INTEGRAL DOMAINS IN WHICH ANY TWO \( v \)-COPRIME ELEMENTS ARE COMAXIMAL

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Abstract. Domains in which the star operations \( d \) (the trivial star operation) and \( w \) coincide have received a good deal of attention recently. These are exactly the domains \( D \) in which \( I = D \) whenever \( I \) is a finitely generated ideal of \( D \) with \( I^v = D \). In this work, we study what happens when “finitely generated” is replaced by “two-generated.” It turns out that these are precisely the domains in which \( d = F \), where \( F \) is a certain star operation closely connected to, but more complicated than, the \( w \)-operation.

Introduction

Throughout this work, \( D \) denotes a domain, and \( K \) denotes its quotient field. We recall the \( v \)-operation: For a nonzero fractional ideal \( I \) of \( D \), we set \( I^{-1} = (D : I) = \{ u \in K \mid uI \subseteq D \} \) and then \( I^v = (I^{-1})^{-1} \). (The map \( I \mapsto I^v \) is an example of a star operation; we review pertinent definitions below as needed.) We say that nonzero elements \( a, b \in D \) are \( v \)-coprime if \( (a, b)^v = D \) and comaximal if \( (a, b) = D \). It is easy to see that \( a \) and \( b \) are \( v \)-coprime if and only if \( (a, b) \cap (a, b)^{-1} = D \) if and only if \( (a, b) = (ab) \). The primary purpose of this work is to study \( DF \)-domains, domains \( D \) in which \( a, b \in D \) are comaximal whenever \( a, b \) are \( v \)-coprime. The terminology arises as follows. In [3] H. Adams studied \( F \)-prime (shortened from factorization-prime) ideals. These are primes that contain no pair of \( v \)-coprime elements. She called an ideal \( I \) of \( D \) an \( F \)-ideal if whenever \( a, b, x \in D \) with \( (a, b)^v = D \) and \( x(a, b) \subseteq I \) we have \( x \in I \). As is pointed out in [16], an \( F \)-ideal is precisely an ideal \( I \) satisfying \( I^F = I \) for a certain star operation \( F \) on \( D \), and we shall show that \( DF \)-domains are precisely those domains for which the \( d \)-operation (the identity star operation) is identical to the \( F \)-operation.

Examples of \( DF \)-domains include Prüfer domains and one-dimensional domains. If fact, these are examples of \( DW \)-domains, that is, domains in which the two star operations \( d \) and \( w \) (reviewed below) coincide. DW-domains were introduced (but not named) in [7] and further studied in [8] (where they were called \( t \)-linkative domains), [26], [28], and [29]. It is easy to see that \( D \) is a DW-domain if and only if \( I \) is principal for each finitely generated ideal \( I \) of \( D \) such that \( I^v \) is principal (see [28, Proposition 2.1]). Hence DW-domains are DF-domains, but we shall show (Proposition 5.2) that DF-domains form a properly larger class.

Recall that GCD-domains may be characterized as those domains \( D \) in which \( (a, b)^v \) is principal for all nonzero \( a, b \in D \). Now, it is well known that if \( (a, b)^v = (d) \) for a given pair of elements \( a, b \) in a domain \( D \), then gcd\((a, b) \) exists and is equal to \( d \), but the converse is false. Thus domains \( D \) in which \( (a, b) \) is principal whenever...
1. The $F$- and $t_2$-operations

We begin by recalling some basic facts about star operations. Denote by $F(D)$ (resp., $f(D)$) the set of nonzero fractional (resp., nonzero finitely generated fractional) ideals of $D$. A star operation on $D$ is then a mapping $I \mapsto I^*$ of $F(D)$ into $F(D)$ such that for all nonzero $a \in K$ and $I, J \in F(D)$,

(1) $(aD)^* = aD$ and $aI^* = (aI)^*$;
(2) $I \subseteq I^*$, and $I \subseteq J$ implies $I^* \subseteq J^*$; and
(3) $(I^*)^* = I^*$.

For any star operation $*$ on $D$, two new star operations $*_f$ and $*_w$ can be constructed by setting, for $I \in F(D)$, $I^*_f = \bigcup\{J^* \mid J \subseteq I \text{ and } J \in f(D)\}$ and $I^*_w = \{x \in K \mid xJ \subseteq I \text{ for some } J \in f(D) \text{ with } J^* = D\}$. A star operation $*$ on $D$ is said to be of finite type if $* = *_f$; hence $*_f$ and $*_w$ are of finite type. An ideal $I \in F(D)$ is said to be a $*$-ideal if $I^* = I$, and a $*$-ideal is called a maximal $*$-ideal if it is maximal among proper integral $*$-ideals. We denote by $\text{Max}^*(D)$ the set of maximal $*$-ideals of $D$. Assuming $D$ is not a field, it is known that each maximal $*$-ideal is prime, that $*_f$-maximal ideals are plentiful in the sense that each nonzero $*_f$-ideal (and hence each nonzero element) of $D$ is contained in a maximal $*_f$-ideal, that a prime ideal minimal over a $*_f$-ideal is itself a $*_f$-ideal, and that $*_f-\text{Max}(D) = *_w-\text{Max}(D)$ [5, Theorem 2.16]. Also, if $I \in F(D)$, then

$a, b$ are elements of $D$ such that $\gcd(a, b)$ exists might be expected to form a strictly smaller class that the class of DF-domains. This is indeed the case. In fact the property just mentioned is easily seen to be equivalent to $(a, b) = D$ whenever $a, b$ are elements of $D$ for which $\gcd(a, b) = 1$, and domains with this property were called pre-Bézout domains by Cohn [6]. Interestingly, the “finitely generated version” of this property has recently been studied by Park and Tartarone: they call a domain $D$ GCD-Bézout if $(a_1, \ldots, a_n) = (d)$ whenever $a_1, \ldots, a_n \in D$ and $\gcd(a_1, \ldots, a_n) = d$.

In Section 1 we review terminology of star operations and study two particular star operations, the $F$- and $t_2$-operations, both defined in [16]. In Section 2 we give several characterizations of DF-domains, study their properties, compare and contrast the class of DF-domains with the other classes mentioned above, and explore what happens when we combine the DF-property with other well-studied properties (such as GCD, Krull). Section 3 is devoted to studying localization. We prove that a domain $D$ for which $D_M$ is a DF-domain for each maximal ideal $M$ of $D$ is a DF-domain, but we also give an example of a DF- (in fact, a DW-) domain $D$ with a maximal ideal $M$ such that $D_M$ is not DF, thus answering a question left open in [28]. We also consider other properties locally, proving, for example, that a domain $D$ is a Prüfer domain if and only if it is a DF-domain that is locally a GCD-domain and is such that $F$-primes localize (to $F$-primes). We devote a brief Section 4 to connections with regular sequences. Our main result here is a generalization of the fact that in a Noetherian domain $D$, an ideal $I$ has (classical) grade at least 2 if and only if $I^{-1} = D$ [25, Exercise 2, page 102]. In Section 5 we analyze an example of Uda [30] to show that the DF-property is weaker than the DW-property. We also study the behavior of the DF-property in pullbacks, yielding many more examples of DF-domains (that are not DW-domains). Finally, in Section 6, we consider polynomial and Nagata rings. We show, for example, that $D[X]$ is a DF-domain if and only if $D$ is a field.
\[ I^* = \bigcap_{P \in \text{Max}(D)} ID_P \] (5, Corollary 2.10), and hence \( I^* \cap \text{ID}_P = \text{ID}_P \) for each \( P \in \text{Max}(D) \). The best-known star operations are the \( d \)-, \( \vdash \) (defined above), \( t \)-, and \( w \)-operations. The \( d \)-operation is just the identity function on \( F(D) \), so that \( d = d_f = d_w \). The \( t \)-operation (resp., \( w \)-operation) is given by \( t = v_f \) (resp., \( w = v_w \)). For two star operations \( \ast_1 \) and \( \ast_2 \) on \( D \), we write \( \ast_1 \leq \ast_2 \) when \( I^{\ast_1} \subseteq I^{\ast_2} \) for all \( I \in F(D) \) (and \( \ast_1 \leq \ast_2 \) when \( \ast_1 \leq \ast_2 \) but \( \ast_1 \neq \ast_2 \)). It is known that \( d \leq \ast_w \leq \ast_f \leq \ast \leq v \), \( \ast_w \leq w \), and \( \ast_f \leq t \) for any star operation \( \ast \) on \( D \).

