LCM-SPLITTING SETS IN SOME RING EXTENSIONS

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ABSTRACT. Let S be a saturated multiplicative set of an integral domain. Call S an lcm splitting set if $dD_S \cap D$ and $dD \cap sD$ are principal idelas for every $d \in D$ and $s \in S$. We show that if R is an R_2-stable overring of D (that is, if whenever $a, b \in D$ and $aD \cap bD$ is principal, it follows that $(aD \cap bD)R = aR \cap bR$) and if S is an lcm splitting set of D, then the saturation of S in R is an lcm splitting set in R. Consequently, if D is Noetherian and $p \in D$ is a (nonzero) prime element, then p is also a prime element of the integral closure of D. Also, if D is Noetherian, S is generated by prime elements of D and if the integral closure of D_S is a UFD, then so is the integral closure of D.

Let D be an integral domain with quotient field K. By an overring of D we mean a ring between D and K. A saturated multiplicative set S of D is called an *lcm splitting set* if (a) for all $s \in S$ and for all $d \in D$, $sD \cap dD$ is principal and (b) for all $d \in D\backslash\{0\}$ we have $d = st$, where $s \in S$ and $(t) \cap (\sigma) = (t\sigma)$ for all $\sigma \in S$ (this definition is equivalent to the one given in the abstract cf. [**AAZ1**, Lemma 1.2]). The notion of lcm splitting sets was studied in [**AAZ**], where it was used to prove several Nagata-like theorems, i.e., theorems of the form: if S is an lcm splitting multiplicative set of D and if D_S has (a suitable multiplicative) property P, then so does D (see section 4 of [**AAZ**]). Following Uda [**U**], we call a ring extension E of D R_2*-stable* if $aD \cap bD = cD$ with $a, b, c \in D$ implies $aE \cap bE = cE$ (that is, whenever c is an LCM for $a, b \in D$, the same is true for c, a, b in E). The purpose of this note is to record the consequences of the following result: Let $D \subseteq E$ be an R_2-stable extension of domains, where E is an overring of D. Let S be a saturated multiplicative set of D and let S' be the saturation of S in E. If S is a splitting (resp. lcm splitting) multiplicative set in D, then so is S' in E respectively. Thus, to take a familiar example, if E is a flat overring of D, then the saturation, in E, of an lcm splitting set S of D is lcm splitting in E. According to a result of Beck cited in [**F**, Lemma 4.5], the integral closure D' of a Noetherian domain D is an R_2-stable extension of D. Consequently, if D is Noetherian, every principal prime

1991 Mathematics Subject Classification. Primary 13A05, 13A15; Secondary 13B02, 13B22.
Key words and phrases. lcm-splitting set, R_2-stable overring, Noetherian domain.
The first author gratefully acknowledges the hospitality of the University of Arkansas at Fayetteville during his three-day visit in April, 2000.
The authors gratefully acknowledge the referee's interest in improving the presentation of this paper.

of D is a principal prime of D'. Using this we show, for example, that if S is a saturated multiplicative set generated by nonzero principal primes of a Noetherian domain D, and if $(D_S)'$ is a UFD, then so is D'. Which is yet another variation of Nagata's theorem.

