq M$ for $j = 1, \ldots, k$, the desired contradiction. $\square$

The assumption on the $\star$-dimension in Theorem 5.3 is necessary; for example, the Prüfer domain $\mathbb{Z} + X\mathbb{Q}[[X]]$ is d-(super) potent but does not have finite d-character (note that $d = t$ here).

Theorem 5.3 is, at first glance, a generalization of part of [4, Corolllary 1.7], which states that a t-potent domain of t-dimension one has finite t-character. In fact, Theorem 5.3 actually follows from [4, Corollary 1.7]. Indeed, if R is as in Theorem 5.3, then Lemma 5.1 shows that t-$\mathrm{Max}(R) = \star$-$\mathrm{Max}(R)$. (However, it is not generally the case that $\star = t$.) We have included the proof given above, since it seems much more conceptual than the one given in [4].

As mentioned in the paragraph following Proposition 1.4, it is possible to have finite-type star operations $\star_1 \leq \star_2$ on a domain R with R $\star_1$-potent but not $\star_2$-potent. Indeed, this phenomenon can occur in a 2-dimensional PvMD. We are grateful to the referee for suggesting the following construction.

Example 5.4. Let T be the absolute integral closure of $\mathbb{Z}[X]$. Since $\mathbb{Z}[X]$ is a Krull domain, T is a PvMD, as was shown by H. Prüfer [24] (see also the more recent paper by F. Lucius [22]). Moreover, it follows (Krull [21, Satz 9]) that, since $\mathbb{Z}[X]$ has t-dimension one, so does T. Now let R be the localization of T at a maximal ideal lying over $(2, X)$ in $\mathbb{Z}[X]$. Then R is a (local and hence) d-potent domain of t-dimension one. However, R does not have finite t-character (since the ring of algebraic integers does not have finite character [14, Proposition 42.8]), and hence R is not t-potent by Theorem 5.3 (or [4, Corollary 1.7]).

In [13], Gilmer introduced the notion of sharpness. The definition amounts to the following. Call a maximal ideal M of a domain R *sharp* if $\bigcap\{R_N \mid N \in \mathrm{Max(R), N \neq M}\} \nsubseteq \mathrm{R_M}$, and call R *sharp* if each maximal ideal of R is sharp. In [13] Gilmer focussed on one-dimensional domains and proved that a sharp almost Dedekind is a Dedekind domain. (In a later paper, Gilmer and Heinzer [10] extended the ideas to higher dimensions, primarily in the setting of Prüfer domains.) The notion of sharpness was extended to star operations $\star$ of finite type in [12, Remark 1.4]: a maximal $\star$-ideal M of a domain R is $\star$-sharp if $\bigcap R_N \nsubseteq R_M$, where the intersection is taken over all maximal $\star$-ideals $N \neq M$. Hence, for our purposes it is convenient to relabel "sharp" as "d-sharp." It is relatively easy to prove that a t-potent maximal t-ideal must be t-sharp (see below), but this cannot be extended to arbitrary finite-type star operations. In particular, it is not true for the d-operation, as can be seen by observing that maximal ideals of $k[x, y]$ are d-potent (as are the maximal ideals of any Noetherian domain) but are not d-sharp: if M is maximal in $R := k[x, y]$ and $u \in \bigcap\{R_N \mid N \in \mathrm{Max(R), N \neq M}\}$, then we must have $u \in R$, lest $(R :_R Ru)$ be contained in a height one prime and hence in infinitely many maximal ideals.

Proposition 5.5. *Let R be a domain.*

1. *If M is a t-potent maximal t-ideal of R, then M is t-sharp.*
2. *If $\star$ is a finite-type star operation on R and M is a $\star$-super potent maximal $\star$-ideal of R, then M is $\star$-sharp.*

Proof. (1) This follows easily from [12, proof of Theorem 1.2]. Here is a direct proof: Choose a t-rigid ideal of R contained in M. Since $I^v = I^t \subseteq M$, $I^{-1} \neq R$. Choose $u \in I^{-1} \setminus R$. Then $I \subseteq (R :_R Ru)$, and hence $(R :_R Ru) \nsubseteq N$ for each $N \in t\text{-Max}(R)$ with $N \neq M$. On the other hand, since $u \notin R$, we must have $(R :_R Ru) \subseteq M$. Hence $u \in \bigcap\{R_N \mid N \in t\text{-Max}(R), N \neq M\} \setminus R_M$, as desired.

(2) Let $\star$ be a finite-type star operation on R, M be a $\star$-super potent maximal $\star$-ideal, and I a $\star$-super rigid ideal contained in M. Since M is also a (maximal) t-ideal (Theorem 1.5), we have $I^{-1} \neq R$. Then, as in the proof of (1), if we choose $u \in I^{-1} \setminus R$, then $u \in \bigcap\{R_N \mid N \in \star\text{-Max}(R), N \neq M\} \setminus R_M$. □

We observe that a t-sharp maximal t-ideal need not be t-potent ([12, Example 1.5]). To force t-sharpness to imply t-potency, we add a finiteness condition. Recall that a fractional t-ideal I of a domain R has *finite type* if $I = J^v$ for some finitely generated fractional ideal J. We then say that R is *v-coherent* if I^{-1} has finite type for each finitely generated fractional ideal I of R. (The notion of v-coherence, with a different name, was introduced by El Abidine [8].) We then have from [12, Theorem 1.6] that a v-coherent t-sharp domain is t-potent. The next result is immediate.