We next recall the definitions of the \( t_2 \)- and \( F \)-operations.

**Definition 1.1.** Let \( J \subseteq K \) and \( I \in F(D) \).

1. For the \( t_2 \)-operation: Set \( J' = \bigcup \{(a,b)^v : a, b \in J\} \). Then set \( I_0 = I \), \( I_n = (I_{n-1})' \) for \( n > 0 \), and \( I^{t_2} = \bigcup_{k=0}^{\infty} I_k \). The \( t_2 \)-operation was shown in [16] to be a finite-type star operation.

2. For the \( F \)-operation: Set \( J' = \{ x \in K : x(a,b) \subseteq J \text{ for some } a, b \in J \text{ with } (a,b)^v = D \} \). Then set \( I_0 = I \), \( I_n = (I_{n-1})' \) for \( n > 0 \), and \( I^F = \bigcup_{k=0}^{\infty} I_k \). It was observed in [16] that this defines a finite-type star operation on \( D \) (but most of the necessary details were already present in [3]).

Observe that the \( t_2 \)- and \( F \)-operations are similar to the \( t \)- and \( w \)-operations, the differences being that finite subsets are replaced by two-element subsets and iteration is required. Clearly, we have \( F \leq t_2 \), \( F \leq w \), and \( t_2 \leq t \). In [16], an example was given showing that it is possible to have \( F < t_2 \); in fact, in that example, it is easy to see that we have \( d = F = w < t_2 \). In Example 5.1 below, we show that it is possible to have \( F < w \) and \( t_2 < t \), answering questions posed in [16].

Although the \( t_2 \)- and \( F \)-operations are defined inductively, only one step is needed to determine whether a given ideal is a \( t_2 \)- or \( F \)-ideal.

**Lemma 1.2.** Let \( I \) be a nonzero ideal of a domain \( D \). Then the following statements hold.

1. \( I \) is a \( t_2 \)-ideal if \( (a,b)^v \subseteq I \) whenever \( a, b \in I \).

2. \( I \) is an \( F \)-ideal if \( x \in I \) whenever \( x(a,b) \subseteq I \) with \( x, a, b \in D \) and \( (a,b)^v = D \).

3. \( I \) is a prime \( F \)-ideal (\( F \)-prime) if \( I \) does not contain any pair of \( v \)-coprime elements.

**Proof.** Statements (1) and (2) follow easily from the definitions. For (3), suppose that \( I \) is as hypothesized and that \( x(a,b) \subseteq I \) with \( (a,b)^v = D \). Then, \( (a,b) \notin I \), so that we must have \( x \in I \). Hence \( I \) is an \( F \)-ideal by (2). \( \square \)

As has already been mentioned, for any star operation \( \ast \) on \( D \), we may define \( \ast_w \) by \( I^{\ast_w} = \bigcup \{(I : J) : J \text{ is a finitely generated subideal of } I \text{ and } J^* = D \} \), and we have \( v_w = t_w = w \).

**Proposition 1.3.** For any domain \( D \), the \( F \)- and \( F_w \)-operations on \( D \) are identical.

**Proof.** Since \( F_w \leq F \) by definition, it suffices to show that each \( F_w \)-ideal is also an \( F \)-ideal. Accordingly, let \( I \) be an \( F_w \)-ideal of \( D \), and suppose that \( x, a, b \in D \) are such that \( (a,b)^v = D \) and \( x(a,b) \subseteq I \). Since \( 1(a,b) \subseteq (a,b) \) and \( (a,b)^v = D \), we have \( (a,b)^F = D \) and hence \( x \in I^{F_w} = I \). The result now follows from Lemma 1.2. \( \square \)
For any ∗-operation on \( D \), it is known that if \( P \) is a \( w \)-prime of \( D \), then every prime ideal contained in \( P \) is also a \( w \)-prime. Hence we have the following:

**Corollary 1.4.** If \( P \) is an \( F \)-prime of \( D \), then so is every nonzero prime of \( D \) contained in \( P \) .

**Questions 1.5.** Let \( D \) be a domain.

1. Must we have \( F\text{-Max}(D) \subseteq t_2\text{-Max}(D) \)?
2. Must we have \( F\text{-Max}(D) = t_2\text{-Max}(D) \)?
3. If \( I \) is an ideal of \( D \) with \( I^{t_2} = D \), do we necessarily have \( I^F = D \)?
4. Do we have \( t_2\text{-Max}(D) \subseteq F\text{-Max}(D) \)?
5. What conditions on \( D \) ensure \( t_2 = t \)?
6. In general, we have \( F = F_w \leq (t_2)_w \leq w \). When do we have \( F = (t_2)_w \) or \( (t_2)_w = w \) ?

It is not difficult to show that Questions (1)-(3) are equivalent:

**Lemma 1.6.** Suppose that \( *_1 \leq *_2 \) are finite-type star operations on \( D \). Then the following statements are equivalent.

1. \( *_1\text{-Max}(D) \subseteq *_2\text{-Max}(D) \).
2. \( *_1\text{-Max}(D) = *_2\text{-Max}(D) \).
3. If \( I \) is an ideal of \( D \) with \( I^{*_2} = D \), then \( I^{*_1} = D \).

**Proof.** Assume (1), and let \( M \in *_2\text{-Max}(D) \). Since \( *_1 \leq *_2 \), we have \( M^{*_1} \neq D \). Hence \( M \) is contained in a maximal \( *_1 \)-ideal \( N \) of \( D \). However, by assumption, this yields \( N \in *_2\text{-Max}(D) \), and we must therefore have \( M = N \), that is, \( M \in *_1\text{-Max}(D) \). Thus (1) \( \Rightarrow \) (2). Assume (2), and let \( I \) satisfy \( I^{*_1} \neq D \). Then \( I \subseteq M \) for some \( M \in *_1\text{-Max}(D) = *_2\text{-Max}(D) \), and we have \( I^{*_2} \subseteq M \not\subseteq D \). Hence (2) \( \Rightarrow \) (3). Finally, assume (3), and let \( M \in *_1\text{-Max}(D) \). Then \( M^{*_1} \neq D \), whence, by assumption, \( M^{*_2} \neq D \). Since \( M^{*_2} \) is a \( *_1 \)-ideal and \( M \subseteq M^{*_2} \), this yields \( M = M^{*_2} \). Thus \( M \) is a \( *_2 \)-ideal. Since every \( *_2 \)-ideal is also a \( *_1 \)-ideal, \( M \) cannot be contained in a larger \( *_2 \)-ideal, i.e., \( M \in *_2\text{-Max}(D) \).

Recall that if \( * \) is a star operation on \( D \), then we say that \( D \) has finite \( * \)-character if each nonzero element of \( D \) is contained in only finitely many maximal \( * \)-ideals of \( D \). (When \( * = d \), one says that \( D \) has finite character.)

**Proposition 1.7.** If \( D \) has finite \( t_2 \)-character, then \( t_2\text{-Max}(D) = F\text{-Max}(D) \).

**Proof.** Suppose that \( D \) has finite \( t_2 \)-character, and let \( M \in F\text{-Max}(D) \). If \( M \) is not a \( t_2 \)-ideal, then, since every \( t_2 \)-ideal is a \( F \)-ideal, we have \( M^{t_2} = D \). Choose a nonzero element \( a \in M \). Then \( a \) is in only finitely many maximal \( t_2 \)-ideals, and, since \( M^{t_2} = D \), we may use prime avoidance to find \( b \in M \) with \( (a, b) \) in no maximal \( t_2 \)-ideal, that is, \( (a, b)^{t_2} = D \). However, this yields \( (a, b)^v = D \), contradicting that \( M \) is a maximal \( F \)-ideal. Thus \( M \) must be a \( t_2 \)-ideal and hence a maximal \( t_2 \)-ideal. The result now follows from Lemma 1.6.