The proof of the main result and the statement of the consequences would become easier if we bring in some auxiliary terminology. We shall call $r,\ s \in D\backslash\{0\}$, *v-coprime* if $(r) \cap (s) = (rs)$. The reason for this terminology will become apparent after the next few lines. Let $r,\ s \in D\backslash\{0\}$. Then $(r) \cap (s) = (rs)(r,s)^{-1}$ is principal $\Leftrightarrow ((r,s)^{-1})^{-1} = (r,s)_v$ is principal and obviously, in this event, every generator of $(r,s)_v$ is a GCD of r and s. Thus $(r) \cap (s) = (rs) \Leftrightarrow (r,s)_v = D$, and in this case we say that r and s are v-coprime in D. Obviously r and s being v-coprime in D does not mean that r and s are v-coprime in a ring extension E of D (for instance, take $2, X$ in $D = \mathbf{Z}[X]$ and $E = \mathbf{Z}[X/2]$). Yet it is easy to see that $D \subseteq E$ is an R_2-stable extension of domains if, and only if, the event of r, s being v-coprime in D implies that r, s are v-coprime in E (that is if $aE :_E b = aE$ for each $a, b \in D \setminus \{0\}$ with $aD :_D b = aD$). Therefore, $D \subseteq E$ is an R_2-stable extension of domains if and only if every regular sequence of length two of D is a regular sequence of E. Using a construction of [**DHLRZ**], we show that if a, b are two v-coprime elements of a domain D such that $aD + bD \neq D$, there exists a domain extension E of D such that a, b are v-coprime in E, but a, b are not v-coprime in E'. Let us recall from [**AAZ1**], that a saturated multiplicative set S of D is called a *splitting set* if for all $d \in D\backslash\{0\}$ we have $d = st$, where $s \in S$ and t is v-coprime to S (i.e. is v-coprime to every member of S). If S is a splitting set of D, then the set $T = \{t \in D; t \text{ is } v\text{-coprime to } S\}$ is a splitting set called the *m-complement* of S. Let us also recall from [**AZ**], that $d \in D\backslash\{0\}$ is an *(lcm) extractor* if for all $x \in D$, $(d) \cap (x)$ is principal, and that divisors of products of extractors are again extractors. Therefore, an lcm splitting set is a splitting set consisting of extractors. Moreover, by Corollary 2.5 of [**AZ1**], a saturated multiplicative set S, of D, generated by extractors is an lcm splitting set if and only if every prime ideal that is disjoint with S contains an element σ that is v-coprime to S.

Theorem 1. *Let $D \subseteq E$ be an R_2-stable extension of domains, where E is an overring of D. Let $S \subseteq D$ be a saturated multiplicative set and S' its saturation in E. If S is splitting (resp. lcm splitting) in D, then so is S' in E respectively.*

The proof follows from the following technical lemma, which goes slightly farther than the theorem.

Lemma 1. *Let D, E, S, S' be as above, assume that S is splitting in D, let T be the m-complement of S in D and T' the saturation of T in E. Then the following assertions hold.*

(a) Every nonzero element x of E is expressible as $x = (s/s')(t/t')$, where $s, s' \in S$ and $t, t' \in T$, with $s/s' \in E$ and $t/t' \in E$.

(b) $S' = \{ws/s';\ s, s' \in S, s/s' \in E, w \in U(E)\}$ and $T' = \{wt/t';\ t, t' \in T, t/t' \in E, w \in U(E)\}$.

(c) S' is splitting in E with m-complement T'.

(d) If S is lcm splitting in D, then $S' = SU(E)$.

(e) If S is lcm splitting in D, then so is S' in E, hence E_T is a GCD-domain.

Proof. (a) Let $x \in E \setminus \{0\}$. We can write $x = st/s't'$ with $s, s' \in S$ and $t, t' \in T$. So t' divides st in E, hence t' divides t in E, that is $t/t' \in E$, because t',s are v-coprime not only in D but also in E, by R_2-stableness. Similarly, $s/s' \in E$.

(b), (c) follow from (a), because whenever $s/s', t/t' \in E$, with $s, s' \in S$, $t, t' \in T$, they are v-coprime in E.

(d) follows from (b), because in this case every two elements in S have an LCM belonging to S.