Corollary 5.6. *A v-coherent t-sharp domain of t-dimension one has finite t-character.* $\square$

The above-mentioned theorem of Gilmer follows easily:

Corollary 5.7. [13, Theorem 3] *Let R be a d-sharp almost Dedekind domain. Then R is a Dedekind domain.*

Proof. Any Prüfer domain is v-coherent. Moreover, the d- and t-operations coincide in a Prüfer domain. Hence R has finite character by Corollary 5.6, and it is well known that this implies that R is a Dedekind domain. $\square$

We now turn to the characterization of generalized Krull domains. Since these domains are completely integrally closed, the next result will prove useful. (Recall that a domain R with quotient field K is *completely integrally closed* if, whenever $a \in R$ and $u \in K$ are such that $au^n \in R$ for each positive integer n, then $u \in R$.)

Lemma 5.8. *Let R be a completely integrally closed domain and M a t-super potent maximal t-ideal of R. Then* $\mathrm{ht}(M) = 1$.

Proof. We proceed contrapositively. Suppose that M is a t-super potent maximal t-ideal of R and that P is a nonzero prime properly contained in M. Choose a t-super rigid ideal $I \subseteq M$ with $I \nsubseteq P$. By Theorem 1.11, $P \subseteq \bigcap (I^n)^\star$, and hence $\bigcap (I^n)^\star \neq (0)$. Therefore, R is not completely integrally closed by [5, Corollary 3.4]. $\square$

Recall that a domain R is a *Prüfer v-multiplication domain* (PvMD) if each nonzero finitely generated ideal of R is t-invertible; it is well known that R is a PvMD if and only if each maximal t-ideal of R is essential (note that the set of maximal t-ideals is always a defining family).

Theorem 5.9. *The following statements are equivalent for a domain R.*

(1) *R is a generalized Krull domain.*
(2) *R is a t-potent essential domain of t-dimension one.*
(3) *R is a t-potent PvMD of t-dimension one.*
(4) *R is a completely integrally closed t-super potent domain.*
(5) *R is a t-super potent domain of t-dimension one.*

Proof. (1) $\Rightarrow$ (2): Assume (1), and let $\mathcal{P}$ be a finite character defining family of height-one essential primes. By Lemma 5.1, $\mathcal{P}$ is in fact the set of maximal t-ideals of R. (This also follows from [13, Corollary 43.9]). Hence R has t-dimension one. Also, R is t-potent since $\mathcal{P}$ has finite character.

(2) $\Rightarrow$ (3): Assume (2), and let $\mathcal{P}$ be a defining family of essential primes. For $P \in \mathcal{P}$, PR_P is a t-prime in the valuation domain R_P, and it is well known (see, e.g., [20, Lemma 3.17]) that this implies that P is a t-prime of R. Then, since R has finite t-character by Theorem 5.3, $\mathcal{P}$ also has finite character and is therefore the entire set of t-primes (Lemma 5.1). Therefore, R_P is a valuation domain for each t-prime P, that is, R is a PvMD.

(3) $\Rightarrow$ (4): Let R be a t-potent PvMD of t-dimension one. Then R_P is a rank-one valuation domain for each t-prime P, and hence $R = \bigcap R_P$ is completely integrally closed. Also, since R is t-potent and t-locally d-super potent, R is t-super potent by Theorem 1.10.

(4) $\Rightarrow$ (5): This follows from Lemma 5.8.

(5) $\Rightarrow$ (1): Assume (5). Then R has finite t-character by Theorem 5.3, and R_M is a valuation domain for each maximal t-ideal M by Corollary 1.12. Hence R is a generalized Krull domain. □

One upshot of Theorem 5.9 is that a t-super potent domain of t-dimension one must be completely integrally closed. Note that without the restriction on the t-dimension, a t-super potent domain need not even be integrally closed (Example 4.6).

We close with a brief discussion regarding the connection between PvMDs and t-super potent domains. Observe that a t-potent PvMD is automatically t-super potent, but a PvMD need not be t-potent. (For example, a non-Dedekind almost Dedekind domain is a PvMD but is not t-potent (note that $d = t$ in this situation).) In a PvMD, all nonzero finitely generated ideals are t-invertible, while in a t-super potent domain one has t-invertibility only "above" t-super rigid ideals. Since, as is well known, if I is a t-invertible ideal in a domain R, then both I and I^{-1} have finite type, a natural question arises: if R is both t-super potent and v-coherent, must R be a PvMD? Even if we add the condition that R be integrally closed, the answer is "no," as is shown by the ring $R := \mathbb{Z} + x\mathbb{Q}(z)[[x]]$ of Example 4.6 (with z an indeterminate): R is t-super potent by Theorem 4.4, is v-coherent by [10, Theorem 3.5], is integrally closed by standard pullback results, but is not a PvMD [9, Theorem 4.1].

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