# $*$-finite ideals contained in infinitely many maximal $*_s$-ideals

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Abstract. We give a new characterization of $*$-finite ideals that are contained in infinity many $*_s$-maximal ideals.

Assuming familiarity with star operations, for now, we give a new proof of Theorem 1 below. This result was proved in [**1**] using reductio ad absurdum, essentially proving the equivalent statement, given within the parentheses in the statement of Theorem 1. In this note we offer (a) a direct proof of it and (b) the equivalent statement mentioned above in a more general form. The purpose here is ($\alpha$) to show that with a suitable approach the infinitely many sets involved can be "displayed" and ($\beta$) to give a more general characterization of domains of finite $*_s$-character.

Theorem 1. *Let $D$ be an integral domain, $*$ a finite character star operation on $D$, and $\Gamma$ a set of proper $*$-finite $*$-ideals of $D$ such that every proper $*$-finite $*$-ideal of $D$ is contained in some member of $\Gamma$. Let $A$ be a nonzero finitely generated ideal of $D$ with $A^* \neq D$. Then $A$ is contained in an infinite number of maximal $*$-ideals if and only if there exists an infinite family of mutually $*$-comaximal ideals in $\Gamma$ containing $A$. (Equivalently, with the same assumptions on $A$, $A$ is contained in at most a finite number of maximal $*$-ideals if and only if $A$ is contained in at most a finite number of mutually $*$-comaximal members of $\Gamma$.)*

The reader may consult Sections 32 and 34 of [**2**] for results on star operations. We include below a working introduction.

Let $D$ denote an integral domain with quotient field $K$ and let $F(D)$ be the set of nonzero fractional ideals of $D$. A *star operation* $*$ on $D$ is a function $*\colon F(D) \to F(D)$ such that for all $A, B \in F(D)$ and for all $0 \neq x \in K$:

(a) $(x)^* = (x)$ and $(xA)^* = xA^*$,

(b) $A \subseteq A^*$ and $A^* \subseteq B^*$ whenever $A \subseteq B$, and

(c) $(A^*)^* = A^*$.

For $A, B \in F(D)$ we have $(AB)^* = (A^*B)^* = (A^*B^*)^*$ and $(A+B)^* = (A^*+B)^* = (A^*+B^*)^*$. A fractional ideal $A \in F(D)$ is called a $*$-*ideal* if $A = A^*$ and a $*$-*ideal of finite type* if $A = B^*$ where $B$ is a finitely generated fractional ideal. Also, $A \in F(D)$ is called $*$-*finite* if $A^*$ is of finite type. To each star operation $*$ we can associate another star operation $*_s$ by defining $A^{*_s} = \bigcup\{B^* \mid 0 \neq B$ is a finitely generated subideal of $A\}$ for all $A \in F(D)$. It is easy to see that for a finitely

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generated $A \in F(D)$ we have $A^* = A^{*_s}$ and that $(*_s)_s = *_s$. A star operation $*$ is said to be of *finite character* if $* = *_s$; so $*_s$ is of finite character. If $*$ is a finite character star operation on $D$, then by Zorn's Lemma a proper integral $*$-ideal is contained in a proper integral ideal maximal with respect to being a $*$-ideal, called a *maximal $*$-ideal*, and each such maximal $*$-ideal is prime. We call proper ideals $A$ and $B$ of $D$ *$*$-comaximal* if $(A+B)^* = D$. If $*$ is of finite character, then $(A+B)^* = D$ if and only if $A$ and $B$ share no maximal $*$-ideals if and only if there exist finitely generated ideals $\mathbf{a} \subseteq A$, $\mathbf{b} \subseteq B$ such that $(\mathbf{a}, \mathbf{b})^* = D$. An integral domain $D$ is said to be of *finite $*$-character*, for a finite character star operation $*$, if every nonzero nonunit of $D$ is contained in at most a finite number of maximal $*$-ideals of $D$. Obviously, $D$ is of finite $*$-character if and only if every nonzero finitely generated ideal $I$ with $I^* \neq D$ is contained in at most a finite number of maximal $*$-ideals. So $D$ is not of finite character if and only if $D$ contains a finitely generated ideal $I$ with $I^* \neq D$ such that $I$ is contained in infinitely many maximal $*$-ideals.

One part of the proof of Theorem 1 is to show that if a $*$-finite ideal $A$ is contained in infinitely many maximal $*$-ideals then $A$ must be contained in an infinite number of mutually $*$-comaximal members of $\Gamma$. For this we need to prepare a little. Let $A$ be as in the statement and let $F$ be the (infinite) set of maximal $*$-ideals containing $A$. Let us call a subset $S = \{A_1, A_2, ..., A_n\}$ of $\Gamma$ a *state $S_n$ of height $n$ for $A$* if $|S| = n$, each $A_i \supseteq A$, and the elements of $S$ are mutually $*$-comaximal. Note that every proper nonempty subset of a state is also a state. Call a state $S_n$ *unsaturated* if we can find at least one $M \in F$ such that $M$ does not contain any member of $S_n$. So a state $S_n$ is *saturated* if for every $M \in F$, $M$ contains some member of $S_n$. Note also that any member of $F$ can contain at most one member of a state because the members of a state are mutually $*$-comaximal.