(e) follows from (d), [**AAZ**, Proposition 2.4] and the fact that every two elements in S have an LCM in D (and E). □

When $S = D \setminus \{0\}$, the last part of the previous lemma gives the following

Corollary 1. *Let $D \subseteq E$ be an R_2-stable extension of domains, where E is an overring of D. If D is a GCD domain, then so is E.*

Now to see that the above theorem does apply to some of the familiar types of overrings of D, we need to recall some deønitions. If $D \subseteq E$ is an extension of domains, E is called *t-linked* over D, if for all ønitely generated nonzero ideals I of D, $I^{-1} = D$ implies that $(IE)^{-1} = E$ [**DHLZ**] (see also [**U**], where the t-linked extensions are called G_2-stable extensions). Clearly every t-linked overring of D is R_2-stable over D, but what is interesting is that, according to [**DHLZ**, Corollary 2.3], the complete integral closure D^* of D is t-linked over D and so is, among many others, the pseudo integral closure of D. Here, by the pseudo integral closure of D we mean the ring $\overline{D} = \bigcup(I_v : I_v)$, where I ranges over nonzero ønitely generated ideals of D. It is easy to see that $D' \subseteq \overline{D} \subseteq D^*$. The ring $\overline{D}$ was called pseudo integral closure in [**AHZ**], where it was studied somewhat systematically. (The pseudo integral elements have been studied before under some less impressive names such as regular integral elements [**B**, Ch. 7, Sect. 1, Exercise 30], and in not so much detail.) For other t-linked overrings of D the reader may consult [**DHLZ**], where it is also mentioned that if D is Noetherian, then the integral closure D' of D is a t-linked overring of D (see Proposition 2 for another case when D' of D is t-linked over D). To justify all the above deønitions we bring in the following statement.

Proposition 1. *Let D be a domain, $p \in D$ a prime element such that $\cap_n p^n D = 0$ and E an R_2-stable overring of D. Then p is a prime element or a unit of E.*

Proof. Assume that p is a nonunit of E. By [**AAZ1**, Proposition 1.6], the set $S = \{wp^n;\ w \in U(D),\ n \geq 0\}$ is an lcm splitting set of D. By Lemma 1, its saturation in E, $S' = \{wp^n;\ w \in U(E),\ n \geq 0\}$, is also lcm splitting. So p is an irreducible extractor in E, that is a prime element. □

Corollary 2. *If D is a domain and $p \in D$ a prime element such that $\cap_n p^n D = 0$, then p is also prime in $\overline{D}$ and D^*. In particular, if D is a Noetherian domain, then any nonzero principal prime of D extends to a principal prime to D'.*

Proof. Let E be $\overline{D}$ and D^*. As D_{pD} is a DVR, it contains E, hence p is a nonunit of E. Apply Proposition 1 and the fact that E is t-linked over D. □

Another case (that caused this investigation) when D' is R_2-stable over D, is that of the almost GCD domains. Let us recall from [**Z**], that an integral domain D is called an *almost GCD* (AGCD) domain if for each pair $x, y \in D$, there exists a positive integer $n = n(x, y)$ such that $(x^n) \cap (y^n)$ is principal. It was shown in Lemma 3.5 and Theorem 3.1 of [**Z**], that if D is an AGCD domain, then D' is

R_2-stable over D and $D \subseteq D'$ is a root extension, that is for each $x \in D'$, $x^n \in D$ for some n. Let us recall from [**ADR**], that an integral domain D is called *root closed* (in its quotient field K), if whenever $x \in K$ and $x^n \in D$ for some n, then $x \in D$. Also, the *root closure* of a domain D is the smallest root closed overring of D. In particular, if $D \subseteq E$ is a root extension (that is, every element of E has some power in D), where E is a root closed overrring of D, then E is the root closure of D. In this case, we have

Proposition 2. *Let D be a domain and E its root closure. If $D \subseteq E$ is a root extension, then it is R_2-stable. In particular, D' is R_2-stable over D, if D is an AGCD domain.*

Proof. If $x, y \in D \setminus \{0\}$ are v-coprime in D and $a \in E$ is a common multiple of them in E, there exists n such that $a^n \in D$ (because E is a root extension of D) and a^n is a common multiple of x^n, y^n in D. Then $x^n y^n$ divides a^n in D, hence xy divides a in E, because $(a/xy)^n \in E$ and E is root closed. □

It was also shown in [**Z**, Theorem 3.9], that D is an integrally closed AGCD domain if and only if for each finitely generated nonzero ideal I there is a positive integer n such that $(I^n)_v$ is principal. Thus an almost factorial domain of Storch [**S**], which is nothing but a Krull domain with torsion divisor class group, is an almost GCD domain.