**Proposition 1.8.** If \( D \) has finite \( t \)-character, then \( t\text{-Max}(D) = t_2\text{-Max}(D) = F\text{-Max}(D) = w\text{-Max}(D) \). In particular, finite \( t \)-character implies both finite \( t_2 \)- and finite \( F \)-character.

**Proof.** Assume that \( D \) has finite \( t \)-character, and let \( M \) be a maximal \( t_2 \)-ideal of \( D \). If \( M \) is not a \( t \)-ideal, then \( M^t = D \), and, as in the proof of Proposition 1.7, we
can find $a, b \in M$ with $(a, b)$ in no maximal $t$-ideal of $D$. But then $(a, b)^v = D$, a contradiction. Hence $t$-$\text{Max}(D) = t_2$-$\text{Max}(D)$, and $D$ also has finite $t_2$-character. A similar conclusion for maximal $F$-ideals now follows from Proposition 1.7. Finally, it is well know that $t$-$\text{Max}(D) = \omega$-$\text{Max}(D)$ in general ([5, Theorem 2.16]). □

It follows from Proposition 1.7 that finite $t_2$-character implies finite $F$-character. However, it does not imply finite $t$-character — see Proposition 5.2 below.

In [22] the authors introduced the class of TV-domains, domains in which the $t$-operation coincides with the $v$-operation. By [22, Theorem 1.3], TV-domains have finite $t$-character, so that Proposition 1.8 applies to this class of domains. Now recall that a domain is a Mori domain if it satisfies the ascending chain condition on divisorial ideals. It was observed in [22] that the class of TV-domains includes (but is properly larger than) the class of Mori domains. In particular, Proposition 1.8 applies to Noetherian domains. Actually, for Mori domains, we can say a good deal more:

**Proposition 1.9.** Let $D$ be a Mori domain. Then every $t_2$-prime of $D$ is a $t$-prime.

**Proof.** Let $P$ be a $t_2$-prime of $D$, and let $a$ be a nonzero element of $P$. By [19, Theorem 2.1], $a$ is contained in only finitely many $t$-primes of $D$. Use prime avoidance to choose $b \in P$ with $b$ in no $t$-prime $Q$ of $D$ for which $a \in Q$ and $Q \subseteq P$. Since $P$ is a $t_2$-prime, $(a, b)^v \subseteq P$. Shrink $P$ to a prime $P_0$ minimal over $(a, b)^v$. Then $P_0$ is a $t$-prime, and by construction we must have $P = P_0$. □

We suspect that Questions (1) - (4) above have negative answers in general. With respect to Question 5, we do not even know whether $t_2 < t$ in a one-dimensional local Noetherian domain. (We do know from Proposition 5.2 below that $t_2 < t$ can occur (albeit in a domain that is far from being Noetherian).)

2. DF-domains

We begin this section with several characterizations of DF-domains. We recall the definition: The domain $D$ is a DF-domain if for $a, b \in D$ with $(a, b)^v = D$, we have $(a, b) = D$. Now recall from [7] that an overring $E$ of a domain $D$ is $t$-linked over $D$ if $(E : I_E) = E$ whenever $I$ is a finitely generated ideal of $D$ with $I^{-1} = D$, equivalently, if $(J E)^{v_E} = E$ whenever $J$ is an ideal of $D$ with $J^d = D$. It was shown that every overring of $D$ is $t$-linked over $D$ if and only if every maximal ideal of $D$ is a $t$-ideal, i.e., if and only if $D$ is a DW-domain. In [9] the notion of $t$-linkedness was extended as follows. Given $D$ and an overring $E$ and star operations $*$ on $D$ and $*1$ on $E$, $E$ is $(*, *1)$-linked over $D$ if $(J E)^{*1} = E$ whenever $J$ is an ideal of $D$ with $J^* = D$.

**Theorem 2.1.** The following statements are equivalent for a domain $D$.

1. $D$ is a DF-domain.
2. $a, b \in D$ with $(a, b)^v = (d)$ implies $(a, b) = (d)$.
3. $a, b \in D$ with $(a, b)^v$ principal implies $(a, b)$ principal.
4. Each nonzero ideal of $D$ is an $F$-ideal; equivalently, the $d$- and $F$-operations on $D$ are identical.
5. Each maximal ideal of $D$ is an $F$-prime.
6. For every overring $E$ of $D$, $E$ is $(F, F_E)$-linked over $D$. 
Proof. (1) ⇒ (2): Let $D$ be a DF-domain, and let $a, b \in D$ with $(a, b)^v = dD$ for some $d \in D$. Then $(a/d, b/d)^v = (1/d)(a, b)^v = D$. Since $D$ is a DF-domain, this yields $(a/d, b/d) = D$ and, therefore, $(a, b) = dD$.

(2) ⇒ (3): Trivial.

(3) ⇒ (4): Assume (3). Let $I$ be a nonzero ideal of $D$, and suppose that $x(a, b) \subseteq I$ with $(a, b)^v = D$. By (3) $(a, b) = (c)$ for some $c \in D$. Hence $D = (a, b)^v = (c) = (a, b)$, and we have $x \in I$. Therefore, $IF = I$.

(4) ⇒ (5): Trivial.

(5) ⇒ (6): Assume (5), let $E$ be an overring of $D$, and let $I$ be an ideal of $D$ with $IF = D$. If $(IE)^{Fr} \neq E$, then $IE$ is contained in a maximal $F$-ideal $Q$ of $E$. Let $M$ be a maximal ideal of $D$ containing $Q \cap D$. Then $M$ is an $F$-prime. However, $IF = D$ and $I \subseteq M$, a contradiction.

(6) ⇒ (1): Assume (6), and let $a, b \in D$ with $(a, b)^v = D$. Then $(a, b)^F = D$ also. Suppose, by way of contradiction, that $(a, b)$ is a proper ideal of $D$, and let $M$ be a maximal ideal containing $(a, b)$. Then there is a valuation overring $V$ of $D$ whose maximal ideal $N$ satisfies $N \cap D = M$. By assumption, we have $((a, b)V)^{Fr} = V$. However, every ideal of $V$ is a $t$-ideal and hence also an $F_V$-ideal, and this yields $((a, b)V)^{Fr} = (a, b)V \subseteq N$, a contradiction. □

Since $F \subseteq w$ for all domains, the following is immediate.

**Corollary 2.2.** A DW-domain is a DF-domain. □

We consider another property stronger than DF. Recall that in [6] Cohn defined a pre-Bézout domain to be a domain $D$ satisfying the following property: $a, b \in D$ with $\gcd(a, b) = 1$ implies $(a, b) = D$. We list a few equivalent conditions:

**Lemma 2.3.** The following statements are equivalent for a domain $D$.

1. $(a, b) \in D$ with $\gcd(a, b) = d$ implies $(a, b)$ principal.
2. $(a, b) \in D$ with $\gcd(a, b) = d$ implies $(a, b) = (d)$.
3. $D$ is a pre-Bézout domain.
4. Each proper 2-generated ideal of $D$ is contained in a proper principal ideal.

**Proof.** Assume (1), and let $a, b \in D$ with $\gcd(a, b) = d$. Then $(a, b)$ is principal, say $(a, b) = (c)$. Since $c | a$ and $c | b$, we have $(d) \subseteq (c)$. On the other hand, $(c) = (a, b) \subseteq (d)$. Hence (1) ⇒ (2). That (2) ⇒ (3) is trivial. Assume (3), and let $a, b \in D$ be such that $(a, b)$ is contained in no proper principal ideal. Then $\gcd(a, b) = 1$, and we have $(a, b) = D$ (i.e., $(a, b)$ is not a proper ideal) by (3). Thus (3) ⇒ (4). Finally, assume (4), and let $a, b \in D$ with $\gcd(a, b) = d$. A standard argument yields $\gcd(a/d, b/d) = 1$, so that $(a/d, b/d)$ is not contained in a proper principal ideal. Thus $(a/d, b/d) = D$ by (4) and hence $(a, b) = (d)$. □

Now suppose that $D$ is pre-Bézout, and let $a, b \in D$ with $(a, b)^v = D$. Then, as we have already observed, $\gcd(a, b)$ exists and is equal to 1 and hence $(a, b) = D$. This yields:

**Corollary 2.4.** A pre-Bézout domain is a DF-domain. □

The converse of Corollary 2.4 is false. Let $L \subseteq k$ be fields, $X$ a set of indeterminates over $k$ with $|X| \geq 2$, $M$ the maximal ideal of $k[X]$ generated by $X$, and $D = L + Mk[X]_M$. It is well known that $D$ is then a local domain whose maximal ideal is divisorial and hence a $t$-ideal. Since DW-domains are characterized as domains each of whose maximal ideals is a $t$-ideal ([26, Proposition 2.2] and [7,
Lemma 2.1), $D$ is a DW-domain and hence a DF-domain. However, for $x \neq y \in X$, we have \( \gcd(x,y) = 1 \), but \( (x,y) \subseteq D \).