Lemma 1. *Let $S_n = \{A_1, A_2, ..., A_n\}$ be an unsaturated state for $A$ and let $M$ be a maximal $*$-ideal in $F$ such that $M$ does not contain any member of $S_n$. Then there is a state $S_{n+1} = \{A_1, A_2, ..., A_n, A_{n+1}\}$ for $A$.*

Proof. Let $S_n = \{A_1, A_2, ..., A_n\}$ and let $M$ be as in the statement of the lemma. Since $A_i \nsubseteq M$ we have $(M + A_i)^* = D$ and so there are finitely generated ideals $m_i \subseteq M$ such that $(m_i + A_i)^* = D$ for $i = 1, 2, ..., n$. Let $B = A + m_1 + m_2 + \cdots + m_n$. Clearly as $A$ and $m_i$ are all contained in $M$, we have $B^* \neq D$. Hence according to the condition, $B$ is contained in a $*$-ideal $A_{n+1}$ of finite type in $\Gamma$. Also, because of the presence of $m_i \subseteq A_{n+1}$ we have $(A_{n+1} + A_i)^* = D$. Thus $S_{n+1} = \{A_1, A_2, ..., A_n, A_{n+1}\}$ is a state for $A$. □

Lemma 2. *Let $A$ be a proper $*$-ideal of finite type. If $A$ is contained in two or more maximal $*$-ideals of $D$, then $A$ is contained in at least two proper $*$-comaximal $*$-ideals of finite type in $\Gamma$.*

Proof. Let $P, Q$ be two distinct maximal $*$-ideals containing $A$. Since $(P + Q)^* = D$ we have finitely generated ideals $p \subseteq P$ and $q \subseteq Q$ such that $(p+q)^* = D$. Since $A$, $p \subseteq P$ and $A$, $q \subseteq Q$ are such that $D \neq (A+p)^*$, $(A+q)^*$ are of finite type there exist $*$-ideals of finite type, $H$ and$K$, in $\Gamma$ with $H \supseteq A+ p$ and $K \supseteq A + q$. Because $p \subseteq H$ and $q \subseteq K$, $H$ and$K$ are $*$-comaximal. □

Lemma 3. *Let $S_n$ be a saturated state for $A$ with infinite $F$. Then there is a state $S_{n+1}$ for $A$.*

PROOF. In this case if $S_n = \{A_1, A_2, ..., A_n\}$, then $T = \{M \in F|\ M$ contains some $A_i\}$ is precisely $F$. So $A_j$, for some $j$, is contained in an infinite subset of $F$. By Lemma 2 $A_j$ has a state $\{H_j, K_j\} \subseteq \Gamma$. Since $A_j$ is $*$-comaximal with each of $A_i$ for $i \neq j$ we conclude that $H_j$ and $K_j$ are $*$-comaximal with each $A_i$ for $i \neq j$. Since $A_j$ contains $A$ the collection $\{A_1, A_2, ..., A_n\}\backslash\{A_j\}) \cup \{H_j, K_j\}$ is a state of height $n+1$ for $A$. □