Theorem 1 can lead to a number of Nagata type theorems, but here we shall be concerned with only those that are expressible in most general terms. Yet before we start even that, we need to recall a few facts. First let us note that over a coherent domain every pseudo integral element is in fact integral. This follows from the fact that if I is a nonzero finitely generated ideal of a coherent domain D, then $I_v = (I^{-1})^{-1}$ is finitely generated. Thus the integral closure of a coherent domain is its pseudo integral closure and hence is t-linked over D.

Corollary 3. (Nagata type Theorems) *Let S be an lcm splitting multiplicative set in a coherent domain D. If $(D_S)'$ is a GCD domain, then so is D'. Consequently, if D is Noetherian and S is a saturated multiplicative set generated by nonzero principal primes of D and if $(D_S)'$ is a UFD, then so is D'.*

Proof. The proof depends upon the fact that, in each case $(D_S)' = D'_{S'}$, where S' is the saturation of S in D', and that, in each case, S' is an lcm splitting multiplicative set in D'. In the coherent case the result follows, say from [**GP**, Theorem 3.1] or [**AAZ**, Theorem 4.3]. In the Noetherian case the actual theorem of Nagata can be used, once we note that by Corollary 2, every nonzero principal prime of Noetherian D extends to a principal prime of D'. □

Corollary 4. *Let S be an lcm splitting multiplicative set in a coherent domain D. If $(D_S)'$ is an AGCD domain, then so is D'. Consequently, if D is Noetherian, S is a saturated multiplicative set of D generated by nonzero principal primes and $(D_S)'$ is an almost factorial domain, then so is D'.*

Proof. We apply Theorems 4.1 and 4.3 of [**AAZ**]. Now the second part follows, because D' is a Krull domain. □

We close this paper giving some examples.

Example 1. The extension of UFDs $\mathbf{Z}[X] \subseteq \mathbf{Z}[X/2]$ satisfies the conclusion of Theorem 1 for every multiplicative set (cf. [**CMZ**, Corollary 1.2]), but, obviously,

$\mathbf{Z}[X/2]$ is not R_2-stable over $\mathbf{Z}[X]$. Also, X is prime in $\mathbf{Z}[X]$, but not prime in $\mathbf{Z}[X/2]$.

The hypothesis $\cap_n p^n D = 0$ is essential in Proposition 1, as the following example shows.

EXAMPLE 2. Let $D = \mathbf{Z} + X\mathbf{Q}(i)[X]$ and $p = 2$. Then, $D' = \mathbf{Z}[i] + X\mathbf{Q}(i)[X] = A[i]$, $D/pD \simeq \mathbf{Z}/2\mathbf{Z}$ and $D'/pD' \simeq \mathbf{Z}[i]/2\mathbf{Z}[i]$. So, p is prime in D, but not prime in D'. Also, D' is a Bezout domain, cf. [**CMZ**, Corollary 4.13], so D' is R_2-stable over D, cf. [**DHLRZ**, Theorem 2.4]. Moreover, $D^* = \mathbf{Q}(i)[X]$, so $pD^* = D^*$.