In fact, we can characterize pre-Bézout domains among DF-domains. In \cite{28} the authors call a domain $D$ a GCD-Bézout domain if \( (a_1, a_2, \ldots, a_n) \) is principal whenever $a_1, \ldots, a_n$ are elements of $D$ with a greatest common divisor, and they show that a GCD-Bézout domain is a DW-domain. Indeed, in \cite[Corollary 2.12]{28}, they characterize GCD-Bézout domains as DW-domains that satisfy the PSP-property of Arnold-Sheldon \cite{2}. A domain $D$ satisfies PSP (for primitive implies superprimitive) if for each finitely generated ideal $I$ that is not contained in a proper principal ideal we have $I^v = D$. (This terminology arises as follows: an element $f \in D[X]$ is called primitive (resp. superprimitive) if $c(f)$, the ideal of $D$ generated by the coefficients of $f$, is not contained in a proper principal ideal of $D$ (resp., satisfies $c(f)^v = D$).) Following \cite{27}, let us say that a domain satisfies LPSP–for linear PSP–if each two-generated ideal $I$ not contained in a proper principal ideal satisfies $I^v = D$ (that is, if each primitive linear polynomial is superprimitive). Then we have the following:

**Proposition 2.5.** A domain $D$ is pre-Bézout if and only if it is a DF-domain with LPSP.

**Proof.** Suppose that $D$ is pre-Bézout. Then $D$ is a DF-domain by Corollary 2.4. Also, for $a, b \in D$ with $(a, b)$ not contained in a proper principal ideal, we have $\gcd(a, b) = 1$, whence $(a, b) = D$ and then $(a, b)^v = D$. Therefore, $D$ also satisfies LPSP. Now suppose that $D$ is DF and satisfies LPSP, and let $a, b \in D$ with $\gcd(a, b) = 1$. Then $(a, b)$ cannot be contained in a proper principal ideal, whence $(a, b)^v = D$ by LPSP. Since $D$ is a DF-domain, we then have $(a, b) = D$. Therefore, $D$ is pre-Bézout. \( \square \)

We don’t know whether a pre-Bézout domain must be GCD-Bézout (but we doubt it). However:

**Proposition 2.6.** A local pre-Bézout domain is GCD-Bézout.

**Proof.** Let $(D, M)$ be a local pre-Bézout domain. Then each proper 2-generated ideal of $D$ is contained in a proper principal ideal by Lemma 2.3. We show that (in the local case) this extends to all finitely generated ideals. Thus let $(a_1, \ldots, a_n)$, $n > 2$, be a proper finitely generated ideal. By induction, we may assume that $(a_1, \ldots, a_{n-1}) \subseteq (b)$ for some $b \in M$. We also have $(a_n, b) \subseteq (c)$ for some $c \in M$, and hence $(a_1, \ldots, a_n) \subseteq (c) \subseteq M$, as desired. By \cite[Proposition 2.6]{28}, $D$ is a GCD-Bézout domain. \( \square \)

We next consider what happens when the DF-property is combined with other commonly considered properties.

**Proposition 2.7.** A DF-domain of finite t-character is a DW-domain.

**Proof.** Let $D$ be a DF-domain of finite $t$-character, and let $M$ be a maximal ideal of $D$. Then $M$ is a maximal $F$-ideal and hence a maximal $t$-ideal by Proposition 1.8. Thus each maximal ideal of $D$ is a maximal $t$-ideal, whence $D$ is a DW-domain. \( \square \)

In particular, the DF- and DW-properties coincide for Noetherian domains. Noetherian DW-domains of arbitrary dimension (including $\infty$) exist—see \cite[Examples 2.1 and 2.7]{21}.
Let * be a star operation on a domain $D$. Then the *-dimension of $D$ is the length of a longest chain of *-primes in $D$ (where, for the purposes of this definition, $(0)$ is counted as a *-prime). The next two results strengthen [26, Corollary 2.3].

**Proposition 2.8.** Let $D$ have $F$-dimension one. Then the following statements are equivalent.

1. dim($D$) = 1.
2. $D$ is a DW-domain.
3. $D$ is a DF-domain.

**Proof.** Since height-one primes are $t$-primes, we obtain (1) $\Rightarrow$ (2) immediately, and (2) $\Rightarrow$ (3) is easy (Corollary 2.2). Assume (3). Then, since each maximal ideal of $D$ is an $F$-prime and primes within an $F$-prime are $F$-primes by Corollary 1.4, $D$ must have dimension one. □

Observe that if $D$ has finite $t$-character, then the $t$- and $F$-dimensions are the same by Proposition 1.8. It is well known that a Krull domain is a Dedekind domain if and only if it has dimension one. Then, since a Krull domain has finite $t$-character and has $t$-dimension one, we obtain:

**Corollary 2.9.** Let $D$ be a Krull domain. Then the following statements are equivalent.

1. $D$ is a DW-domain.
2. $D$ is a DF-domain.
3. $D$ is a Dedekind domain.

□

Since a Dedekind domain is a PID if and only if it is a UFD, we have:

**Corollary 2.10.** Let $D$ be a UFD. The following statements are equivalent.

1. $D$ is a DW-domain.
2. $D$ is a DF-domain.
3. $D$ is a PID.

□

We next give a direct proof of a result of Mott and the second author [27, Corollary 6.6]. Recall that a domain $D$ is atomic if each nonzero, nonunit of $D$ factors as a product of atoms (irreducible elements).

**Corollary 2.11.** An atomic pre-Bézout domain is a PID.

**Proof.** Let $D$ be an atomic pre-Bézout domain. By Corollary 2.10 it suffices to show that $D$ is a UFD, and for this it suffices to show that each atom is prime. Thus let $a$ be an atom, and suppose that $a \mid bc$ for some $b, c \in D$. If $(a, b)$ is not contained in a proper principal ideal, then by assumption, we may write $1 = ar + bs$ with $r, s \in D$; multiplication by $c$ then yields that $a \mid c$. Suppose that $(a, b) \subseteq (d)$ for some nonunit $d$. Then $a = dt$, $t \in D$. Since $a$ is an atom, and $d$ is not a unit, $t$ must be a unit. Therefore, since $d \mid b$, we have that $a \mid b$, as desired. □

We have GCD and AGCD versions of Corollary 2.10; the latter strengthens [26, Corollary 2.6]. The other properties will be considered later.
Proposition 2.12. Let $D$ be a GCD-domain (resp., AGCD-domain). Then the following statements are equivalent.

(1) $D$ is a DW-domain.
(2) $D$ is a DF-domain.
(3) $D$ is a Bézout domain (resp., AB-domain).

Proof. We give the proof for the AGCD case; the proof for the GCD case is similar (and easier). That statement (1) implies statement (2) is trivial. Assume statement (2), and let $a, b \in D$. Since $D$ is an AGCD-domain, we have $(a^n, b^n)$ principal for some positive integer $n$. The DF-assumption then yields that $(a^n, b^n)$ is principal. Hence $D$ is an AB-domain. This gives (2) $\Rightarrow$ (3). Finally, if $D$ is an AB-domain, then $D$ is a DW-domain by [26, Corollary 2.6].