PROOF. (of Theorem 1) Let us first consider a $*$-ideal $A$ of finite type for which we can find a sequence of unsaturated states $S_1, S_2, ..., S_n, ...$ such that $S_i \subsetneq S_{i+1}$. Then $S = \bigcup S_i$ contains an infinite set of mutually $*$-comaximal members of $\Gamma$. Call such ideals $A$ terminal. Note that a $*$-ideal $C$ of finite type contained in a terminal member of $\Gamma$ is itself a terminal ideal and hence is contained in infinitely many mutually $*$-comaximal members of $\Gamma$. This leaves us with ideals $B$ that, are contained in infinitely many maximal $*$-ideals, yet are not contained in any terminal ideals. Call such ideals nonterminal. This means that any member of $\Gamma$ that contains $B$ is either contained in only a finite number of maximal $*$-ideals or is such that after a finite number of applications of Lemma 1 there is a saturated state $T_1 = \{B_{11}, B_{12}, ..., B_{1n_1}\}$. Now because $n_1$ is finite, at least one of the $B_{1i}$ is contained in infinitely many maximal $*$-ideals and so is nonterminal. Let us, by a relabeling, assume that $B_{1n_1}$ is nonterminal. By Lemma 3, $B_{1n_1}$ is contained in at least two $*$-comaximal $H, K \in \Gamma$ which are $*$-comaximal with each $B_{1i}$ for $i = 1, 2, ..., n_1 - 1$; in fact, any proper ideal containing $B_{1n_1}$ is $*$-comaximal with each $B_{1i}$ for $i = 1, 2, ..., n_1 - 1$. So again by a finite number of applications of Lemma 1 we get a saturated state $T_2 = \{B_{21}, B_{22}, ...B_{2(n_2-1)}, B_{2n_2}\}$. We can select $B_{2n_2}$ as the nonterminal member and continue as before. Now suppose that we have thus reached the $rth$ state, emanating from nonterminal $B_{(r-1)n_{r-1}}$, $T_r = \{B_{r1}, B_{r2}, ...B_{r(n_r-1)}, B_{rn_r}\}$. Since the number of steps taken is finite and since $n_r$ is finite we have at least one of the $B_{ri}$ is contained in infinitely many maximal $*$-ideals and so is nonterminal. Selecting as before $B_{rn_r}$ as the nonterminal member of the state $T_r$, using Lemmas 1 and 2, we can generate the state $T_{r+1} = \{B_{(r+1)1}, B_{(r+1)2}, ...B_{(r+1)(n_{(r+1)}-1)}, B_{(r+1)n_{r+1}}\}$. Since each of $B_{(r+1)j}$ contains $B_{rn_r}$ and since $T_r$ is a state, each $B_{(r+1)j}$ is $*$-comaximal with $B_{ri}$ for $i = 1, 2, ..., n_r - 1$.

By induction we have an infinite sequence $T_1, T_2, ..., T_r, T_{r+1}, ....$ of states containing $B$. By Lemma 2, each of $T_r$ contains at least two elements. Of these exactly one element, $B_{rn_r}$, is taken to generate the next state. So essentially we have from state $T_r = \{B_{r1}, B_{r2}, ...B_{r(n_r-1)}, B_{rn_r}\}$ the state $U_r = \{B_{r1}, B_{r2}, ...B_{r(n_r-1)}\}$, and from $T_{r+1}$ the state $U_{r+1} = \{B_{(r+1)1},\ B_{(r+1)2}, ...\ B_{(r+1)(n_{(r+1)}-1)}\}$. By our earlier comments each member of $U_{r+1}$ is $*$-comaximal with each member of $U_r$ and for each $r$, $U_r$ has at least one element, which we can select to be $B_{r1}$. This gives us an infinite sequence of members of $\Gamma$, $B_{11}, B_{21}, B_{31}, ..., B_{r1}....$ To see that the $B_{i1}$ are mutually $*$-comaximal, take a pair of distinct elements of the sequence, say $B_{i1}, B_{j1}$, where $i \neq j$. We can assume that $i < j$. Now argue as follows: $B_{j1} \supseteq B_{(j-1)n_{j-1}} \supseteq B_{(j-2)n_{j-2}} \supseteq ... \supseteq B_{in_i}$ and $B_{in_i}$ and $B_{i1}$ are $*$-comaximal. Thus $B_{11}, B_{21}, B_{31}, ..., B_{r1}...$ is an infinite sequence of mutually $*$-comaximal members of $\Gamma$ containing the nonterminal ideal $B$. Now as a $*$-ideal $A$ of finite type that is contained in an infinite number of maximal $*$-ideals can either be terminal or nonterminal the proof is complete. The converse is obvious because no maximal $*$-ideal can contain two $*$-comaximal ideals. □

As indicated in the introduction, we can associate a star operation $*_s$ of finite character with every star operation $*$. The fact that for every finitely generated $A \in F(D)$ we have $A^* = A^{*_s}$, allows us to state Theorem 1 in the following form.

THEOREM 2. *Let $D$ be an integral domain, $*$ a star operation on $D$, and $\Gamma$ a set of proper $*$-ideals of finite type of $D$ such that every proper $*$-finite $*$-ideal of $D$ is contained in some member of $\Gamma$. Let $A$ be a nonzero finitely generated ideal of $D$ with $A^* \neq D$. Then $A$ is contained in an infinite number of maximal $*_s$-ideals if and only if there exists an infinite family of mutually $*_s$-comaximal ideals in $\Gamma$ containing $A$. (Equivalently, with the same assumptions on $A$, $A$ is contained in at most a finite number of $*_s$-maximal ideals if and only if $A$ is contained in at most a finite number of mutually $*_s$-comaximal members of $\Gamma$.)*

## References

[1] T. Dumitrescu and M. Zafrullah, Characterizing domains of finite $*$-character, J. Pure Appl. Algebra **214** (2010), 2087-2091.
[2] R. Gilmer, Multiplicative Ideal Theory, Dekker, 1972.

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