In [**DHLRZ**], a domain E such that E' is not R_2-stable over E is constructed. So, there exist two v-coprime elements $a, b \in E$ which are not v-coprime in E'. To introduce an ad-hoc terminology, let us say that the elements a, b are *terminating v-coprime* (tv-coprime) in E, if a, b are v-coprime in E but they are not v-coprime in E'. More precisely, in [**DHLRZ**, Example 4.1], it is shown that if D is a domain of characteristic $\neq 2$ and Y_1, Y_2 are indeterminates, then Y_1, Y_2 are tv-coprime in some domain extension of $D[Y_1, Y_2]$. Note that Y_1, Y_2 are v-coprime but not comaximal in $D[Y_1, Y_2]$ (we say that two elements a, b of a domain B are comaximal if $aB + bB = B$). Using the construction of [**DHLRZ**, Example 4.1], we may give the following generalization.

PROPOSITION 3. *Let D be a domain and a, b two v-coprime nonzero elements of D. Then a, b are not comaximal in D if and only if a, b are tv-coprime in some domain extension of D.*

Proof. The "if" part is obvious. For proving the converse, assume that $aD + bD \neq D$. Now, we follow the pattern of [**DHLRZ**, Example 4.1], that is, we consider the subring $E = D + XI$ of $D[X]$, where $I = aD[X] + (1 - bX)D[X]$ (note that in [**DHLRZ**], the ring $E = D + X(a, 2 - bX)D[X]$ was used). We notice that E has the following pullback description. Let us consider the direct product ring $G = D \times (D/aD)[1/\bar{b}]$, where $\bar{b}$ is the class of b modulo aD. Since $\bar{b}$ is a non zerodivisor in D/aD, $D/aD \subseteq D/aD[1/\bar{b}]$ and this inclusion is proper because a, b are not comaximal. From $D[X]$ to G, we consider the epimorphism q with kernel XI, given by $q(f(X)) = (f(0), f(1/\bar{b}))$. This epimorphism is obtained applying the Chinese Remainder Theorem for the comaximal ideals $XD[X]$ and I of $D[X]$. Let us consider D as a subring of G via the monomorphism r obtained restricting q to D. Now, E is the pullback of the diagram composed of $D[X], q, G, D$, that is, $E = q^{-1}(D)$. In particular, $D[X]$ is an overring of E. For showing that a, b are tv-coprime in E, we follow the plan of [**DHLRZ**, Example 4.1], making the computations in the pullback. So, we claim that: 1) a, b are v-coprime in E, 2) $aX, bX \in E'$ and 3) $X \notin E'$. To prove 1), let $f \in (aE + bE)^{-1}$. Since $D[X]$ is R_2-stable (flat) over D, $f \in D$. Since $bf \in E$, $q(bf) = (0, \bar{b}f(1/\bar{b})) \in D$ (via r !), that is $\bar{b}f(1/\bar{b}) = 0$, so $f(1/\bar{b}) = 0$, hence $f \in E$. To prepare for 2), we notice that an element $h \in D[X]$ is integral over E if and only if $q(h)$ is integral over D. So, for 2), it suffices to see that $q(aX) = (0, 0)$ and $q(bX) = (0, 1)$ are integral over D, which is clear. For 3), we note that $q(X) = (0, 1/\bar{b})$ is not integral over D, so $X \notin E'$. □

Consequently, if $D_0 = \mathbf{Z}[Y] + X(2, 1 - XY)\mathbf{Z}[X, Y]$, then D_0' is not R_2-stable over D_0. We shall use this ring for our next example.

EXAMPLE 3. Let D_0 be as above and W an indeterminate. We set $E = D_0[W]$, $p = 2 + YW$ and $S = \{p^n;\ n \geq 0\}$. As shown in the proof of Proposition 3, p is prime in E and p is not prime in E'. Also, $\cap_n p^n E = 0$. So, S is an lcm splitting set in E (cf. [**AAZ1**, Proposition 1.6]), but not lcm splitting in E'. Indeed, p is irreducible in E', because $U(E') \subseteq U(\mathbf{Z}[X,Y,W]) = \{1,-1\}$, so p is not an extractor in E'.

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