It is clear that a domain $D$ is local if and only if no two nonunits of $D$ are comaximal. Since a local Bézout domain is a valuation domain, we have the following.

Corollary 2.13. A domain $D$ is a valuation domain if and only if it is simultaneously a GCD-domain and a DF-domain in which no two nonunits of $D$ are comaximal.

Recall that a Prüfer $v$-multiplication domain (PVMD) may be characterized as a domain $D$ for which $D_M$ is a valuation domain for each maximal $t$-ideal $M$ of $D$. Examples of PVMDs include Prüfer, Krull, and GCD-domains. It is easy to see that a PVMD that is also a DW-domain is a Prüfer domain (and this was observed in [29, page 1967]), but we do not know whether a domain that is both a PVMD and a DF-domain must be Prüfer. However, recall that a domain is said to be a ring of Krull type if it is a PVMD of finite $t$-character [15]. Then by Proposition 2.7 (and the fact that $d = t$ in a Prüfer domain):

Corollary 2.14. The following statements are equivalent for a domain $D$.

(1) $D$ is of Krull-type and is also a DF-domain
(2) $D$ is of Krull-type and is also a DW-domain.
(3) $D$ is a Prüfer domain of finite character.

3. Localization

In this section, we discuss localization in connection with the DF-property. We begin with some facts about the relation between the $F$-operation on a domain $D$ and the $F$-operation on a ring of quotients of $D$.

Lemma 3.1. Let $D$ be a domain with overring $E$. Let $*$ (resp., $*_1$) be a star operation on $D$ (resp. $E$). For each nonzero fractional ideal $I$ of $D$, set $I^{\delta(*,*_1)} = (IE)^{*_1} \cap I^*$. Then:

(1) $\delta(*,*_1)$ is a star operation on $D$, and $\delta(*,*_1) \leq *$.
(2) If $I^* \subseteq (IE)^{*_1}$ for each fractional ideal $I$ of $D$, then $\delta(*,*_1) = *$; in this case each $*_1$-ideal of $E$ contracts to a $*$-ideal of $D$.
(3) If $S$ is a multiplicatively closed subset of $D$, then $\delta(F,F_{DS}) = F$ and hence $F$-ideals of $DS$ contract to $F$-ideals of $D$.

Proof. (1) That $\delta(*,*_1)$ is a star operation on $D$ follows immediately from [4, Theorem 2]. It is clear that $\delta(*,*_1) \leq *$.
(2) If $I^* \subseteq (IE)^*$, then $I^{(\ast, \ast)} = (IE)^* \cap I^* = I^*$. Now let $A$ be a $*_1$-ideal of $E$. Then, by what was just proved, $(A \cap D)^* = ((A \cap D)E)^* \cap (A \cap D)^* \subseteq A \cap (A \cap D)^* \subseteq A \cap D$.

(3) By (2) we need show only that $I^F \subseteq (ID_S)^F_{DM}$ for each nonzero fractional ideal $I$ of $D$. Let $I$ be a nonzero fractional ideal of $D$, and let $x \in D$ be such that $x(a, b) \subseteq I$ for $a, b \in D$ with $(a, b) = D$. By [31, Lemma 4], $(a, b)D_S = D_S$, whence $x \in (ID_S)^F_{DS}$. It follows that $I^F \subseteq (ID_S)^F_{DS}$, as desired. □

In [26, Theorem 2.9] Mimouni showed that a domain $D$ for which $D_M$ is a DF-domain for each maximal ideal $M$ of $D$ is itself DW. We have a similar result for the DF-property.

**Proposition 3.2.** For a domain $D$, if $D_M$ is a DF-domain for each maximal ideal $M$ of $D$, then $D$ is a DF-domain.

**Proof.** Let $M$ be a maximal ideal of $D$. Under the assumption that $D_M$ is a DF-domain, we have that $MD_M$ is an $F$-prime of $D_M$. By Lemma 3.1 $M$ is an $F$-prime of $D$. Therefore, $D$ is a DF-domain if each localization at a maximal ideal is DF. □

The converse of Proposition 3.2 is false. In fact, we next give an example of a DW-domain $D$ with a maximal ideal $M$ such that $D_M$ is not a DF-domain. Note that this answers a question left open by Park and Tartarone [28, page 60].

**Example 3.3.** In [17] W. Heinzer and J. Ohm present an example of a domain $D$ which is essential ($D = \bigcap D_{P_a}$, where each $P_a$ is a prime ideal of $D$ and $D_{P_a}$ is a valuation domain) but is not a PVMD. As further analyzed in [27] and [12], $D$ has one height-two maximal ideal $M$, with $M$ being a $t$-prime and $D_M$ a regular local ring, and all other maximal ideals of $D$ have height one (and are therefore $t$-primes). (Moreover, $D_P$ is a rank-one discrete valuation domain for each height-one maximal ideal $P$; we use this fact below.) Thus $D$ is a DW-domain, and hence a DF-domain, but, since $MD_M$ is not an $F$-prime (since $MD_M$ is 2-generated and satisfies $(MD_M)^{FDM} = D_M$), $D_M$ is not a DF-domain.

As usual, we say that a domain has a given property locally if each localization at a maximal ideal has the property. Thus the example above is locally a PVMD. In fact, it is also locally a UFD (by the “moreover” statement in the example) and hence locally a GCD-domain and locally a Krull domain. The example “works” because $MD_M$ is not an $F$-ideal. Recall from [32] that a domain $D$ is (conditionally) well behaved if for each prime (maximal) $t$-ideal $P$ of $D$, $PD_P$ is a $t$-prime of $D_P$. Let us now call $D$ (conditionally) $F$-well behaved if for each prime (maximal) $F$-ideal of $D$, $PD_P$ is an $F$-prime of $D_P$. Then the $D$ of the example is neither conditionally well behaved nor conditionally $F$-well behaved.

It is clear that a Prüfer (resp., almost Prüfer) domain is locally GCD (resp., AGCD). We next find conditions that yield a converse.

**Lemma 3.4.** Let $D$ be a local domain. Then the following statements are equivalent.

1. $D$ is an APD.
2. $D$ is an AVD.
3. $D$ is an ABD.
4. $D$ is both an AGCD-domain and a DW-domain.
5. $D$ is both an AGCD-domain and a DF-domain.
Proof. The equivalence of (1), (2), and (3) follows from [1, Theorem 5.8]. Statements (3), (4), and (5) are equivalent by Proposition 2.12 (since an ABD is clearly an AGCD-domain). □

Proposition 3.5. The following statements are equivalent for a domain $D$.

1. $D$ is an APD (resp., Prüfer domain).
2. $D$ is a well-behaved DW-domain that is locally AGCD (resp., locally GCD).
3. $D$ is an $F$-well-behaved DF-domain that is locally AGCD (resp., locally GCD).

Proof. We give the proof for the “non-parenthetical” result. Let $D$ be an APD, and let $M$ be a maximal ideal of $D$. By [1, Theorem 5.8], $D_M$ is an AVD and hence an AGCD-domain. In addition, $PD_P$ is a $t$-prime of $D_P$ for each $t$-prime $P$ of $D$ by [1, Lemma 5.2], i.e., $D$ is well behaved. Finally, $D$ is DW by [26, Corollary 2.11]. This gives (1) ⇒ (2). Now let $D$ satisfy the conditions in (2), let $P$ be a prime ideal (automatically an $F$-prime) of $D$, and let $M$ be a maximal ideal of $D$ containing $P$. By hypothesis $D_M$ is an AGCD DW-domain and hence an AVD by the lemma. Therefore, $D_P$, as an overring of $D_M$, is an AVD, whence $PD_P$ is a $t$- and hence an $F$-prime of $D_P$, as desired. This proves (2) ⇒ (3). Finally, suppose that $D$ is an $F$-well behaved DF-domain that is also locally an AGCD-domain. If $M$ is a maximal ideal of $D$, then $D_M$, being AGCD and DF, is an AVD-domain by the lemma. Hence $D$ is an APD, again by [1, Theorem 5.8]. □

Recall that an almost Dedekind domain is a domain for which each localization at a maximal ideal is a rank-one discrete valuation domain.

Proposition 3.6. The following statements are equivalent for a domain $D$.

1. $D$ is an almost Dedekind domain.
2. $D$ is a well behaved DW-domain that is also locally a Krull domain.
3. $D$ is an $F$-well behaved DF-domain that is also locally a Krull domain.

Proof. It is clear that (1) ⇒ (2). Let $D$ be as in (2). Then for each maximal ideal $M$ of $D$, $D_M$ is DW and Krull and hence, by Corollary 2.9, a Dedekind domain. Thus $D$ is in fact one dimensional, and (3) follows easily. Now let $D$ be as in (3), and let $M$ be a maximal ideal of $D$. Then $D_M$ is both a DF-domain and a Krull domain and hence a (local) Dedekind domain by Corollary 2.9. Hence $D_M$ is a rank-one discrete valuation domain. Therefore, (3) ⇒ (1). □

Similar arguments (using Corollary 2.14) yield the following result.

Proposition 3.7. A domain $D$ is a Prüfer domain if and only if $D$ is an $F$-well behaved DF-domain that is locally a ring of Krull type. □

4. Connections with classical grade

As in [25] we call a sequence $a_1, \ldots, a_n$ of $D$ of elements of $D$ an $R$-sequence if $(a_1, \ldots, a_n) \neq D$ and $a_i$ is not a zero divisor on the module $D/(a_1, \ldots, a_{i-1})$ for $i = 1, \ldots, n$. The classical grade of an ideal $I$ of $D$, denoted by $G(I)$, is then the length of a longest $R$-sequence of elements of $I$. We note that this is “delicate” in the non-Noetherian setting (Kaplansky refrains from defining it there), as Hochster [17] has shown that it is possible for an ideal in a domain to have maximal $R$-sequences of different lengths.
Now recall Exercises 1 and 2 on page 102 of [25]. According to Exercise 1, if an ideal \( I \) of \( D \) satisfies \( G(I) \geq 2 \), then \( I^{-1} = D \). Exercise 2 then provides a converse in case \( D \) is Noetherian. Note that it follows immediately from Exercise 1 that the first two elements of any \( R \)-sequence in \( D \) are \( v \)-coprime. Now suppose that an ideal \( I \) not only satisfies \( I^{-1} = D \) but actually contains two \( v \)-coprime elements \( a, b \). If \( bc \in (a) \) for some \( c \in D \), then one sees immediately that \( c/a \in (a, b)^{-1} = D \) and hence \( c \in (a) \). Therefore, \( a, b \) is an \( R \)-sequence. We state this formally:

**Proposition 4.1.** Let \( I \) be a nonzero proper ideal in a domain \( D \). Then \( G(I) \geq 2 \) if and only if \( I \) contains a pair of \( v \)-coprime elements (and this pair is then an \( R \)-sequence). Thus \( G(I) < 2 \) for every ideal \( I \) of an DF-domain. \( \square \)

**Corollary 4.2.** Let \( I \) be a proper finitely generated ideal of an integral domain \( D \), and suppose that \( I \) contains an element \( a \) which belongs to only finitely many maximal \( t \)-ideals of \( D \). Then \( G(I) \geq 2 \) if and only if \( I^{-1} = D \).

**Proof.** That \( G(I) \geq 2 \) implies \( I^{-1} = D \) has already been discussed. Assume \( I^{-1} = D \). Pick \( a \in I \) with \( a \) contained in only finitely many maximal \( t \)-ideals of \( D \). Since \( I^{-1} = D \), \( I \) is contained in no maximal \( t \)-ideals of \( D \), and we may use prime avoidance to pick \( b \in I \) with \( (a, b) \) contained in no maximal \( t \)-ideal. We then have \( (a, b)^{-1} = (a, b)^{-1} = D \), whence \( a, b \) is an \( R \)-sequence by Proposition 4.1. \( \square \)

**Corollary 4.3.** Let \( D \) be a domain with finite \( t \)-character, and let \( I \) be a proper finitely generated ideal of \( D \). Then \( G(I) \geq 2 \) if and only if \( I^{-1} = D \).

We have the following result, which both generalizes, and provides an easier path to a solution of, Exercise 2 of [25].

**Corollary 4.4.** If \( I \) is an ideal of a TV-domain \( D \), then \( G(I) \geq 2 \) if and only if \( I^{-1} = D \).

**Proof.** Let \( I \) be an ideal in the TV-domain \( D \), and assume that \( I^{-1} = D \). Then \( I^t = I^v = D \), and hence \( J^{-1} = J^v = D \) for some finitely generated subideal \( J \) of \( I \). By Corollary 4.3, we then have \( G(I) \geq G(J) \geq 2 \). \( \square \)

We note that the conclusion of Corollary 4.4 is not valid if \( D \) is only assumed to have finite \( t \)-character, for if \( D \) is a valuation domain with nonprincipal maximal ideal \( M \), then \( D \) has finite \( (t-) \) character, but \( M^{-1} = D \) and \( G(M) = 1 \).

In Proposition 1.8, we saw that in a domain of finite \( t \)-character, we have \( F-\text{Max}(D) = w-\text{Max}(D) \). In fact, by applying the ideas of this section, we can obtain a stronger conclusion (and thereby generalize [16, Proposition 3.3]):

**Corollary 4.5.** In a domain \( D \) of finite \( t \)-character, we have \( F = w \).

**Proof.** Let \( D \) have finite \( t \)-character, and let \( I \) be an \( F \)-ideal of \( D \). Suppose that \( xJ \subseteq I \) for some \( x \in D \) and finitely generated ideal \( J \) with \( J^v = D \). By Corollary 4.3 (and Proposition 4.1), there are elements \( a, b \in J \) with \( (a, b)^v = D \). Since \( x(a, b) \subseteq I \) and \( I \) is an \( F \)-ideal, this yields \( x \in I \). Therefore \( I \) is also a \( w \)-ideal, as desired. \( \square \)

5. Examples

In [30, Section 7], H. Uda presents an example showing that classical grade and polynomial grade can differ. We begin with a review of his example and then proceed to adapt it for our purposes. Specifically, we show that an appropriate localization satisfies \( t_2 < t \) and \( F < w \) and is a DF-domain but not a DW-domain.
Except for a slight change in notation, here is Uda’s example:

**Example 5.1.** Let $k$ be a field and $s, t, u$ indeterminates over $k$. Then set $A = k[s, t, u]_{(s, t, u)}$, and let $P$ denote the maximal ideal of $A$. For each $\alpha, \beta \in P$, let $X_{\alpha \beta}$ be an indeterminate, and let $T = A[[X_{\alpha \beta}]]$. Let $B$ denote the ideal of $T$ generated by the $X_{\alpha \beta}$, and let $J = B^2$. Let $N = PT + B$, so that $N$ is a maximal ideal of $T$, generated by $s, t, u$ and the $X_{\alpha \beta}$. Now for each $\alpha, \beta \in P$, let $P_{\alpha \beta} = (\alpha, \beta)A$, and let $R = A + \sum P_{\alpha \beta} X_{\alpha \beta} + J$. Let $M = N \cap R$. Each $f \in R$ has a unique representation $f = f_0 + \sum f_{\alpha \beta} X_{\alpha \beta} + f_1$ with $f_0 \in A$, $f_{\alpha \beta} \in P_{\alpha \beta}$, and $f_1 \in J$.

**Proposition 5.2.** In Example 5.1:

1. $T$ is integral over $R$.
2. $M$ is a maximal ideal of $R$ and a maximal $t_2$-ideal.
3. $(PR)^t = R$, hence $M$ is not a $t$-ideal.
4. $T^t R_M = T_N$.
5. $R_M$ is not integrally closed.
6. $MR_M$ is a $t_2$-ideal but not a $w$-ideal of $R_M$. Hence in $D := R_M$, $t_2 < t$ and $F < w$.
7. $D$ is a DF-domain but not a DW-domain.
8. $D$ is not a pre-Bézout domain.
9. $D$ does not have finite $t$-character. (Of course, since $D$ is local with maximal ideal a $t_2$-ideal, $D$ does have finite $t_2$-character.)

**Proof.**

1. This follows from the fact that $A \subseteq R$ and $X_{\alpha \beta}^2 \in R$ for each $\alpha, \beta$.
2. By (1) $M$ is a maximal ideal of $R$. Let $f, g \in M$. Write $f = f_0 + \sum f_{\alpha \beta} X_{\alpha \beta} + f_1$ and $g = g_0 + \sum g_{\alpha \beta} X_{\alpha \beta} + g_1$ with $f_0, g_0 \in P$, $f_{\alpha \beta} \in P_{\alpha \beta}$, and $f_1, g_1 \in J$. Then $X_{f_0 g_0} (f, g) R \subseteq R$, and we have $(f, g)^w \subseteq (R :_R X_{f_0 g_0}) \subseteq M$ ([30, Lemma 7.1]).
3. It follows from [30, Proposition 7.3] that $(s, t, u) R^w = R$.
4. Let $f \in T \setminus N$, and write $f = a + g$ with $a \in A$ and $g \in (\{X_{\alpha \beta}\}) R$. Then $f^{-1} = (a - g)/(a^2 - g^2)$ with $a^2 - g^2 \in R \setminus M$.
5. We have $X_{\alpha \beta} \in T \setminus R_M$ for each $\alpha, \beta \in P$.
6. For $f, g \in M$, represent $f, g$ as in (2). Then $X_{f_0 g_0} (f, g) R \subseteq R_M$ with $X_{f_0 g_0} \notin R_M$. It follows that $((f, g) R_M)^w R_M \subseteq MR_M$ and hence that $MR_M$ is a $t_2$-ideal. On the other hand, $M$ is not a $w$-ideal by (3) (since every maximal $w$-ideal is a maximal $t$-ideal); hence $MR_M$ is not a $w$-ideal.
7. Since $MR_M$ is a $t_2$-ideal, it is also an $F$-ideal. Therefore, $D$ is a DF-domain. On the other hand, $D$ is not a DW-domain, since $MR_M$ is not a $t$-ideal.
8. It is clear that $s, t$ are not contained in a proper principal ideal of $T_N$. Hence $(s, t) D$ is not contained in a proper principal ideal of $D$, i.e., $(s, t) D$ is primitive. Of course, $(s, t) D$ is not superprimitive and hence $D$ does not have LPSP. Thus $D$ is not pre-Bézout by Proposition 2.5.
9. Since $D$ is not a DW-domain, $D$ cannot have finite $t$-character by Proposition 2.7.

□

**Remarks/Questions 5.3.** Refer to Proposition 5.2.

1. By (5), $D$ is not integrally closed. Must an integrally closed DF-domain be DW? We doubt that this is true but have no counterexample.
(2) Picozza and Tartarone [29, Theorem 3.7] prove that a DW-domain that is both integrally closed and satisfies the finite-conductor property must be a Prüfer domain. (A domain $E$ is a finite conductor domain if $(a) \cap (b)$ is finitely generated for all $a, b \in E$.) The proof involves two steps: an integrally closed finite conductor domain is a PVMD, and a PVMD that is also a DW-domain must be a Prüfer domain. As we have already remarked, we do not know whether a DF-PVMD must be Prüfer (but we doubt it).

(3) It is clear that $\dim(D) = \infty$: Every one-dimensional domain is a DW-domain. Are there two-dimensional, or at least finite-dimensional, examples of DF-domains that are not DW?

(4) As mentioned in [16], if $n > 2$ and one substitutes $n$-generated ideals for two-generated ideals in the definitions of the $t_2$- and $F$-operations, one obtains new star operations, dubbed the $t_n$- and $F_n$-operations (so that $F_2 = F$). Whether we always have $t_n = t$ or $F_n = w$ were left as open questions. However, by making obvious changes in Example 5.1, one can obtain, for each $n > 1$ a local domain $D_n$ whose maximal ideal is a $t_n$- (and hence also an $F_n$-) ideal but is not an $F_{n+1}$-ideal.

In order to produce more examples of DF-domains that are not DW, we investigate the DF-property in pullback diagrams. Though our results generally parallel those of Mimouni for DW-domains [26], our proofs are somewhat more delicate due to the fact that ideals often must be two-generated. We need several facts about the behavior of $\nu$-ideals, etc., in pullbacks. For this we use [11] as a convenient reference, but the ideas actually come from [10].

Let $T$ be a domain, $M$ a maximal ideal of $T$, $\varphi : T \rightarrow k := T/M$ the natural projection, and $D$ an integral domain contained in $k$. Then let $D = \varphi^{-1}(D)$ be the integral domain arising from the following pullback of canonical homomorphisms.

$$
\begin{array}{ccc}
R & \longrightarrow & D \\
\downarrow & & \downarrow \\
T & \varphi \longrightarrow & T/M = k
\end{array}
$$

We shall refer to this as a diagram of type $\square$.

Proposition 5.5 below allows one to produce many examples of DF-domains.

**Lemma 5.4.** In a pullback of type $\square$:

(1) If $A$ is a $F$-ideal of $D$, then $\varphi^{-1}(A)$ is a $F$-ideal of $R$.

(2) For each nonzero ideal $A$ of $D$, $\varphi^{-1}(A^F) = \varphi^{-1}(A)^F$.

(3) If $Q$ is a maximal $F$-ideal of $T$, then $Q \cap R$ is a maximal $F$-ideal of $R$.

**Proof.** (1) Let $A$ be a $F$-ideal of $D$, and let $I = \varphi^{-1}(A)$. Suppose $r(a, b) \subseteq I$, with $r, a, b \in R$ and $(a, b)^\nu = R$. By [11, Proposition 2.17(2b)], $(\varphi(a), \varphi(b))^{\nu_{\varphi}} = D$. Since $A$ is a $F$-ideal of $D$, this yields $\varphi(r) \in A$ and hence $r \in I$. Thus $I$ is a $F$-ideal of $R$.

(2) Let $A$ be a nonzero ideal of $D$. By (1), we have $\varphi^{-1}(A^F) \supseteq \varphi^{-1}(A)^F$. We now recall the notation of Definition 1.1: For a domain $E$ with quotient field $L$ and a subset $J$ of $L$, we write $J' = \{ y \in L \mid y(e, f) \subseteq J \text{ for some } e, f \in E \text{ with } (e, f)^{n_{\varphi}} = E \}$. To complete the proof, it will suffice to show that $\varphi^{-1}(A') \supseteq \varphi^{-1}(A)^'$. To this end, let $x \in \varphi^{-1}(A')$. Then $\varphi(x)(d_1, d_2) \subseteq A$ for elements $d_1, d_2 \in D$ with
(d_1, d_2)^{\nu_0} = D. According to [20, Lemma 7 and its proof], there are elements r_1, r_2 in R for which \( \varphi(r_i) = d_i \) for \( i = 1, 2 \) and \( \varphi^{-1}(d_1, d_2) = (r_1, r_2) \). By [11, Proposition 2.17(1b)], we have \( R = \varphi^{-1}((d_1, d_2)^{\nu_0}) = (r_1, r_2)^{\nu} \). Since \( x(r_1, r_2) \subseteq \varphi^{-1}(A) \), we have \( x \in \varphi^{-1}(A)' \), as desired.

(3) Let \( Q \) be a maximal \( F \)-ideal of \( T \), and let \( P = Q \cap R \). Suppose that \( P \) is not an \( F \)-prime of \( R \). Then there are elements \( a, b \in P \) for which \( (a, b)^{\nu} = R \). Note that we cannot have \((a, b) \in M \) since \( M \) is divisorial in \( R \). Hence \((a, b)T)^{\nu_T} = T \) by [11, Proposition 2.5(2)], contradicting that \( Q \) is an \( F \)-prime of \( T \).

**Proposition 5.5.** In a pullback of type \( \square \):

1. If \( T, D \) are \( DF \)-domains, then \( R \) is a \( DF \)-domain.
2. If \( T \) is local and \( D \) is a \( DF \)-domain, then \( R \) is a \( DF \)-domain.
3. If \( R \) is a \( DF \)-domain, then \( D \) is a \( DF \)-domain.

**Proof.** (1) Let \( P \) be a maximal ideal of \( R \). If \( P \supseteq M \), then \( P = \varphi^{-1}(p) \) for a maximal ideal \( p \) of \( D \) [11, Theorem 1.9]. Since \( D \) is a \( DF \)-domain, \( p \) is an \( F \)-prime of \( D \) and hence \( P \) is an \( F \)-prime of \( R \) by Lemma 5.4. If \( P = M \), then \( P \) is divisorial (and therefore an \( F \)-prime). If \( P \) is incomparable to \( M \), then \( P = Q \cap T \) for some maximal ideal \( Q \) of \( T \) [11, Theorem 1.9]. Since \( T \) is a \( DF \)-domain, \( Q \) is an \( F \)-prime and hence so is \( P \) by Lemma 5.4.

(2) This follows as in the proof of (1).

(3) Assume that \( R \) is \( DF \), and let \( p \) be a maximal ideal of \( D \). Then \( P := \varphi^{-1}(p) \) is a maximal ideal of \( R \), and, since \( R \) is a \( DF \)-domain, \( P \) is an \( F \)-prime of \( R \). By Lemma 5.4 \( P = P^F = \varphi^{-1}(p^F) \), whence \( p = p^F \), that is, \( p \) is an \( F \)-prime of \( D \). □

According to [26, Theorem 3.1(1)], in a pullback diagram of type \( (\square) \), if \( R \) is \( DW \), then so is \( D \). Hence if we take \( D \) to be a \( DF \)-domain that is not \( DW \) (e.g., the \( D \) of Proposition 5.2) and \( T \) is either local or a \( DF \)-domain, then \( R \) is a \( DF \)-domain that is not \( DW \).

### 6. Polynomial rings

**Proposition 6.1.** Let \( D \) be a domain, and \( Q \) a maximal \( t_2 \)-ideal of \( D[X] \). Then \( Q \) is a maximal \( t \)-ideal of \( D[X] \). Hence \( Q \) is either an upper to zero or the extension of a maximal \( t \)-ideal of \( D \). Moreover, \( t\text{-Max}(D[X]) = t_2\text{-Max}(D[X]) = w\text{-Max}(D[X]) = F\text{-Max}(D[X]) \).

**Proof.** If \( Q \) is an upper to zero, then \( Q \) is a \( t \)-ideal and must therefore be a maximal \( t \)-ideal. Hence we assume that \( Q = Q \cap D \neq (0) \). Suppose, by way of contradiction, that \( Q^t = D[X] \). Then we have \( f_1, \ldots, f_n \in Q \) with \( (f_1, \ldots, f_n) = D[X] \), and it is clear that we must then have \( (c(f_1) + \cdots + c(f_n))^{-1} = D \). By a standard argument, we can then produce \( f \in Q \) with \( c(f) = c(f_1) + \cdots + c(f_n) \) (take \( f = f_1 + X^{k_2}f_2 + \cdots + X^{k_n}f_n \) for appropriately chosen positive integers \( k_2, \ldots, k_n \)), so that \( (c(f))^r = D \). Pick \( a \in P \), \( a \neq 0 \). We claim that \( (a, f)^r = D[X] \). (This is a another standard argument: suppose that \( g \in (a, f)^{-1} \). Since \( ga \in D[X] \), this puts \( g \in K[X] \). We then use the content formula to get \( c(f)^{r+1}c(g) = c(f)^rc(fg) \subseteq D \) for appropriately chosen \( r \) [13, Theorem 28.1]. Since \( c(f)^r = D \), this yields \( g \in D[X] \). Hence \( (a, f)^r = (a, f)^{-1} = D[X] \), as claimed.) However, this contradicts the fact that \( Q \) is a \( t_2 \)-ideal. Hence \( Q^t \neq D[X] \). It follows that \( Q \) must be a maximal \( t \)-ideal of \( D[X] \). The “hence” statement now follows from [22, Proposition 1.1]. As to
the “moreover” statement, we have \( w \)-\( \text{Max}(E) = t \)-\( \text{Max}(E) \) for all domains \( E \) and \( w = F \) on \( D[X] \) [16, Theorem 4.5].

The following corollary strengthens [26, Proposition 2.12].

**Corollary 6.2.** For a domain \( D \), \( D[X] \) is a DF-domain if and only if \( D \) is a field.

**Proof.** Suppose that \( D \) is not a field, and let \( M \) be a maximal ideal of \( D \). Then \( (M, X) \) is a maximal ideal of \( D[X] \) that is not a \( t \)-ideal, hence not a \( F \)-ideal. Thus \( D \) is not a DF-domain. The converse is trivial. \( \square \)

For a prime ideal \( I \) of a domain \( D \), it is well known that \( I \) is a \( t \)-ideal of \( D \) if and only if \( I[X] \) is a \( t \)-ideal of \( D[X] \). This does not hold, however, for \( F \) or \( t_2 \)-ideals, as the next example shows.

**Example 6.3.** In Proposition 5.2, \( M[X] \) is not an \( F \)-ideal of \( D[X] \).

**Proof.** If \( M[X] \) is a \( F \)-ideal of \( D[X] \), then it must be a maximal \( F \)-ideal. (The only primes containing \( M[X] \) are of the form \((M, f)\) with \( f \) monic. Then for any nonzero \( a \in M \), we have \((a, f)^c = D[X]\), so that \((M, f)\) is not an \( F \)-ideal.) However, by Proposition 6.1, this means that \( M \) is a \((\text{maximal})\) \( t \)-ideal of \( D \), a contradiction. \( \square \)

We remark that, although we always have \( F = w \) in \( D[X] \) [16, Theorem 4.5], we do not know whether a \( t_2 \)-prime of \( D[X] \) must be a \( t \)-ideal. (See [16] for some cases where the answer is yes.)

In [24] Kang extended the notion of the Nagata ring as follows. For a star operation \( * \) on \( D \), let \( N_e = \{ g \in D[X] \mid c(g)^* = D \} \) (where \( c(g) \) denotes the content of \( g \), i.e., the ideal of \( D \) generated by the coefficients of \( g \)). The \( * \)-Nagata ring is then \( D[X]_{N_e} \). When \( * = d \), we have the classical Nagata ring, usually denoted by \( D(X) \). In [29], the authors observe that \( D[X]_{N_e} \) is always a DW-domain, and they prove that a domain \( D \) is DW if and only if \( D(X) \) is DW if and only if \( D(X) = D[X]_{N_e} \). This leads to the question: When is \( D[X]_{N_F} \) a DF-domain? We answer this question in the next result. We shall use the fact that the maximal ideals of \( D[X]_{N_e} \) are the ideals \( MD[X]_{N_e} \), where \( M \) is a maximal \( *_f \)-ideal of \( D \) [24, Proposition 2.1].

**Proposition 6.4.** The following statements are equivalent for a domain \( D \).

1. \( D[X]_{N_F} \) is a DF-domain.
2. \( D[X]_{N_F} = D[X]_{N_Y} \).
3. \( D[X]_{N_F} \) is a DW-domain.

**Proof.** Suppose that \( D[X]_{N_F} \) is a DF-domain. Then each maximal ideal of \( D[X]_{N_F} \) is an \( F \)-prime, and, using the fact that the \( F \)-operation has finite type and the above-mentioned description of \( \text{Max}(D[X]_{N_F}) \), we have that \( MD[X]_{N_F} \) is an \( F \)-prime of \( D[X]_{N_F} \) for each maximal \( F \)-ideal \( M \) of \( D \). If follows that \( MD[X] \) is an \( F \)-prime, and hence, by Proposition 6.1, a \( t \)-prime of \( D[X] \) for each such \( M \). Therefore, each maximal \( F \)-ideal of \( D \) is in fact a maximal \( t \)-ideal, and this yields that \( D[X]_{N_F} = D[X]_{N_Y} \). Hence (1) \( \Rightarrow \) (2). It is clear that (2) \( \Rightarrow \) (3) \( \Rightarrow \) (1). \( \square \)

**References**

WHEN ANY TWO $v$-COPRIME ELEMENTS ARE COMAXIMAL


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